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Abstract. Here, we consider the set $\mathbf{F}^n$ of formulas with modal degree $k(\leq n)$ and having only propositional variables $p_1, \cdots, p_m$ in modal logic **S4**. An exact model M for $\mathbf{F}^n$ is one of the simplest Kripke models satisfying $M \models A \Leftrightarrow \mathbf{S4} \vdash A$ for any $A \in \mathbf{F}^n$. Therefore, the model is useful to investigate the provability of formulas in **S4**. Moss [Mos07] constructed a Kripke model which can be shown to be exact for $\mathbf{F}^n$. However, his construction depends on the provability of **S4**. Here, we construct an exact model for $\mathbf{F}^n$ without using the provability of **S4**.

1 Introduction

Formulas are constructed from $\bot$ (contradiction) and the propositional variables $p_1, p_2, \cdots$ by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and $\Box$ (necessitation). We use upper case Latin letters, $A, B, C, \cdots$, with or without subscripts, for formulas. Also, we use Greek letters, $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. The *depth* $d(A)$ of a formula A is defined as

$d(p_i) = d(\bot) = 0,$
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\},$
$d(\Box B) = d(B) + 1.$

Let **ENU** be an enumeration of the formulas. For a non-empty finite set Γ of formulas, the expressions $\bigwedge \Gamma$ and $\bigvee \Gamma$ denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \quad \text{and} \quad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

respectively, where $\{A_1, \cdots, A_n\} = \Gamma$ and A_i occurs earlier than A_{i+1} in **ENU**. Also, the expressions $\bigwedge \emptyset$ and $\bigvee \emptyset$ denote the formulas $\bot \supset \bot$ and $\bot$, respectively.

The set of propositional variables $p_1, \cdots, p_m$ $(m \geq 1)$ is denoted by **V** and the set of formulas constructed from **V** and $\bot$ is denoted by **F**. Also, for any $n = 0, 1, \cdots$, we define $\mathbf{F}^n$ as $\mathbf{F}^n = \{A \in \mathbf{F} \mid d(A) \leq n\}$. In the present paper, we mainly treat the set $\mathbf{F}^n$.

By **S4**, we mean the sequent system defined by Ohnishi and Matsumoto [OM57]. Below, we introduce this system.

A *sequent* is the expression $(\Gamma \to \Delta)$. We often refer to $\Gamma \to \Delta$ as $(\Gamma \to \Delta)$ for brevity and refer to

$$A_1, \cdots, A_i, \Gamma_1, \cdots, \Gamma_j \to \Delta_1, \cdots, \Delta_k, B_1, \cdots, B_\ell$$

as

$$\{A_1, \cdots, A_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \cdots, B_\ell\}.$$

We use upper case Latin letters $X, Y, Z, \cdots$, with or without subscripts, for sequents. The *antecedent* $\mathbf{ant}(\Gamma \to \Delta)$ and the *succedent* $\mathbf{suc}(\Gamma \to \Delta)$ of a sequent $\Gamma \to \Delta$ are defined as

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \to \Delta) = \Delta,$$

respectively. Also, for a sequent X and a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) \neq \emptyset \\ \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) = \emptyset \end{cases}$$

and

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

For a finite set S of formulas or sequents, the expression $\#(S)$ denotes the number of elements in S.

[OM57] defined the system by adding the following two inference rules to the sequent system **LK** given by Gentzen [Gen35] for the classical propositional logic:

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to) \qquad\qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box).$$

Here, we do not use $\neg$ as a primary connective, so we use the additional axiom $\perp \to$ instead of the inference rules $(\neg \to)$ and $(\to \neg)$. We write $X \in \mathbf{S4}$ if X is provable in $\mathbf{S4}$. [OM57] proved that this system enjoys a cut-elimination theorem:

Lemma 1.1 ([OM57]) *If $X \in \mathbf{S4}$, then there exists a cut-free proof figure for X in $\mathbf{S4}$.*

We use $A \equiv B$ instead of $\to (A \supset B) \wedge (B \supset A) \in \mathbf{S4}$. Also, for any two equivalence classes $[A]$ and $[B]$ in $\mathbf{F}/\equiv$, we use $[A] \leq [B]$ instead of $A \to B \in \mathbf{S4}$. Thus, structure $\langle \mathbf{F}^n/\equiv, \leq\rangle$ expresses the mutual relation of formulas.

A *Kripke model* is a structure $\langle W, R, P\rangle$ where W is a non-empty set, R is a binary relation on W, and P is a mapping from the set of propositional variables to 2^W. We extend, as usual, the domain of P to include all formulas. We call P a *valuation* and a member of W a *world*. For a Kripke model $M = \langle W, R, P\rangle$, and for a world $\alpha \in W$, we often write $(M, \alpha) \models A$ and $M \models A$ instead of $\alpha \in P(A)$ and $P(A) = W$, respectively.

Let S be a set of formulas closed under $\supset$ and $\wedge$. We say that a Kripke model $M = \langle W, R, P\rangle$ is *exact for S* if the following two conditions hold:

- for any $A \in S$, $M \models A$ if and only if $\to A \in \mathbf{S4}$,
- $\{P(A) \mid A \in S\} = 2^W$.

This model was introduced in de Bruijn [Bru75]. The following lemma is observed easily; therefore, exact models are useful to investigate the structure $\langle \mathbf{F}^n/\equiv, \leq\rangle$.

Lemma 1.2 *Let $\langle W, R, P\rangle$ be an exact model for $\mathbf{F}^n$. Then the mapping P^* from $\mathbf{F}^n/\equiv$ to 2^W defined as*

$$P^*([A]) = P(A)$$

is an isomorphism and the structure $\langle \mathbf{F}^n/\equiv, \leq\rangle$ is isomorphic to the structure $\langle 2^W, \subseteq\rangle$.

As we mentioned, the Kripke model constructed in [Mos07] is an exact model for $\mathbf{F}^n$. The construction depends on the provability of $\mathbf{S4}$. On the other hand, [Sas09] constructed a way to list all exact models for $\mathbf{F}^n$ by using exact sets for $\mathbf{F}^n$. The construction of the way does not depend on the provability of $\mathbf{S4}$. Here, we directly construct an exact model for $\mathbf{F}^n$ without using the provability of $\mathbf{S4}$.

In the next section, we introduce an exact set and we give the way constructed in [Sas09]. In section 3, we construct an exact set for $\mathbf{F}^n$, and using a result in [Sas09], we obtain an exact model for $\mathbf{F}^n$.

2 Exact sets and exact models for $\mathbf{F}^n$

In the present section, we introduce an exact set for $\mathbf{F}^n$ and a way to list all exact models for $\mathbf{F}^n$ following [Sas09]. Every lemma in the present section has been proved in [Sas09].

First, we introduce the three sets $\mathbf{G}(n)$, $\mathbf{G}^*(n)$, $\mathbf{ED}^n$ as follows. We can see the relation among these three sets in Definition 2.2. Also, we can treat the set $\mathbf{ED}^n$ as the set of formulas, which behave like elementary disjunctions in the classical propositional logic. In other words, $\mathbf{ED}^n$ satisfies the following two conditions:

- $\mathbf{F}^n/\equiv = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{ED}^n\}$,

- for subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}^n$,

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \text{ if and only if } \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}.$$

Definition 2.1 The sets $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ of sequents are defined inductively as follows.
$\mathbf{G}(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,
$\mathbf{G}^*(0) = \emptyset$,
$\mathbf{G}(k+1) = \bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}(X)$,
$\mathbf{G}^*(k+1) = \{X \in \mathbf{G}(k+1) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y))^\Box$ implies $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, for any $Y \in \mathbf{G}(k+1)\}$,
where for any $X \in \mathbf{G}(k)$,
$\mathbf{next}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}(n)), \Gamma \cap \Delta = \emptyset, \mathbf{for}(X) \in \Delta\}$,
$\mathbf{prov}(X) = \{Y \in \mathbf{next}^+(X) \mid Y \in \mathbf{S4}\}$,
$\mathbf{next}(X) = \mathbf{next}^+(X) - \mathbf{prov}(X)$.

Definition 2.2 We define the sets $\mathbf{ED}^n$ and $\mathbf{G}^*$ as follows:

$$\mathbf{ED}^n = \mathbf{G}(n) \cup \bigcup_{i=0}^{n-1} \mathbf{G}^*(i), \quad \mathbf{G}^* = \bigcup_{i=0}^{\infty} \mathbf{G}^*(i).$$

Concerning with $\mathbf{G}(n)$, we have the following lemma.

Lemma 2.3
(1) *None of the members in* $\mathbf{G}(n)$ *is provable in* $\mathbf{S4}$.
(2) *Let* X, Y *and* Z *be sequent in* $\mathbf{G}(n_1)$, $\mathbf{G}(n_2)$ *and* $\mathbf{G}(n_3)$, *respectively. Then*

$$\Box\mathbf{for}(X) \in \mathbf{suc}(Y) \textit{ and } \Box\mathbf{for}(Y) \in \mathbf{suc}(Z) \textit{ imply } \Box\mathbf{for}(X) \in \mathbf{suc}(Z),$$

(3) *For any* $X \in \mathbf{G}(n)$, $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i))$ *and* $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$,
(4) *For any* $X, Y \in \mathbf{G}(n)$, $\mathbf{for}(\mathbf{next}(X)) \to \mathbf{for}(X) \in \mathbf{S4}$.

In Definition 2.1, we use the provability of $\mathbf{S4}$ to define $\mathbf{prov}(X)$ for $X \in \mathbf{G}(n)$. [Sas09] also gave the set without using the provability of $\mathbf{S4}$ as follows.

Definition 2.4 For any $X \in \mathbf{G}(n)$, we define $\mathbf{prov}_1(X)$, $\mathbf{prov}_2(X)$ and $\mathbf{prov}_3(X)$ as follows:
$\mathbf{prov}_1(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box\}$,

$\mathbf{prov}_2(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(Y)$,
$\Box\mathbf{for}(\{Z \in \mathbf{next}(Z_\ominus) \mid (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box\}) \subseteq \Gamma$ for some $Z_\ominus \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)\}$,

$\mathbf{prov}_3(X) = \{(\Box\mathbf{for}(Y), \Gamma \to \Delta, \Box\mathbf{for}(Z)) \in \mathbf{next}^+(X) \mid Y, Z \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(Z))^\Box\}$.

Lemma 2.5 *For any* $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\mathbf{prov}(X) = \mathbf{prov}_1(X) \cup \mathbf{prov}_2(X) \cup \mathbf{prov}_3(X).$$

For a sequent $X \in \mathbf{G}(n+2)$, there exists a sequent $Y \in \mathbf{G}(n+1) - \mathbf{G}^*(n+1)$ such that $X \in \mathbf{next}(Y)$, and similarly, there exists a sequent $Z \in \mathbf{G}(n) - \mathbf{G}^*(n)$ such that $Y \in \mathbf{next}(Z)$. Here, we can see a relation between X and Z. In order to express this relation, we introduce some notions. We note that, by the Lemma 2.3(3), every sequent in $\mathbf{G}(n)$ consists of the members in the set $\mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i))$.

Definition 2.6 For any $X \in \mathbf{G}(n)$ and for any k, we define $X(k)$ as

$$X(k) = (\mathbf{ant}(X) \cap \mathbf{V} \cup \bigcup_{i=0}^{k-1} \Box\mathbf{for}(\mathbf{G}(i)) \to \mathbf{suc}(X) \cap \mathbf{V} \cup \bigcup_{i=0}^{k-1} \Box\mathbf{for}(\mathbf{G}(i))).$$

Definition 2.7 For any $X \in \mathbf{G}(n)$, we define the sets $X\Downarrow$ inductively as follows:
(1) $X \in X\Downarrow$,
(2) if $Y \in \mathbf{next}(Z)$ for some $Z \in X\Downarrow - \mathbf{G}^*$, then $Y \in X\Downarrow$.

Lemma 2.8 *For any $X \in \mathbf{G}(n)$ and for any $Y \in \mathbf{G}(k)$,*
(1) $n \neq 0$ *implies* $X(n-1) \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ *and* $X \in \mathbf{next}(X(n-1))$,
(2) $n > k$ *implies* $X(k) \in \mathbf{G}(k) - \mathbf{G}^*(k)$, $X \in X(k)\Downarrow$ *and* $\Box\mathbf{for}(X(k)) \in \mathbf{suc}(X)$,
(3) *the following three conditions are equivalent:*
(3.1) $\mathbf{ant}(Y) \subseteq \mathbf{ant}(X)$ *and* $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$,
(3.2) $n \geq k$ *and* $Y = X(k)$,
(3.3) $X \in Y\Downarrow$.

Below, we introduce an exact set for $\mathbf{F}^n$ and show results in [Sas09] concerning with exact models for $\mathbf{F}^n$.

Definition 2.9
(1) A set $\mathcal{E}$ is said to be exact for $\mathbf{F}^n$ if the following three conditions hold:
(1.1) $\bigcup_{i=0}^{n} \mathbf{G}^*(i) \subseteq \mathcal{E} \subseteq \mathbf{G}^*$,
(1.2) for any $X \in \mathbf{ED}^n$, $\#(X\Downarrow \cap \mathcal{E}) = 1$,
(1.3) for any $X \in \mathcal{E}$ and for any $Y \in W_{\mathbf{E}}$, $XR_{\mathbf{E}}Y$ implies $Y \in \mathcal{E}$,
where $R_{\mathbf{E}} = \{(X,Y) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y) \text{ or } ((\mathbf{ant}(X))^{\Box}, (\mathbf{suc}(X))^{\Box}) = ((\mathbf{ant}(Y))^{\Box}, (\mathbf{suc}(Y))^{\Box})\}$.

(2) For an exact set $\mathcal{E}$ for $\mathbf{F}^n$, the Kripke model $\mathbf{EM}_{\mathcal{E}}$ is defined as

$$\mathbf{EM}_{\mathcal{E}} = \langle \mathcal{E}, R_{\mathcal{E}}, P_{\mathcal{E}} \rangle,$$

where $R_{\mathcal{E}} = \mathcal{E}^2 \cap R_{\mathbf{E}}$ and $P_{\mathcal{E}}(p_i) = \mathcal{E} \cap \{X \mid p_i \in \mathbf{ant}(X)\}$.

Lemma 2.10
(1) *For any exact set $\mathcal{E}$ for $\mathbf{F}^n$, $\mathbf{EM}_{\mathcal{E}}$ is an exact model for $\mathbf{F}^n$.*
(2) *For any exact model M for $\mathbf{F}^n$, there exists an exact set $\mathcal{E}$ for $\mathbf{F}^n$ such that M is isomorphic to $\mathbf{EM}_{\mathcal{E}}$.*
(3) *Every exact set for $\mathbf{F}^n$ is a subset of* $\bigcup_{i=0}^{n+2\#(\mathbf{ED}^n - W_{\mathbf{E}})} \mathbf{G}^*(i)$.
(4) *Let $\mathcal{E}$ be an exact set for $\mathbf{F}^n$. Then for any $A \in \mathbf{F}^n$,*

$$A \equiv \bigwedge \{\mathbf{for}(X(n)) \mid X \in \mathcal{E}, (\mathbf{EM}_{\mathcal{E}}, X) \not\models A\}.$$

By (2) of the above lemma and the exact model for $\mathbf{F}^n$ in [Mos07], we can see that there exists an exact set for $\mathbf{F}^n$.

3 A construction of an exact set for $\mathbf{F}^n$ without the provability of S4

In the present section, we construct an exact set for $\mathbf{F}^n$ without using the provability of **S4**. As a result, using Lemma 2.10(1), we obtain an exact model for $\mathbf{F}^n$. First, we construct the sequent $X^* \in X\Downarrow \cap \mathbf{G}^*$ for $X \in \mathbf{G}(n)$, and using X^*, we construct an exact set for $\mathbf{F}^n$.

Definition 3.1 Let X and $Y_\oplus$ be sequents in $\mathbf{G}(n)$ and $\mathbf{G}(n+1)$, respectively. Let Δ be a finite set of sequents. Then we define three sequents $\mathbf{n}(X,\Delta)$, $\mathbf{n}(X,Y_\oplus)$ and $\mathbf{n}(Y_\oplus)$ as follows.

$$\mathbf{n}(X,\Delta) = (\Box\mathbf{for}(\mathbf{G}(n) - (\{X\}\cup\Delta)), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\{X\}\cup\Delta)),$$

$$\mathbf{n}(X,Y_\oplus) = \mathbf{n}(X, \{\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_\oplus) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Z))^\Box\}),$$

$$\mathbf{n}(Y_\oplus) = \mathbf{n}(Y_\oplus, \{\mathbf{n}(X,Y_\oplus) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y_\oplus) \cap \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n))\}).$$

We note that if $\Delta \subseteq \mathbf{G}(n)$, then $\mathbf{n}(X,\Delta) \in \mathbf{next}^+(X)$. Also, by the following lemma, we can see that $\mathbf{n}(Y_\oplus) \in \mathbf{next}^+(Y_\oplus)$.

Lemma 3.2 ([Sas09]) *Let X and $Y_\oplus$ be sequents in $\mathbf{G}(n)$ and $\mathbf{G}(n+1)$, respectively. If $\Box\mathbf{for}(X) \in \mathbf{suc}(Y_\oplus)$, then $\mathbf{n}(X,Y_\oplus) \in \mathbf{next}(X)$.*

Definition 3.3 For any $X \in \mathbf{G}(n)$, we define the set $\mathbf{clus}(X)$ as follows.

$$\mathbf{clus}(X) = \{Y \in \mathbf{G}(n) \mid (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box\}.$$

We note that, by Lemma 2.3(3),

$$R_\mathcal{E} = \mathcal{E}^2 \cap \{(X,Y) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y) \text{ or } X \in \mathbf{clus}(Y)\}.$$

Definition 3.4 For a sequent $X \in \mathbf{G}(n)$, we define $\mathbf{mnext}^k(X)$ as follows.
(1) $\mathbf{mnext}^0(X) = X$,
(2) $\mathbf{mnext}^{k+1}(X) = \mathbf{n}(\mathbf{mnext}^k(X), \mathbf{G}(n+k))$.

By Lemma 2.5, we can see that $\mathbf{mnext}^k(X) \in \mathbf{G}(n+k)$ for any $X \in \mathbf{G}(0)$.
Our main purpose is to prove the following theorem.

Theorem 3.5
(1) *$\mathbf{G}^*(1)$ is an exact set for $\mathbf{F}^0$.*
(2) *For any $k \in \{1,2,\cdots\}$ and for any $X \in \mathbf{G}(k)$, we can define the sequent X^* inductively as*

$$X^* = \begin{cases} X & \text{if } X \in \mathbf{G}^*(k) \\ (\mathbf{n}(X))^* & \text{if } X \notin \mathbf{G}^*(k), \end{cases}$$

and the set

$$\mathbf{G}^* \cap (\{Z \mid \Box\mathbf{for}(Z) \in \mathbf{suc}((\mathbf{mnext}^{n+1}(\to \mathbf{V}))^*)\} \cup \mathbf{clus}((\mathbf{mnext}^{n+1}(\to \mathbf{V}))^*))$$

is an exact set for $\mathbf{F}^{n+1}$.

By Lemma 2.5, we have

$$\mathbf{G}^*(1) = \{\mathbf{n}(X,\emptyset) \mid X \in \mathbf{G}(0)\},$$

and therefore, we obtain Theorem 3.5(1). To prove Theorem 3.5(2), we need some lemmas.

Lemma 3.6 *Let X and $Y_\ominus$ be sequents in $\mathbf{G}(n+1)$ and $\mathbf{G}(n) - \mathbf{G}^*(n)$, respectively. Let Y be a sequent in $\mathbf{next}^+(Y_\ominus)$. If $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, then $Y \in \mathbf{next}(Y_\ominus)$.*

Proof. By Lemma 2.5 and Lemma 2.8(1), we have $X \notin \mathbf{prov}_1(X(n)) \cup \mathbf{prov}_2(X(n)) \cup \mathbf{prov}_3(X(n))$. Using $Y \in \mathbf{next}^+(Y_\ominus)$ and $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, it is not hard to see $Y \notin \mathbf{prov}_1(Y_\ominus) \cup \mathbf{prov}_2(Y_\ominus) \cup \mathbf{prov}_3(Y_\ominus)$. Using Lemma 2.5, we obtain $Y \in \mathbf{next}(Y_\ominus)$. ⊣

Lemma 3.7 *Let X be a sequent in $\mathbf{G}(n+1) - \mathbf{G}^*(n+1)$ and let Δ be a subset of $\mathbf{G}(n+1)$ satisfying $\mathbf{n}(X,\Delta) \in \mathbf{next}(X)$. Then*
(1) *$\Delta \subseteq \mathbf{clus}(X)$ implies $\mathbf{n}(X,\Delta) \in \mathbf{G}^*(n+2)$,*
(2) *$\Delta \subseteq \mathbf{G}^*(n+1) \cup \mathbf{clus}(X)$ implies either $\mathbf{n}(X,\Delta) \in \mathbf{G}^*(n+2)$ or $\mathbf{n}(\mathbf{n}(X,\Delta), \mathbf{clus}(\mathbf{n}(X,\Delta)) \in \mathbf{G}^*(n+3)$.*

Proof.

For (1). Suppose that $\mathbf{n}(X,\Delta) \notin \mathbf{G}^*(n+2)$. Then there exists a sequent $Y_{n+2} \in \mathbf{G}(n+2)$ such that $(\mathbf{ant}(\mathbf{n}(X,\Delta)))^{\Box} \subsetneq (\mathbf{ant}(Y_{n+2}))^{\Box}$. Using Lemma 2.3(3) and Lemma 2.8(1), we have either

$$(\mathbf{ant}(X))^{\Box} \subsetneq (\mathbf{ant}(Y_{n+2}(n)))^{\Box} \tag{1.1}$$

or

$$\mathbf{ant}(\mathbf{n}(X,\Delta)) \cap \Box\mathbf{for}(\mathbf{G}(n)) \subsetneq \mathbf{ant}(Y_{n+2}) \cap \Box\mathbf{for}(\mathbf{G}(n)). \tag{1.2}$$

We divide the cases.

The case that (1.1) holds. By (1.1), we have $Y_{n+2}(n) \in \mathbf{G}(n) - \mathbf{clus}(X)$, and therefore,

$$\Box\mathbf{for}(Y_{n+2}(n)) \in \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{clus}(X)) \subseteq (\mathbf{ant}(\mathbf{n}(X,\Delta)))^{\Box} \subsetneq (\mathbf{ant}(Y_{n+2}))^{\Box}. \tag{1.3}$$

On the other hand, by Lemma 2.8(1), we have $Y_{n+2} \in \mathbf{next}(Y_{n+2}(n))$ and $\Box\mathbf{for}(Y_{n+2}(n)) \in \mathbf{suc}(Y_{n+2})$, which is in contradiction with (1.3) and Lemma 2.3(3).

The case that (1.1) does not hold. We have (1.2) and

$$(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y_{n+2}(n)))^{\Box}. \tag{1.4}$$

By (1.2) and Lemma 2.3(3), there exists a sequent $Z \in \{X\} \cup \Delta \subseteq \mathbf{clus}(X)$ such that $\Box\mathbf{for}(Z) \in (\mathbf{ant}(Y_{n+2}))^{\Box}$. By Lemma 2.8(1), we have $Z(n-1) \in \mathbf{G}(n-1)$ and $Z \in \mathbf{next}(Z(n-1))$. Using Lemma 2.3(3), we have $\Box\mathbf{for}(\mathbf{next}(Z(n-1))) \cap \mathbf{clus}(X) = \{\Box\mathbf{for}(Z)\}$. Using $\Box\mathbf{for}(\mathbf{next}(Z(n-1))) - \mathbf{clus}(X) \subseteq (\mathbf{ant}(\mathbf{n}(X,\Delta)))^{\Box}$, we have

$$\Box\mathbf{for}(\mathbf{next}(Z(n-1))) \subseteq (\mathbf{ant}(\mathbf{n}(X,\Delta)))^{\Box} \cup \{\Box\mathbf{for}(Z)\} \subseteq (\mathbf{ant}(Y_{n+2}))^{\Box}. \tag{1.5}$$

On the other hand, by $\Box\mathbf{for}(Z(n-1)) \to \Box\mathbf{for}(Z) \in \mathbf{S4}$, $\Box\mathbf{for}(Z) \in \{X\} \cup \Delta \subseteq \mathbf{suc}(X)$, and Lemma 2.3(1), we have $\Box\mathbf{for}(Z(n-1)) \in \mathbf{suc}(X)$. Using Lemma 2.3(3) and (1.4) we have $\Box\mathbf{for}(Z(n-1)) \in (\mathbf{suc}(Y_{n+2}(n)))^{\Box} \subseteq (\mathbf{suc}(Y_{n+2}))^{\Box}$. Using (1.5) and Lemma 2.3(4), we have $Y_{n+2} \in \mathbf{S4}$, which is in contradiction with $Y_{n+2} \in \mathbf{G}(n+2)$ and Lemma 2.4(1).

For (2). For brevity's sake, we refer to X_{n+2} as $\mathbf{n}(X,\Delta)$. We note that $X_{n+2} = \mathbf{n}(X,\Delta) \in \mathbf{next}(X) \subseteq \mathbf{G}(n+2)$. Suppose that $X_{n+2} \notin \mathbf{G}^*(n+2)$. Then by (1),

$$\mathbf{n}(X_{n+2}, \mathbf{clus}(X_{n+2})) \in \mathbf{next}(X_{n+2}) \text{ implies } \mathbf{n}(X_{n+2}, \mathbf{clus}(X_{n+2})) \in \mathbf{G}^*(n+3).$$

Using Lemma 2.5, we have only to show

$$\mathbf{n}(X_{n+2}, \mathbf{clus}(X_{n+2})) \in \mathbf{next}^+(X_{n+2}) - (\mathbf{prov}_1(X_{n+2}) \cup \mathbf{prov}_2(X_{n+2}) \cup \mathbf{prov}_3(X_{n+2})).$$

It is not hard to see that

$$\mathbf{n}(X_{n+2}, \mathbf{clus}(X_{n+2})) \in \mathbf{next}^+(X_{n+2}) - (\mathbf{prov}_1(X_{n+2}) \cup \mathbf{prov}_3(X_{n+2})).$$

We show

$$\mathbf{n}(X_{n+2}, \mathbf{clus}(X_{n+2})) \notin \mathbf{prov}_2(X_{n+2}). \tag{2.1}$$

Suppose that (2.1) does not hold. Then there exist sequents $Y_{n+2} \in \mathbf{G}(n+2)$ and $Z \in \mathbf{G}(n) - \mathbf{G}^*(n)$ such that

$$Y_{n+2} \in \mathbf{clus}(X_{n+2}), \tag{2.2}$$

$$\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_{n+2}), \tag{2.3}$$

$$\Box\mathbf{for}(\{Z_{n+2} \in \mathbf{next}(Z) \mid (\mathbf{ant}(Y_{n+2}))^{\Box} \subseteq (\mathbf{ant}(Z_{n+2}))^{\Box}\}) \subseteq \mathbf{ant}(\mathbf{n}(X_{n+2}, \mathbf{clus}(X_{n+2}))). \tag{2.4}$$

By (2.2) and Lemma 2.3(3), we have $(\mathbf{suc}(Y_{n+2}))^{\Box} = (\mathbf{suc}(X_{n+2}))^{\Box}$. Using (2.3), we have

$$\Box\mathbf{for}(Z) \in (\mathbf{suc}(X_{n+2}))^{\Box} \cap \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)) \subseteq \Box\mathbf{for}((\{X\} \cup \Delta) - \mathbf{G}^*(n)) \subseteq \Box\mathbf{for}(\mathbf{clus}(X));$$

and thus, $Z \in \mathbf{clus}(X)$. We define Z_{n+2} as

$$Z_{n+2} = \mathbf{n}(Z, \{X\} \cup \Delta).$$

Then by $\Box\mathbf{for}(Z) \in (\mathbf{suc}(X_{n+2}))^{\Box}$ and $Z \in \mathbf{clus}(X)$, we have

$$(\mathbf{ant}(Z_{n+2}))^{\Box} = (\mathbf{ant}(X_{n+2}))^{\Box}. \tag{2.5}$$

Using (2.2), we have $(\mathbf{ant}(Z_{n+2}))^{\Box} = (\mathbf{ant}(Y_{n+2}))^{\Box}$, and using Lemma 3.6, we have $Z_{n+2} \in \mathbf{next}(Z_n)$. Using (2.4), we have

$$\Box\mathbf{for}(Z_{n+2}) \in \mathbf{ant}(\mathbf{n}(X_{n+2}, \mathbf{clus}(X_{n+2}))).$$

Using Lemma 2.3(3), we have $Z_{n+2} \notin \mathbf{clus}(X_{n+2})$, which is in contradiction with (2.5). ⊣

Lemma 3.8 *For any* $X \in \mathbf{G}^*(n+1)$,

(1) $\Box\mathbf{for}(Y_{\ominus}) \in \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n))$ *implies* $\#(\mathbf{next}(Y_{\ominus}) \cap \mathbf{clus}(X)) = 1$,

(2) $\mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n)) \subseteq \Box\mathbf{for}(\mathbf{clus}(X(n)) \cup \mathbf{G}^*(n))$,

(3) *for any* $Y, Z \in \mathbf{G}^*$, $\Box\mathbf{for}(Y) \in \mathbf{suc}(X) \cup \Box\mathbf{for}(\mathbf{clus}(X))$ *and* $Y R_{\mathbf{E}} Z$ *imply* $\Box\mathbf{for}(Z) \in \mathbf{suc}(X) \cup \Box\mathbf{for}(\mathbf{clus}(X))$.

Proof.

For (1). By Lemma 3.2, we have $\mathbf{n}(Y_{\ominus}, X) \in \mathbf{next}(Y_{\ominus}) \cap \mathbf{G}(n+1)$. Using Lemma 2.4(3), we have $(\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(\mathbf{n}(Y_{\ominus}, X)))^{\Box}$. Using $X \in \mathbf{G}^*(n+1)$, we have $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(\mathbf{n}(Y_{\ominus}, X)))^{\Box}$, and therefore, $\mathbf{n}(Y_{\ominus}, X) \in \mathbf{clus}(X)$. Using Lemma 2.9, we obtain (1).

For (2). We note that $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(\mathbf{n}(Y_{\ominus}, X)))^{\Box}$ implies $(\mathbf{ant}(X(n)))^{\Box} = (\mathbf{ant}(Y_{\ominus}))^{\Box}$. Hence, using the proof of (1), we obtain (2).

For (3). If $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$, then by Lemma 2.3(2) and Lemma 2.5, we have $\Box\mathbf{for}(Z) \in \mathbf{suc}(X)$. If $Y \in \mathbf{clus}(X)$, then we have that $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y)$ implies $\Box\mathbf{for}(Z) \in \mathbf{suc}(X)$ and that $Z \in \mathbf{clus}(Y)$ implies $Z \in \mathbf{clus}(X)$. ⊣

Definition 3.9 Let X be sequents in $\mathbf{G}(n+1)$. Two sets $\mathbf{suc}(X)^{\circ}$ and $\mathbf{suc}(X)^*$ and a number $\#(X)$ are defined as follows:

$$\mathbf{suc}(X)^{\circ} = \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)), \quad \mathbf{suc}(X)^* = \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}^*(n)).$$

$$\#(X) = 2\#(\mathbf{suc}(X_{n+1})^{\circ}) + \#(\mathbf{suc}(X_{n+1})^*).$$

Lemma 3.10 *For any* $X \in \mathbf{G}(n+1)$,

(1) $\mathbf{n}(X) \in \mathbf{next}(X)$,

(2) $\#(X) > \#(\mathbf{n}(X))$.

Proof.

For (1). It is easily seen that $\mathbf{n}(X) \in \mathbf{next}^+(X)$. By Lemma 2.5, it is not hard to see that, for any $Y_{\ominus}$, $\Box\mathbf{for}(Y_{\ominus}) \in \mathbf{suc}(X)^{\circ}$ implies $(\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(\mathbf{n}(Y_{\ominus}, X)))^{\Box}$, and therefore, we have $\mathbf{n}(X) \notin \mathbf{prov}_1(X)$. Also, it is not hard to see that $\mathbf{n}(X) \notin \mathbf{prov}_2(X)$. Therefore, by Lemma 2.5, it is sufficient to show $\mathbf{n}(X) \notin \mathbf{prov}_3(X)$.

Suppose that $\mathbf{n}(X) \in \mathbf{prov}_3(X)$. Then there exist two sequents Y' and Y'' in $\mathbf{G}^*(n+1)$ such that $\Box\mathbf{for}(Y') \in \mathbf{ant}(\mathbf{n}(X))$, $\Box\mathbf{for}(Y'') \in \mathbf{suc}(\mathbf{n}(X))$ and $(\mathbf{ant}(Y'))^{\Box} = (\mathbf{ant}(Y''))^{\Box}$. By Lemma 2.8(2), we have $\Box\mathbf{for}(Y'(n)) \in (\mathbf{suc}(Y'))^{\Box} = (\mathbf{suc}(Y''))^{\Box}$. Also, by $\Box\mathbf{for}(Y'') \in \mathbf{suc}(\mathbf{n}(X))$, we have $\mathbf{n}(Y''(n), X) = Y''$. Therefore, we have $\Box\mathbf{for}(Y'(n)) \in (\mathbf{suc}(X))^{\Box}$, and thus,

$$\Box\mathbf{for}(\mathbf{n}(Y'(n), X)) \in (\mathbf{suc}(\mathbf{n}(X)))^{\Box}. \tag{1.1}$$

By Lemma 2.3(3), we have $(\mathbf{suc}(Y'(n)))^{\Box} = (\mathbf{suc}(Y''(n)))^{\Box}$, and thus,

$$(\mathbf{suc}(\mathbf{n}(Y'(n), X)))^{\Box} = (\mathbf{suc}(\mathbf{n}(Y''(n), X)))^{\Box} = (\mathbf{suc}(Y''))^{\Box} = (\mathbf{suc}(Y'))^{\Box}.$$

Hence, $\mathbf{n}(Y'(n), X) = Y'$. Using (1.1),

$$\Box\mathbf{for}(Y') \in (\mathbf{suc}(\mathbf{n}(X)))^{\Box},$$

which is in contradiction with Lemma 2.3(3) and $\Box\mathbf{for}(Y') \in \mathbf{ant}(\mathbf{n}(X))$.

For (2). By (1), we have

$$\#(\mathbf{suc}(X)^\circ) = \#(\mathbf{suc}(\mathbf{n}(X))^\circ) + \#(\mathbf{suc}(\mathbf{n}(X))^*).$$

Therefore, if $\mathbf{suc}(X)^* \neq \emptyset$, then

$$\#(X) = 2\#(\mathbf{suc}(X)^\circ) + \#(\mathbf{suc}(X)^*) > 2\#(\mathbf{suc}(X)^\circ) \geq 2\#(\mathbf{suc}(\mathbf{n}(X))^\circ) + \#(\mathbf{suc}(\mathbf{n}(X))^*) = \#(\mathbf{n}(X)).$$

Suppose that $\mathbf{suc}(X_{n+1})^* = \emptyset$. Then we note that there exits a sequent $Y_\ominus \in \mathbf{G}(n) - \mathbf{G}^*(n)$ such that $\Box\mathbf{for}(Y_\ominus) \in \mathbf{suc}(X)^\circ$ and

$$\{Z_\ominus \in \mathbf{G}(n) \mid \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(X), (\mathbf{ant}(Y_\ominus))^\Box \subseteq (\mathbf{ant}(Z_\ominus))^\Box\}$$

$$= \{Z_\ominus \in \mathbf{G}(n) \mid \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(X), (\mathbf{ant}(Y_\ominus))^\Box = (\mathbf{ant}(Z)_\ominus)^\Box\}.$$

By $\{Z_\ominus \in \mathbf{G}(n) \mid \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(X), (\mathbf{ant}(Y_\ominus))^\Box = (\mathbf{ant}(Z_\ominus))^\Box\}) \subseteq \mathbf{clus}(Y_\ominus)$ and Lemma 3.7(1), we have $\mathbf{n}(Y_\ominus, X) \in \mathbf{G}^*(n+1)$. Hence,

$$\#(X) = 2\#(\mathbf{suc}(X)^\circ) + \#(\mathbf{suc}(X)^*) = 2\#(\mathbf{suc}(X)^\circ) > 2\#(\mathbf{suc}(\mathbf{n}(X))^\circ) + \#(\mathbf{suc}(\mathbf{n}(X))^*) = \#(\mathbf{n}(X)).$$

$\dashv$

By the above lemma, for any $X \in \mathbf{G}(n+1) - \mathbf{G}^*(n+1)$, we have $\mathbf{n}(X) \in \mathbf{G}(n+2)$, and therefore, we can define the sequent X^* as in Theorem 3.5(2).

Lemma 3.11 *For any* $X \in \mathbf{G}(n+1)$,

(1) $X^* \in X\Downarrow \cap \mathbf{G}^*$,

(2) *for any* $Y \in \mathbf{ED}^n$, $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ *implies* $\#(Y\Downarrow \cap \mathcal{E}(X^*)) = 1$,

(3) *for any* $Y \in \mathbf{ED}^n$, $\Box\mathbf{for}(Y) \in \mathbf{ant}(X)$ *implies* $\#(Y\Downarrow \cap \mathcal{E}(X^*)) = 0$,

where $\mathcal{E}(X^*) = \mathbf{G}^* \cap (\{Z \mid \ \Box\mathbf{for}(Z) \in \mathbf{suc}(X^*)\} \cup \mathbf{clus}(X^*))$.

Proof. We use an induction on $\#(X)$. Basis($\#(X) = 2$) is included Induction step by the following two reasons:

- in Induction step, we treat the case that $X \in \mathbf{G}^*(n+1)$,
- by Lemma 3.7(2), we have that $\mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n)) = \{\Box\mathbf{for}(X(n))\}$ implies $X \in \mathbf{G}^*(n+1)$.

Induction step. We divide the cases.

The case that $X \in \mathbf{G}^*(n+1)$. We have $X^* = X$ and (1). Suppose that $Y \in \mathbf{ED}^n$. If $Y \notin \mathbf{G}(n) - \mathbf{G}^*(n)$, then we have $\{Y\} = Y\Downarrow$, and therefore, we obtain the following two conditions:

- $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ implies $\#(Y\Downarrow \cap \mathcal{E}(X)) = 1$,
- $\Box\mathbf{for}(Y) \in \mathbf{ant}(X)$ implies $\#(Y\Downarrow \cap \mathcal{E}(X)) = 0$.

We assume that $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$.

We show (2). Suppose that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$. By $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and Lemma 3.8(2), there exists Y_{n+1} such that $\{Y_{n+1}\} = \mathbf{next}(Y) \cap \mathbf{clus}(X)$. Using the definition of $\mathbf{G}(n)$ and $X \in \mathbf{G}^*(n+1)$, we have $Y_{n+1} \in \mathbf{G}^*(n+1)$, and using Lemma 2.9, $\{Y_{n+1}\} = Y\Downarrow \cap \mathcal{E}(X)$.

We show (3). Suppose that $\Box\mathbf{for}(Y) \in \mathbf{ant}(X)$ and $Z \in Y\Downarrow \cap \mathcal{E}(X)$. If $Z \in Y\Downarrow \cap \mathbf{G}^* \cap \{Z \mid \Box\mathbf{for}(Z) \in \mathbf{suc}(X)\}$, then by $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$, we have $Z \in Y\Downarrow \cap \mathbf{G}^*(n) = \emptyset$. We assume that $Z \in Y\Downarrow \cap \mathbf{G}^* \cap \mathbf{clus}(X)$. Then using the definition of $\mathbf{G}(n)$, we have $Z \in \mathbf{G}^*(n+1)$. Using Lemma 2.8, $\Box\mathbf{for}(Y) = \Box\mathbf{for}(Z(n)) \in (\mathbf{suc}(Z))^\Box = (\mathbf{suc}(X))^\Box$, which is in contradiction with $\Box\mathbf{for}(Y) \in \mathbf{ant}(X)$ and Lemma 2.3(3).

The case that $X \notin \mathbf{G}^*(n+1)$. By Lemma 3.10, we have $\mathbf{n}(X) \in \mathbf{G}(n+2)$ and $\#(X) > \#(\mathbf{n}(X))$. Also, we have $X^* = (\mathbf{n}(X))^*$. Therefore, using the induction hypothesis, the following three conditions hold:

(4) $X^* \in \mathbf{n}(X)\Downarrow \cap \mathbf{G}^*$,

(5) for any $Y \in \mathbf{ED}^{n+1}$, $\Box\mathbf{for}(Y) \in \mathbf{suc}(\mathbf{n}(X))$ implies $\#(Y\Downarrow \cap \mathcal{E}(X^*)) = 1$,

(6) for any $Y \in \mathbf{ED}^{n+1}$, $\Box\mathbf{for}(Y) \in \mathbf{ant}(\mathbf{n}(X))$ implies $\#(Y\Downarrow \cap \mathcal{E}(X^*)) = 0$.
By Lemma 3.10(1) and (4), we have (1).

We show (2). Suppose that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{ED}^n)$. If $Y \in \mathbf{G}^*$, then we have $\Box\mathbf{for}(Y) \in \mathbf{suc}(\mathbf{n}(X)) \cap \Box\mathbf{for}(\mathbf{ED}^{n+1})$, and using (5), we obtain (2). We assume that $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$. Then we have $\mathbf{next}(Y) \cap \{Z \mid \Box\mathbf{for}(Z) \in \mathbf{suc}(\mathbf{n}(X))\} = \{\mathbf{n}(Y, X)\}$, and using (5), we obtain $\#(\mathbf{n}(Y, X)\Downarrow \cap\mathcal{E}(X^*)) = 1$. On the other hand, by Lemma 2.3(3), we have

$$Y\Downarrow \cap\mathcal{E}(X^*) = \Big(\bigcup_{Y_{n+1}\in\mathbf{next}(Y),\Box\mathbf{for}(Y_{n+1})\in\mathbf{ant}(\mathbf{n}(X))} (Y_{n+1}\Downarrow \cap\mathcal{E}(X^*))\Big) \cup (\mathbf{n}(Y, X)\Downarrow \cap\mathcal{E}(X^*)).$$

Using (6), we obtain (2).

We show (3). Suppose that $\Box\mathbf{for}(Y) \in \mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{ED}^n)$. Then similarly to the proof of (2), we can assume that $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$. Then we have $\Box\mathbf{for}(\mathbf{next}(Y)) \subseteq \mathbf{ant}(\mathbf{n}(X))$, and using (6), we obtain (3).

$\dashv$

From Lemma 3.10, Lemma 3.8(3) and Lemma 3.11, we obtain Theorem 3.5(2).

References

[Bru75] N. G. de Bruijn, *Exact finite models for minimal propositional calculus over a finite alphabet*, Technological University Eindhoven, Report 75-WSK-02, 1975.

[Gen35] G. Gentzen, *Untersuchungen über das logisch Schliessen*, Mathematische Zeitschrift, 39, 1934–35, pp. 176–210, 405–431.

[Mos07] L. S. Moss, *Finite models constructed from canonical formulas*, Journal of Philosophical Logic, 36, 2007, pp. 605–640.

[OM57] M. Ohnishi and K. Matsumoto, *Gentzen method in modal calculi*, Osaka Mathematical Journal, 9, 1957, pp. 113–130.

[Sas09] K. Sasaki, *Formulas in modal logic S4*, manuscript, 2009.