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Abstract. We construct the m-universal model $\langle W, R, P\rangle$ for the modal logic **S4**, using an induction on the depth of the modal connective $\Box$. Main difference from the construction given by Shehtman [She78] is that our construction gives an effective description of the image of the valuation P, $\{P(A) \mid A \in \mathbf{S}\}(\subsetneq 2^W)$, where $\mathbf{S}$ is the set of formulas constructed from $\bot$ and propositional variables $p_1, \cdots, p_m$. Also this description gives the formulas in **S4** that behave like principal disjunctive normal forms in (non-modal) classical propositional logic; and clarifies a relation between the model and the depth of $\Box$ of formulas.

1. Introduction

In this paper, we treat the formulas constructed from $\bot$ (contradiction) and the propositional variables $p_1, \cdots, p_m$, by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation); and study mutual provability for modal logic **S4**. The set of these propositional variables is denoted by $\mathbf{V}$ and the set of these formulas is denoted by $\mathbf{S}(\mathbf{V})$. We use upper case Latin letters, $A, B, C, \cdots$, possibly with suffixes, for formulas.

A Kripke model is a structure $\langle W, R, P\rangle$, where W is a non-empty set, R is a binaly relation on W, and P is a mapping from the set of propositional variables to 2^W. We extend, as usual, the domain of P to the set of formulas, and call P a valuation.

The m-universal model for **S4** is the Kripke model such that the dual of it is the free interior algebra of rank m, and is isomorphic to the algebra $\langle \mathbf{S}(\mathbf{V})/\equiv, \wedge, \vee, \supset, \bot, \Box\rangle$, the restriction from Tarski-Lindenbaum algebra for **S4** into $\mathbf{S}(\mathbf{V})$. So, the m-universal model has most complete information of behavor of formulas in $\mathbf{S}(\mathbf{V})$ in **S4**, and treated in several articles(cf. [She78], Bellissima [Bel85], Chagrov and Zakharyaschev [CZ97]). The dual is m-universal model $\langle W, R, P\rangle$ is $\langle\{P(A) \mid A \in \mathbf{S}(\mathbf{V})\}, \cap, \cup, \supset, \emptyset, \Box\rangle$. So, to know the restriction of Tarski-Lindenbaum algebra in detail, we need to clarify the universe $\{P(A) \mid A \in \mathbf{S}(\mathbf{V})\}$. It is known that W is not finite. So, the universe is not 2^W since the quotient set $\mathbf{S}(\mathbf{V})/\equiv$ is countable, while 2^W is not. From this it seems difficult to clarify the universe.

Our purpose is to clarify the algebra $\langle \mathbf{S}(\mathbf{V})/\equiv, \wedge, \vee, \supset, \bot, \Box\rangle$, especially its universe $\mathbf{S}(\mathbf{V})/\equiv$. In the next three sections, we consider the finite parts of $\mathbf{S}(\mathbf{V})/\equiv$. More precisely, we consider the quotient set from the set

$$\mathbf{S}^n(\mathbf{V}) = \{A \in \mathbf{S}(\mathbf{V}) \mid d(A) \leq n\},$$

for any n, where $d(A)$, the depth of $\Box$ of A, is defined as

$d(p_i) = d(\bot) = 0$,
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
$d(\Box B) = d(B) + 1$.

Also we give a list of formulas such that each two of them are non-equivalent and each equivalent class in $\mathbf{S}^n(\mathbf{V})/\equiv$ has one of such formulas as a representative. In section 5, using the list, we constuct m-universal model, and give an effective description of the image of valuation. In section 6, we show that the result in section 5 gives the formulas in **S4** that behave like principal disjunctive normal forms in (non-modal) classical propositional logic; and clarifies a relation between the model and the depth of $\Box$ of formulas.

In the following notations, we use a sequent system for the modal logic **S4**. We introduce it following Ohnishi and Matsumoto [OM57]. We use Greek letters, Γ and Δ, possibly with suffixes, for finite sets of

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formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. By a sequent, we mean the expression $(\Gamma \to \Delta)$. We often write $\Gamma \to \Delta$ instead of the expression with the parenthesis. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

We use upper case Latin letters $X, Y, Z, \cdots, X_0, X_1, X_2, \cdots$ for sequents. For a sequent $\Gamma \to \Delta$, we define $\mathbf{ant}(\Gamma \to \Delta)$ and $\mathbf{suc}(\Gamma \to \Delta)$ as follows:

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma, \qquad \mathbf{suc}(\Gamma \to \Delta) = \Delta.$$

By **S4**, we mean the system obtained from the sequent system **LK** for classical propositional logic by adding

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to) \qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box).$$

By [OM57], this system enjoys a cut-elimination theorem:

Lemma 1.1 ([OM57]). *If $\Gamma \to \Delta \in$ **S4**, then there exists a cut-free proof figure for $\Gamma \to \Delta$ in* **S4**.

Also we use the following. Let **ENU** be an enumeration of the formulas. For a non-empty finite set S of formulas, the expressions

$$\bigwedge S \qquad \text{and} \qquad \bigvee S$$

denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \qquad \text{and} \qquad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

respectively, where $\{A_1, \cdots, A_n\} = S$ and A_i occurs earlier than A_{i+1} in **ENU**. Also the expressions

$$\bigwedge \emptyset \qquad \text{and} \qquad \bigvee \emptyset$$

denote the formulas $\bot \supset \bot$ and $\bot$, respectively. Also for a sequent X and for a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as follows:

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \Gamma \neq \emptyset \\ \bigvee \Delta & \text{if } \Gamma = \emptyset, \end{cases}$$

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

2. A construction of non-equivalent formulas

In section 2, section 3 and section 4, we consider the quotient set $\mathbf{S}^n(\{p_1 \cdots, p_m\})$. It is known the algebra $\langle \mathbf{S}^n(\{p_1 \cdots, p_m\}), \wedge, \vee, \supset, \bot, \Box$ is Boolean. So, we have only to consider its generators. First, we define a list of formulas, which will be proved to be representatives of the generators.

Definition 2.1. The sets $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ $(n = 0, 1, 2, \cdots)$ of sequents, and the mappings $\mathbf{next}^+$, $\mathbf{prov}$, $\mathbf{next}$ are defined inductively as follows:

$\mathbf{G}(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,

$\mathbf{G}^*(0) = \emptyset$,

$\mathbf{next}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}(k)), \Gamma \cap \Delta = \emptyset, \mathbf{for}(X) \in \Delta\}$, for $X \in \mathbf{G}(k)$,

$\mathbf{prov}(X) = \{Y \in \mathbf{next}^+(X) \mid Y \in \mathbf{S4}\}$, for $X \in \mathbf{G}(k)$,

$\mathbf{next}(X) = \mathbf{next}^+(X) - \mathbf{prov}(X)$, for $X \in \mathbf{G}(k)$,

$$\mathbf{G}(k+1) = \bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}(X),$$

$\mathbf{G}^*(k+1) = \{X \in \mathbf{G}(k+1) \mid (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box}$ implies $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$, for any $Y \in \mathbf{G}(k+1)\}$.

Here we use the provability of **S4**, but in section 4, this provability will be replaced another conditions concerning only the structure of sequents.

DEFINITION 2.2. We define $\mathbf{G}^n$ as follows:

$$\mathbf{G}^n = \mathbf{G}(n) \cup \bigcup_{k=0}^{n-1} \mathbf{G}^*(k).$$

In the following theorem, it is shown that the above $\mathbf{G}^n$ is the set of representatives for the generators of the Boolean.

THEOREM 2.3.
(1) $\mathbf{S}^n(\mathbf{V})/\equiv = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{G}^n\}$.
(2) *For subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathbf{G}^n$, $\mathcal{S}_1 = \mathcal{S}_2$ *iff* $[\bigwedge \mathbf{for}(\mathcal{S}_1)] = [\bigwedge \mathbf{for}(\mathcal{S}_2)]$.

(1) will be proved in the next section, Here we prove (2). To prove (2), we need some lemmas.

LEMMA 2.4.
(1) $\mathbf{G}(n) \subseteq \mathbf{S}^n(\mathbf{V}) - \mathbf{S}^{n-1}(\mathbf{V})$.
(2) *every member of* $\mathbf{G}(n)$ *is not provable in* **S4**.

Proof. By an induction on n. ⊣

LEMMA 2.5. *For any* $X, Y \in \mathbf{G}^n$, $X \neq Y$ *implies* $\mathbf{for}(X) \vee \mathbf{for}(Y) \in \mathbf{S4}$.

PROOF. We use an induction on n.

Basis($n = 0$). We have $X, Y \in \mathbf{G}^0 = \mathbf{G}(0)$. So, there exist subsets V_1, V_2 of $\mathbf{V}$ such that $X = (\mathbf{V} - V_1 \to V_1)$, $Y = (\mathbf{V} - V_2 \to V_2)$ and $V_1 \neq V_2$. By $V_1 \neq V_2$, we have either $V_1 \cap (\mathbf{V} - V_2) \neq \emptyset$ or $V_2 \cap (\mathbf{V} - V_1) \neq \emptyset$. Hence either $\mathbf{suc}(X) \cap \mathbf{ant}(Y) \neq \emptyset$ or $\mathbf{suc}(Y) \cap \mathbf{ant}(X) \neq \emptyset$, and so, we obtain the lemma.

Induction step($n \geq 1$). We divide the cases.

The case that $\{X, Y\} \subseteq \mathbf{G}(n)$. There exist sequents $X_0, Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_0)$ and $Y \in \mathbf{next}(Y_0)$. So, there exist sets $\Gamma_X, \Gamma_Y, \Delta_X, \Delta_Y$ of formulas such that
(1) $X = (\Box\Gamma_X, \mathbf{ant}(X_0) \to \mathbf{suc}(X_0), \Box\Delta_X)$, $Y = (\Box\Gamma_Y, \mathbf{ant}(Y_0) \to \mathbf{suc}(Y_0), \Box\Delta_Y)$,
(2) $\Gamma_X \cup \Delta_X = \Gamma_Y \cup \Delta_Y = \mathbf{for}(\mathbf{G}(n-1))$,
(3) $\Gamma_X \cap \Delta_X = \Gamma_Y \cap \Delta_Y = \emptyset$,
(4) $\mathbf{for}(X_0) \in \Delta_X, \mathbf{for}(Y_0) \in \Delta_Y$.

If $X_0 \neq Y_0$, then by the induction hypothesis, $\mathbf{for}(X_0) \vee \mathbf{for}(Y_0) \in \mathbf{S4}$, and so, we obtain the lemma. Suppose that $X_0 = Y_0$. Then by $X \neq Y$, we have either $\Gamma_X \neq \Gamma_Y$ or $\Delta_X \neq \Delta_Y$, and using (2) and (3), we have both. Without loss of generality, we can suppose that $\Gamma_X \not\subseteq \Gamma_Y$. So, there exists a formula $A \in \Gamma_X - \Gamma_Y$, and using (2) and (3), $A \in \Gamma_X \cap \Delta_Y$. So, we have $\Box\Gamma_X \to \Box\Delta_Y \in \mathbf{S4}$. We note $\Box\Gamma_X, \mathbf{for}(X) \in \mathbf{S4}$ and $\Box\Delta_Y \to \mathbf{for}(Y) \in \mathbf{S4}$. Using (*cut*), possibly several times, we obtain $\to \mathbf{for}(X), \mathbf{for}(Y) \in \mathbf{S4}$, and hence we obtain the lemma.

The case that $\{X, Y\} \not\subseteq \mathbf{G}(n)$. There exists $Z \in \{X, Y\} - \mathbf{G}(n)$. Without loss of generality, we can suppose that $Z = Y \notin \mathbf{G}(n)$, and then $Y \in \bigcup_{k=0}^{n-1} \mathbf{G}^*(k) \subseteq \mathbf{G}(n-1)$. If $X \notin \mathbf{G}(n)$, then $X \in \bigcup_{k=0}^{n-1} \mathbf{G}^*(k) \subseteq \mathbf{G}(n-1)$. Using the induction hypothesis, we obtain the lemma. So, we assume that $X \in \mathbf{G}(n)$. Then there exist $X_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_0)$. By $Y \in \bigcup_{k=0}^{n-1} \mathbf{G}^*(k)$ and Lemma 2.4(1), we have $Y \neq X_0$. By the induction hypothesis, we have $\mathbf{for}(X_0) \vee \mathbf{for}(Y) \in \mathbf{S4}$. We note that $\mathbf{for}(X_0) \vee \mathbf{for}(Y) \to \mathbf{for}(X) \vee \mathbf{for}(Y) \in \mathbf{S4}$. Using (*cut*), we obtain the lemma. ■

Proof of Theorem 2.3(2). The "if part" is clear. We show the "only if" part. Suppose that $\mathcal{S}_1 \not\subseteq \mathcal{S}_2$. Then there exists a sequent $X \in \mathcal{S}_1 - \mathcal{S}_2$. By Lemma 2.5, we have $\mathbf{for}(X) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \in \mathbf{S4}$. By Lemma 2.4(2), we have $\mathbf{for}(X) \notin \mathbf{S4}$. Also we have $\bigwedge \mathbf{for}(\mathcal{S}_1) \to \mathbf{for}(X) \in \mathbf{S4}$. Hence considering the figure we obtain $\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \notin \mathbf{S4}$. Similarly, we can show that $\mathcal{S}_2 \not\subseteq \mathcal{S}_1$ implies $\bigwedge \mathbf{for}(\mathcal{S}_1) \to \bigwedge \mathbf{for}(\mathcal{S}_2) \notin \mathbf{S4}$. ⊣

3. Representatives of the equivalent classes in $\mathbf{S}^n(\mathbf{V})/\equiv$

Here we prove the following theorem.

THEOREM 3.1. *For any $A \in \mathbf{S}^n(\mathbf{V})$, there exists a subset $\mathcal{S}$ of $\mathbf{G}^n$ such that $A \equiv \bigwedge \mathbf{for}(\mathcal{S})$.*

From the above theorem, we obtain Theorem 3.1(1), and that every equivalent class in $\mathbf{S}^n(\mathbf{V})/\equiv$ has a representative $\bigwedge \mathbf{for}(\mathcal{S})$ for some subset $\mathcal{S}$ of $\mathbf{G}^n$. To prove Theorem 3.1, we need some lemmas.

LEMMA 3.2. *For any subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{G}^n$,*
(1) $\bigwedge \mathcal{S}_1 \wedge \bigwedge \mathcal{S}_2 \equiv \bigwedge(\mathcal{S}_1 \cup \mathcal{S}_2)$,
(2) $\bigwedge \mathcal{S}_1 \vee \bigwedge \mathcal{S}_2 \equiv \bigwedge(\mathcal{S}_1 \cap \mathcal{S}_2)$.

Proof. (1) is clear. By Lemma 2.5, we have (2). ⊣

LEMMA 3.3. *Let $\Sigma, \Gamma, \Gamma_1, \Delta, \Delta_1$ be finite sets of formulas. Then for any subset $\Sigma' \subseteq \Sigma$,*

$$\Box\Sigma', \{\mathbf{for}(\Box\Gamma, \Box\Phi, \Gamma_1 \to \Delta_1, \Box\Psi, \Box\Delta) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Proof. We define $\mathcal{S}$ as follows:

$$\mathcal{S} = \{\mathbf{for}(\Box\Gamma, \Box\Phi, \Gamma_1 \to \Delta_1, \Box\Psi, \Box\Delta) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\},$$

and prove

$$\Box\Sigma', \mathcal{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

We use an induction on $\#(\Sigma - \Sigma')$.
Basis($\Sigma' = \Sigma$). We note that

$$\mathbf{for}(\Box\Gamma, \Box\Sigma, \Gamma_1 \to \Delta_1, \Box\Delta) \in \mathcal{S}$$

and

$$\Box\Sigma, \mathbf{for}(\Box\Gamma, \Box\Sigma, \Gamma_1 \to \Delta_1, \Box\Delta), \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Using weakening rule, we obtain the lemma.
Induction step($\Sigma' \neq \Sigma$)). By the induction hypothesis, for any $A \in \Sigma - \Sigma'$,

$$\Box(\Sigma' \cup \{A\}), \mathcal{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Using $(\vee \to)$, possibly several times,

$$\Box\Sigma', \bigvee(\Box(\Sigma - \Sigma')), \mathcal{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Using $(\vee \to)$, possibly several times,

$$\Box\Sigma', \bigvee(\Delta_1 \cup \Box\Delta \cup \Box(\Sigma - \Sigma')), \mathcal{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Using $(\supset\to)$, possibly several times,

$$\Box\Sigma', \mathbf{for}(\Box\Gamma, \Gamma_1, \Box\Sigma' \to \Delta_1, \Box\Delta, \Box(\Sigma - \Sigma')), \mathcal{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

We note that

$$\mathbf{for}(\Box\Gamma,\Gamma_1,\Box\Sigma' \to \Delta_1,\Box\Delta,\Box(\Sigma-\Sigma')) \in \mathcal{S},$$

and so,

$$\Box\Sigma',\mathcal{S},\Box\Gamma,\Gamma_1 \to \Delta_1,\Box\Delta \in \mathbf{S4}.$$

⊣

COROLLARY 3.4. *Let X be a sequent in $\mathbf{G}(n)$ and let Y be a sequent in $\mathbf{G}_\ell$. Then*
(1) $\mathbf{for}(\mathbf{next}(X)) \to \mathbf{for}(X) \in \mathbf{S4}$,
(2) $\bigwedge\mathbf{for}(\mathbf{next}(X)) \equiv \mathbf{for}(X)$,
(3) $\{\mathbf{for}(Z) \mid Z \in \mathbf{next}(X), \Box\mathbf{for}(Y) \in \mathbf{suc}(Z)\} \to \mathbf{for}(X), \Box\mathbf{for}(Y) \in \mathbf{S4}$.

DEFINITION 3.5. We define $\mathbf{BG}_\ell$ as follows:

$$\mathbf{BG}_\ell = \mathbf{V} \cup \bigcup_{i=0}^{\ell-1} \Box\mathbf{for}(\mathbf{G}(i)).$$

LEMMA 3.6. *Let X be a sequent in $\mathbf{G}(n)$. Then*
(1) $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{BG}_n$,
(2) $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$.

Proof. By Lemma 2.4(1) and an induction on n. ⊣

LEMMA 3.7. *Let X and Y be sequents in $\mathbf{G}(n)$. Then*

$$(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box \textit{ implies } (\to \mathbf{for}(X), \Box\mathbf{for}(Y)) \in \mathbf{S4}.$$

Proof. By $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$, there exists a formula $\Box A \in (\mathbf{ant}(X))^\Box - (\mathbf{ant}(Y))^\Box$. Using Lemma 3.6, we have $\Box A \in (\mathbf{ant}(X))^\Box \cap (\mathbf{suc}(Y))^\Box$. So,

$$\Box A \to \mathbf{suc}(Y) \in \mathbf{S4}.$$

Hence

$$\Box A \to \mathbf{for}(Y) \in \mathbf{S4}.$$

Using $(\to \Box)$,

$$\Box A \to \Box\mathbf{for}(Y) \in \mathbf{S4}.$$

Using weakening rule,

$$\mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y) \in \mathbf{S4}.$$

Hence we obtain the lemma. ⊣

LEMMA 3.8. *Let X be a sequent in $\mathbf{G}^*(n)$ and let Y be a sequent in $\mathbf{G}(n)$ satisfying $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. Then*

$$\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in \mathbf{S4}.$$

Proof. If $n = 0$, then the lemma is clear from $\mathbf{G}^*(0) = \emptyset$. Also, if $X = Y$, then the lemma is clear. So, we assume $n > 0$ and $X \neq Y$. By $X \in \mathbf{G}^*(n)$ and $Y \in \mathbf{G}(n)$, there exist sequents $X_0, Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_0)$ and $Y \in \mathbf{next}(Y_0)$. So, there exist four sets $\Gamma_X, \Gamma_Y, \Delta_X$ and Δ_Y such that
(1) $X = (\Box\Gamma_X, \mathbf{ant}(X_0) \to \mathbf{suc}(X_0), \Box\Delta_X)$, $Y = (\Box\Gamma_Y, \mathbf{ant}(Y_0) \to \mathbf{suc}(Y_0), \Box\Delta_Y)$,
(2) $\Gamma_X \cup \Delta_X = \Gamma_Y \cup \Delta_Y = \mathbf{for}(\mathbf{G}(n))$,
(3) $\Gamma_X \cap \Delta_X = \Gamma_Y \cap \Delta_Y = \emptyset$,
(4) $\mathbf{for}(X_0) \in \Delta_X$, $\mathbf{for}(Y_0) \in \Delta_Y$.
Also we have

(5) $X \notin \mathbf{S4}$, $Y \notin \mathbf{S4}$.
By $Y \in \mathbf{G}(n)$ and Corollary 3.4(1),

$$\mathbf{for}(\mathbf{next}(Y_0)) \to \mathbf{for}(Y_0) \in \mathbf{S4}.$$

Using $(\Box \to)$ and $(\to \Box)$,

$$\Box\mathbf{for}(\mathbf{next}(Y_0)) \to \Box\mathbf{for}(Y_0) \in \mathbf{S4}.$$

By $\mathbf{ant}(X)^{\Box} = \mathbf{ant}(Y)^{\Box}$, (1) and Lemma 2.4(1), we have $\Gamma_X = \Gamma_Y$. Using (2),(3) and (4), we have $\Box\mathbf{for}(Y_0) \in \Box\Delta_Y = \Box\Delta_X$, and so, $\Box\mathbf{for}(Y_0) \to \mathbf{for}(X) \in \mathbf{S4}$. Using (cut),

$$\Box\mathbf{for}(\mathbf{next}(Y_0)) \to \mathbf{for}(X) \in \mathbf{S4},$$

that is,

$$\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box} \text{ or } (\mathbf{ant}(X))^{\Box} \not\subseteq (\mathbf{ant}(Z))^{\Box}\}) \to \mathbf{for}(X) \in \mathbf{S4}.$$

By $X \in \mathbf{G}^*(n)$, we have that $(\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box}$ if and only if $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Z))^{\Box}$, and so,

$$\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Z))^{\Box} \text{ or } (\mathbf{ant}(X))^{\Box} \not\subseteq (\mathbf{ant}(Z))^{\Box}\}) \to \mathbf{for}(X) \in \mathbf{S4}.$$

By $\mathbf{ant}(X)^{\Box} = \mathbf{ant}(Y)^{\Box}$, (1) and Lemma 3.6, we have $\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Z))^{\Box}\} = \{Y\}$, and so,

$$\Box\mathbf{for}(Y), \Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(X))^{\Box} \not\subseteq (\mathbf{ant}(Z))^{\Box}\}) \to \mathbf{for}(X) \in \mathbf{S4}.$$

Using $(\vee \to)$, possibly several times,

$$\Box\mathbf{for}(Y), \{\mathbf{for}(X) \vee \Box\mathbf{for}(Z) \mid Z \in \mathbf{next}(Y_0), (\mathbf{ant}(X))^{\Box} \not\subseteq (\mathbf{ant}(Z))^{\Box}\} \to \mathbf{for}(X) \in \mathbf{S4}.$$

Using Lemma 3.7, and (cut), possibly several times, we obtain the lemma. ⊣

LEMMA 3.9. *Let X and Y be sequents in $\mathbf{G}(n)$ satisfying $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$. Then $X \in \mathbf{G}^*(n)$ if and only if $Y \in \mathbf{G}^*(n)$.*

Proof. From the definition of $\mathbf{G}^*(n)$,

$$X \in \mathbf{G}^*(n) \text{ if and only if } (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box} \text{ implies } (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Z))^{\Box}, \text{ for any } Z \in \mathbf{G}(n),$$
$$Y \in \mathbf{G}^*(n) \text{ if and only if } (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box} \text{ implies } (\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(Z))^{\Box}, \text{ for any } Z \in \mathbf{G}(n).$$

Using $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$, we obtain the lemma. ⊣

DEFINITION 3.10. We define a mapping $\mathbf{cf}$ as follows:

$$\mathbf{cf}(X) = \begin{cases} \bigwedge\mathbf{for}(\{Y \in \mathbf{G}(n) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}\}) & \text{if } X \in \mathbf{G}^*(n) \\ \bot \supset \bot & \text{if } X \in \mathbf{G}(n) - \mathbf{G}^*(n) \end{cases}$$

LEMMA 3.11. *Let X be a sequent in $\mathbf{G}(n)$ and let Σ be a subset of $(\mathbf{ant}(X))^{\Box}$. Then*

$$\Sigma, \mathbf{cf}(X), \Phi \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

where $\Phi = \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n) - \mathbf{G}^*(n), (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(X))^{\Box}\}$.

Proof. We use an induction on $\omega n + \#((\mathbf{ant}(X))^{\Box} - \Sigma)$.

Basis($n = 0$). We note that $\mathbf{ant}(X)^{\Box} = \emptyset$ and for any $Y \in \mathbf{G}(0) - \mathbf{G}^*(0) = \mathbf{G}^*(0)$, $\mathbf{ant}(Y)^{\Box} = \emptyset$. Hence $\Phi = \mathbf{G}(0)$. So, it is not hard to see that $\Phi \to \in \mathbf{S4}$. Hence we obtain the lemma.

Induction step($n > 0$). By $n > 0$, there exists a sequent $X_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_0)$. By the induction hypothesis,

$$\bot \supset \bot, \{\mathbf{for}(Y_0) \mid Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1), (\mathbf{ant}(Y_0))^{\Box} \subseteq (\mathbf{ant}(X_0))^{\Box}\} \to \Box\mathbf{for}(X_0) \in \mathbf{S4}.$$

Since $(\Box\mathbf{for}(X_0) \to \Box\mathbf{for}(X)), (\bot \to \bot) \in \mathbf{S4}$, using (cut), twice,

$$\{\mathbf{for}(Y_0) \mid Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1), (\mathbf{ant}(Y_0))^\Box \subseteq (\mathbf{ant}(X_0))^\Box\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using weakening rule,

$$\Sigma, \Phi, \{\mathbf{for}(Y_0) \mid Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)\} \to \Box\mathbf{for}(X) \in \mathbf{S4}. \tag{i}$$

On the other hand, by the induction hypothesis,

$$\Sigma, \mathbf{cf}(X), \Phi, A \to \Box\mathbf{for}(X) \in \mathbf{S4}, \tag{ii}$$

for any formula $A \in (\mathbf{ant}(X))^\Box - \Sigma$. (ii) also holds for any $A \in (\mathbf{suc}(X))^\Box$, and so, for any $A \in \mathbf{G}(n-1) - \Sigma$. Let Y be a sequent in $\mathbf{G}(n)$ such that $(\mathbf{ant}(Y))^\Box = \Sigma$. Then (ii) holds for any $A \in \mathbf{G}(n-1) - (\mathbf{ant}(Y))^\Box = (\mathbf{suc}(Y))^\Box$. We note that $\mathbf{suc}(Y) = \{\mathbf{for}(Y_0)\} \cup (\mathbf{suc}(Y))^\Box$ if $Y \in \mathbf{next}(Y_0)$, so using (1) and $(\vee \to)$, possibly several times,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\bigvee \mathbf{suc}(Y) \mid Y \in \bigcup_{Y_0 \in \mathbf{G}(n-1)-\mathbf{G}^*(n-1)} \mathbf{next}(Y_0), (\mathbf{ant}(Y))^\Box = \Sigma\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Also we have that $(\mathbf{ant}(Y))^\Box = \Sigma$ implies $\Sigma \to \bigwedge \mathbf{ant}(Y) \in \mathbf{S4}$, for any $Y \in \mathbf{G}(n)$; so using $(\supset\to)$,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(Y))^\Box = \Sigma\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using $(w \to)$, possibly several times,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using the definition of Φ,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using the definition of $\mathbf{G}^*(n)$,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(X))^\Box\} \to \Box\mathbf{for}(X) \in \mathbf{S4}. \tag{iii}$$

If $X \notin \mathbf{G}^*(n)$, then by Lemma 3.9, $(\mathbf{ant}(Y))^\Box = (\mathbf{ant}(X))^\Box$ implies $Y \notin \mathbf{G}^*(n)$, and so,

$$\{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(X))^\Box\} = \emptyset \subseteq \mathbf{cf}(X).$$

If $X \in \mathbf{G}^*(n)$, then from Definition 3.10, we also have

$$\{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(X))^\Box\} \subseteq \mathbf{cf}(X).$$

So, the above condition also holds in any case. Using (iii), we obtain the lemma. ⊣

Lemma 3.12. *Let X be a sequent in $\mathbf{G}(n)$. Then*

$$\Box\mathbf{for}(X) \equiv \mathbf{cf}(X) \wedge \bigwedge\{\mathbf{for}(X_1) \mid X_1 \in \mathbf{G}(n+1), \Box\mathbf{for}(X) \in \mathbf{suc}(X_1)\}.$$

Proof. By Lemma 3.8 and $(\to \wedge)$, possibly several times,

$$\Box\mathbf{for}(X) \to \mathbf{cf}(X) \in \mathbf{S4}.$$

Also we note that

$$\Box\mathbf{for}(X) \to \bigwedge\{\mathbf{for}(X_1) \mid X_1 \in \mathbf{G}(n+1), \Box\mathbf{for}(X) \in \mathbf{suc}(X_1)\} \in \mathbf{S4}.$$

Using $(\to \wedge)$,

$$\Box\mathbf{for}(X) \to \mathbf{cf}(X) \wedge \bigwedge\{\mathbf{for}(X_1) \mid X_1 \in \mathbf{G}(n+1), \Box\mathbf{for}(X) \in \mathbf{suc}(X_1)\} \in \mathbf{S4}.$$

We show the converse. By Corollary 3.4(3), for any $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\{\mathbf{for}(Y_1) \mid Y_1 \in \mathbf{next}(Y), \Box\mathbf{for}(X) \in \mathbf{suc}(Y_1)\} \to \mathbf{for}(Y), \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using $(\to \wedge)$, possibly several times,

$$\{\mathbf{for}(Y_1) \mid Y_1 \in \bigcup_{Y \in \mathbf{G}(n)-\mathbf{G}^*(n)} \mathbf{next}(Y), \Box\mathbf{for}(X) \in \mathbf{suc}(Y_1)\} \to \bigwedge \mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)), \Box\mathbf{for}(X) \in \mathbf{S4}.$$

On the other hand, by Lemma 3.11,

$$\mathbf{cf}(X), \bigwedge \mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)) \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using (cut),

$$\mathbf{cf}(X), \{\mathbf{for}(Y_1) \mid Y_1 \in \bigcup_{Y \in \mathbf{G}(n)-\mathbf{G}^*(n)} \mathbf{next}(Y), \Box\mathbf{for}(X) \in \mathbf{suc}(Y_1)\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Hence we obtain the lemma. ⊣

Lemma 3.13.

$$\bot \equiv \bigwedge \mathbf{for}(\mathbf{G}^n).$$

Proof. By an induction on n and Corollary 3.4(2). ⊣

Lemma 3.14. *For a subset $\mathcal{S}$ of $\mathbf{G}^n$*

$$\bigwedge \mathbf{for}(\mathcal{S}) \supset \bot \equiv \bigwedge \mathbf{for}(\mathbf{G}^n - \mathcal{S}).$$

Proof. By Lemma 3.13 and Lemma 2.5. ⊣

Proof of Theorem 3.1. We use an induction on n.

Basis($n = 0$). The theorem follows from the results in Classical propositional logic.

Induction step($n > 0$). We use an induction on A.

If $A = \bot$, then from Lemma 3.13, we obtain the lemma.

If A is a propositional variable p_i, then by the induction hypothesis, there exists a subset $\mathcal{S} \subseteq \mathbf{G}^{n-1}$ such that $p_i \equiv \bigwedge \mathbf{for}(\mathcal{S})$. So,

$$p_i \equiv \bigwedge \mathbf{for}((\mathcal{S} \cap (\mathbf{G}(n-1) - \mathbf{G}^*(n-1))) \cup (\mathcal{S} \cap (\bigcup_{k=0}^{n-1} \mathbf{G}^*(k))).$$

Using Corollary 3.4(2),

$$p_i \equiv \bigwedge \mathbf{for}((\bigcup_{X \in \mathcal{S} \cap (\mathbf{G}(n-1)-\mathbf{G}^*(n-1))} \mathbf{next}(X)) \cup (\mathcal{S} \cap (\bigcup_{k=0}^{n-1} \mathbf{G}^*(k)))).$$

We note that

$$(\bigcup_{X \in \mathcal{S} \cap (\mathbf{G}(n-1)-\mathbf{G}^*(n-1))} \mathbf{next}(X)) \cup (\mathcal{S} \cap (\bigcup_{k=0}^{n-1} \mathbf{G}^*(k))) \subseteq \mathbf{G}^n.$$

If $A = B \wedge C$, then by the induction hypothesis, there exist subsets $\mathcal{S}_B$ and $\mathcal{S}_C$ of $\mathbf{G}^n$ such that

$$B \equiv \bigwedge \mathbf{for}(\mathcal{S}_B), \quad \text{and} \quad C \equiv \bigwedge \mathbf{for}(\mathcal{S}_C).$$

Using Lemma 3.2,

$$B \wedge C \equiv \bigwedge \mathbf{for}(\mathcal{S}_B) \wedge \bigwedge \mathbf{for}(\mathcal{S}_C) \equiv \bigwedge \mathbf{for}(\mathcal{S}_B \cup \mathcal{S}_C).$$

Similarly, if $A = B \vee C$, then

$$B \vee C \equiv \bigwedge \mathbf{for}(\mathcal{S}_B \cap \mathcal{S}_C).$$

Also, if $A = B \supset C$, then using Lemma 3.13,

$$B \supset C \equiv (B \supset \bot) \vee C \equiv \bigwedge \mathbf{for}((\mathbf{G}^n - \mathcal{S}_B) \cap \mathcal{S}_C).$$

If $A = \Box B$, then $B \in \mathbf{S}^{n-1}(\mathbf{V})$, using the induction hypothesis, there exists a subset $\mathcal{S}$ of $\mathbf{G}^{n-1}$ such that

$$B \equiv \bigwedge \mathbf{for}(\mathcal{S}).$$

Hence

$$A = \Box B \equiv (\bigwedge \Box\mathbf{for}(\mathcal{S} \cap \mathbf{G}(n-1))) \wedge (\bigwedge \Box\mathbf{for}(\mathcal{S} \cap (\bigcup_{k=0}^{n-2} \mathbf{G}^*(k)))).$$

By Lemma 3.12 and Lemma 3.9,

$$\bigwedge \Box\mathbf{for}(\mathcal{S} \cap \mathbf{G}(n-1)) \equiv \bigwedge \bigcup_{X \in \mathcal{S} \cap \mathbf{G}(n-1)} (\mathbf{cf}(X) \wedge \bigwedge\{\mathbf{for}(X_1) \mid X_1 \in \mathbf{G}(n), \Box\mathbf{for}(X) \in \mathbf{suc}(X_1)\}).$$

$$\equiv \bigwedge \mathbf{for}(\mathbf{S}_1 \cup \mathbf{S}_2),$$

where

$$\mathbf{S}_1 = \bigcup_{X \in \mathcal{S} \cap \mathbf{G}^*(n-1)} \{Y \in \mathbf{G}^*(n-1) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{suc}(Y))^{\Box}\},$$

$$\mathbf{S}_2 = \bigcup_{X \in \mathcal{S} \cap \mathbf{G}(n-1)} \{X_1 \in \mathbf{G}(n) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(X_1)\}.$$

On the other hand, by the induction hypothesis, there exists a subset $\mathcal{T}$ of $\mathbf{G}^{n-1}$ such that

$$\bigwedge \Box\mathbf{for}(\mathcal{S} \cap (\bigcup_{k=0}^{n-2} \mathbf{G}^*(k))) \equiv \bigwedge \mathbf{for}(\mathcal{T}).$$

Using Corollary 3.4(2),

$$\bigwedge \Box\mathbf{for}(\mathcal{S} \cap (\bigcup_{k=0}^{n-2} \mathbf{G}^*(k))) \equiv \bigwedge \mathbf{for}(\mathcal{T}) \equiv \bigwedge \mathbf{for}(\mathbf{S}_3),$$

where

$$\mathbf{S}_3 = (\bigcup_{X \in \mathcal{T} \cap (\mathbf{G}(n-1) - \mathbf{G}^*(n-1))} \mathbf{next}(X)) \cup (\mathcal{T} \cap \bigcup_{k=0}^{n-1} \mathbf{G}^*(k)).$$

Hence

$$A = \Box B \equiv \bigwedge \mathbf{for}(\mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3)$$

and we note that $\mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3 \subseteq \mathbf{G}^n$. ⊣

4. Provability of formulas in $\mathbf{next}(X)$

In Definition 4.1, we use the provability of **S4** to define $\mathbf{prov}(X)$ for $X \in \mathbf{G}(n)$. In this section, we give the set without using the provability of **S4**.

Definition 4.1. For $X \in \mathbf{G}(n)$, we define $\mathbf{prov}_1(X)$, $\mathbf{prov}_2(X)$ and $\mathbf{prov}_3(X)$ as follows:

$\mathbf{prov}_1(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box\}$,

$\mathbf{prov}_2(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Y_0))) \in \mathbf{next}^+(X) \mid Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1),$
$(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Y_0)) \in \mathbf{G}(n), \Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid \Gamma_0^\Box \subseteq (\mathbf{ant}(Z))^\Box\}) \subseteq \Gamma \cap \Box\mathbf{for}(\mathbf{G}(n))\}$,

$\mathbf{prov}_3(X) = \{(\Box\mathbf{for}(Y), \Gamma \to \Delta, \Box\mathbf{for}(Z)) \in \mathbf{next}^+(X) \mid Y, Z \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(Z))^\Box\}$.

The purpose in this section is to prove

Theorem 4.2. *For* $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\mathbf{prov}(X) = \mathbf{prov}_1(X) \cup \mathbf{prov}_2(X) \cup \mathbf{prov}_3(X).$$

To prove the theorem above, we need some lemmas.

Lemma 4.3. *For* $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\mathbf{prov}_1(X) \subseteq \mathbf{prov}(X).$$

Proof. Let X_1 be in $\mathbf{prov}_1(X)$. Then $X_1 \in \mathbf{next}^+(X)$ and there exist finite sets Γ and Δ and a sequent $Y \in \mathbf{G}(n)$ such that
(1) $X_1 = (\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta, \Box\mathbf{for}(Y))$,
(2) $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$.
Using Lemma 3.7, we have $X_1 \in \mathbf{S4}$, and hence, we obtain the lemma. ⊣

Lemma 4.4. *For* $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\mathbf{prov}_2(X) \subseteq \mathbf{prov}(X).$$

Proof. Let X_1 be in $\mathbf{prov}_2(X)$. Then $X_1 \in \mathbf{next}^+(X)$ and there exist finite sets Γ, Δ, Γ_0 and Δ_0 and a sequent $Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that
(1) $X_1 = (\Gamma \to \Delta, \Box\mathbf{for}(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Y_0)))$,
(2) $(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Y_0)) \in \mathbf{G}(n)$,
(3) $\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid \Gamma_0^\Box \subseteq (\mathbf{ant}(Z))^\Box\}) \subseteq \Gamma \cap \Box\mathbf{for}(\mathbf{G}(n))$.
By Corollary 3.4(1), we have

$$\mathbf{for}(\mathbf{next}(Y_0)) \to Y_0 \in \mathbf{S4}.$$

Using $(\Box \to)$ and $(\to \Box)$, possibly several times,

$$\Box\mathbf{for}(\mathbf{next}(Y_0)) \to \Box Y_0 \in \mathbf{S4}.$$

We define Y as $Y = (\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Y_0))$. Then $\mathbf{ant}(Y) = \Gamma_0$ and

$$\Box\mathbf{for}(\mathbf{next}(Y_0)) \to Y \in \mathbf{S4}.$$

So,

$$\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid \Gamma_0^\Box \subseteq (\mathbf{ant}(Z))^\Box\}), \Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(Y))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\}) \to \mathbf{for}(Y) \in \mathbf{S4}.$$

Using (3),

$$\Gamma, \Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(Y))^{\Box} \not\subseteq (\mathbf{ant}(Z))^{\Box}\}) \to \mathbf{for}(Y) \in \mathbf{S4}.$$

Using $(\vee \to)$, possibly several times,

$$\Gamma, \{\mathbf{for}(Y) \vee \Box\mathbf{for}(Z) \mid Z \in \mathbf{next}(Y_0), (\mathbf{ant}(Y))^{\Box} \not\subseteq (\mathbf{ant}(Z))^{\Box}\} \to \mathbf{for}(Y) \in \mathbf{S4}.$$

Using Lemma 3.7 and (cut), possibly several times,

$$\Gamma \to \mathbf{for}(Y) \in \mathbf{S4}.$$

So, we have $X_1 \in \mathbf{S4}$, and hence, we obtain the lemma. ⊣

LEMMA 4.5. *For* $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\mathbf{prov}_3(X) \subseteq \mathbf{prov}(X).$$

Proof. By Lemma 3.8, we obtain the lemma. ⊣

LEMMA 4.6. *Let* X *be a sequent in* $\mathbf{G}(n+1)$ *and let* X_0 *be a sequent in* $\mathbf{G}(n)$. *Then*

$$X \in \mathbf{next}(X_0) \textit{ if and only if } X_0 = (\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n).$$

Proof. By Lemma 3.6 and Definition 2.1, we obtain the lemma. ⊣

LEMMA 4.7. *Let* X *be a sequent in* $\mathbf{G}(n+k)$. *Then*

(1) *for any* $k \geq 0$, $(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{G}(n)$,

(2) *for any* $k \geq 1$, $\Box\mathbf{for}(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{suc}(X)$.

(3) *for any* $k \geq 1$ *and for any* $X_0 \in \mathbf{G}(n)$, $\mathbf{ant}(X_0) \subseteq \mathbf{ant}(X)$ *and* $\mathbf{suc}(X_0) \subseteq \mathbf{suc}(X)$ *imply* $\mathbf{ant}(X) \cap \mathbf{BG}_n = \mathbf{ant}(X_0)$, $\mathbf{suc}(X) \cap \mathbf{BG}_n = \mathbf{suc}(X_0)$ *and* $\Box\mathbf{for}(X_0) \in \mathbf{suc}(X)$.

Proof. For (1). We use an induction on k.

Basis($k = 0$). By $X \in \mathbf{G}(n)$ and Lemma 3.6, $(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) = X \in \mathbf{G}(n)$.

Induction step($k > 0$). By $X \in \mathbf{G}(n+k)$, there exists a sequent $X_0 \in \mathbf{G}(n+k-1)$ such that $X \in \mathbf{next}(X_0)$. By the induction hypothesis, we have

$$(\mathbf{ant}(X_0) \cap \mathbf{BG}_n \to \mathbf{suc}(X_0) \cap \mathbf{BG}_n) \in \mathbf{G}(n).$$

On the other hand, by Lemma 4.6,

$$\mathbf{ant}(X_0) = \mathbf{ant}(X) \cap \mathbf{BG}_{n+k-1} \text{ and } \mathbf{suc}(X_0) = \mathbf{suc}(X) \cap \mathbf{BG}_{n+k-1}.$$

So,

$$(\mathbf{ant}(X) \cap \mathbf{BG}_{n+k-1} \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_{n+k-1} \cap \mathbf{BG}_n) \in \mathbf{G}(n).$$

Since $k \geq 1$, $\mathbf{BG}_{n+k-1} \supseteq \mathbf{BG}_n$. Hence we obtain (1).

For (2). We use an induction on k.

Basis($k = 1$). By (1),

$$(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{G}(n).$$

Using Lemma 4.6,

$$X \in \mathbf{next}(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n),$$

and using Definition 2.1, we obtain (2).

Induction step($k > 1$). By $X \in \mathbf{G}(n+k)$, there exists a sequent $X_0 \in \mathbf{G}(n+k-1)$ such that $X \in \mathbf{next}(X_0)$. By the induction hypothesis, we have

$$\Box\mathbf{for}(\mathbf{ant}(X_0) \cap \mathbf{BG}_n \to \mathbf{suc}(X_0) \cap \mathbf{BG}_n) \in \mathbf{suc}(X_0).$$

Similarly to (1), we have

$$\mathbf{ant}(X_0) \cap \mathbf{BG}_n = \mathbf{ant}(X) \cap \mathbf{BG}_{n-k-1} \cap \mathbf{BG}_n = \mathbf{ant}(X) \cap \mathbf{BG}_n,$$

$$\mathbf{suc}(X_0) \cap \mathbf{BG}_n = \mathbf{suc}(X) \cap \mathbf{BG}_{n-k-1} \cap \mathbf{BG}_n = \mathbf{suc}(X) \cap \mathbf{BG}_n,$$

and so,

$$\Box\mathbf{for}(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{suc}(X_0).$$

By $X \in \mathbf{next}(X_0)$, we have $\mathbf{suc}(X_0) \subseteq \mathbf{suc}(X)$, and so, we obtain (2).

For (3). By $\mathbf{ant}(X_0) \subseteq \mathbf{ant}(X)$ and $\mathbf{suc}(X_0) \subseteq \mathbf{suc}(X)$, we have

$$\mathbf{ant}(X_0) \cap \mathbf{BG}_n \subseteq \mathbf{ant}(X) \cap \mathbf{BG}_n \text{ and } \mathbf{suc}(X_0) \cap \mathbf{BG}_n \subseteq \mathbf{suc}(X) \cap \mathbf{BG}_n.$$

Using $X_1 \in \mathbf{G}(n)$ and Lemma 3.6, we have

$$\mathbf{ant}(X_0) \subseteq \mathbf{ant}(X) \cap \mathbf{BG}_n \text{ and } \mathbf{suc}(X_0) \subseteq \mathbf{suc}(X) \cap \mathbf{BG}_n.$$

On the other hand, by (1) and Lemma 3.6, we have

$$\mathbf{ant}(X_0) \cup \mathbf{suc}(X_0) = (\mathbf{ant}(X) \cap \mathbf{BG}_n) \cup (\mathbf{suc}(X) \cap \mathbf{BG}_n) = \mathbf{BG}_n,$$

$$\mathbf{ant}(X_0) \cap \mathbf{suc}(X_0) = (\mathbf{ant}(X) \cap \mathbf{BG}_n) \cap (\mathbf{suc}(X) \cap \mathbf{BG}_n) = \emptyset.$$

Hence

$$\mathbf{ant}(X_0) = \mathbf{ant}(X) \cap \mathbf{BG}_n \text{ and } \mathbf{suc}(X_0) = \mathbf{suc}(X) \cap \mathbf{BG}_n.$$

Using (2), we obtain $\Box\mathbf{for}(X_0) = \Box\mathbf{for}(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{suc}(X)$. ⊣

DEFINITION 4.8. For $X \in \mathbf{G}(n)$, the saturation of X, write $\mathbf{sat}(X)$, is defined as follows:

(1) if $n = 0$, then

$$\mathbf{sat}(X) = X,$$

(2) if $n > 0$, then

$$\mathbf{sat}(X) = (\Gamma_d, \Gamma_c, \mathbf{ant}(X), \{A \mid \Box A \in \mathbf{ant}(X))\} \to \mathbf{suc}(X), \Delta_c, \Delta_d, \Delta_f),$$

where

$\Gamma_c = \{\bigwedge S \mid S \subseteq \mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1)), \#(S) > 1\}$,

$\Gamma_d = \{\bigvee S \mid S \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset, S \subseteq \mathbf{BG}_{n-1}, \#(S) > 1\}$,

$\Delta_c = \{\bigwedge S \mid S \cap (\mathbf{suc}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset, S \subseteq \mathbf{BG}_{n-1}, \#(S) > 1\}$,

$\Delta_d = \{\bigvee S \mid S \subseteq \mathbf{suc}(X) - \Box\mathbf{for}(\mathbf{G}(n-1)), \#(S) > 1\}$,

$\Delta_f = \{\mathbf{for}(\mathbf{ant}(X) \cap \mathbf{BG}_\ell \to \mathbf{suc}(X) \cap \mathbf{BG}_\ell) \mid \ell \leq n-1, \mathbf{ant}(X) \cap \mathbf{BG}_\ell \neq \emptyset\}$.

REMARK 4.9. *Let X be a sequent in $\mathbf{G}(n)$. Then*

$$\mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X)) \textit{ and } \mathbf{suc}(X) \subseteq \mathbf{suc}(\mathbf{sat}(X)).$$

LEMMA 4.10. *Let X be a sequent in $\mathbf{G}(n)$. Then*

$$\mathbf{ant}(\mathbf{sat}(X)) \cap \mathbf{suc}(\mathbf{sat}(X)) = \emptyset.$$

Proof. We use $\Gamma_c, \Gamma_d, \Delta_c, \Delta_d.\Delta_f$ as in Definition 4.8. We call a formula of the form $C \wedge D$ a $\wedge$-formula. Similarly, we use $\vee$-formula, $\supset$-formula and $\Box$-formula, We note that

(1) every member of $\Gamma_c \cup \Delta_c$ is a $\wedge$-formula,

(2) every member of $\Gamma_d \cup \Delta_d$ is a $\vee$-formula,

(3) every member of Δ_f is a $\supset$-formula.

Also by Lemma 3.6,

(4) every member of $\mathbf{ant}(X) \cup \mathbf{suc}(X)$ is either a $\Box$-formula or a member of $\mathbf{V}$.
Suppose that $A \in \mathbf{ant}(\mathbf{sat}(X)) \cap \mathbf{suc}(\mathbf{sat}(X))$. Then
(5) $A \in \Gamma_d \cup \Gamma_c \cup \mathbf{ant}(X) \cup \{C \mid \Box C \in \mathbf{ant}(X))\}$,
(6) $A \in \mathbf{suc}(X) \cup \Delta_c \cup \Delta_d \cup \Delta_f$.
By (5), we divide the cases.

The case that $A \in \Gamma_c$. By (1), (2), (3), (4) and (6), we have $A \in \Gamma_c \cap \Delta_c$. So, there exist sets S and S' such that $A = \bigwedge S = \bigwedge S'$, $S \subseteq \mathbf{ant}(X)$ and $S' \cap \mathbf{suc}(X) \neq \emptyset$. By $A = \bigwedge S = \bigwedge S'$, we have $S = S'$. Using the other conditions, there exists a formula $B \in S' \cap \mathbf{suc}(X) = S \cap \mathbf{suc}(X) \subseteq \mathbf{ant}(X) \cap \mathbf{suc}(X)$. This is in contradiction with Lemma 3.6.

The case that $A \in \Gamma_d$ can be shown similarly.

The case that $A \in \mathbf{ant}(X)$. By (1),(2),(3),(4) and (6), we have $A \in \mathbf{ant}(X) \cap \mathbf{suc}(X)$, which is in contradiction with Lemma 3.6.

The case that $A \in \{C \mid \Box C \in \mathbf{ant}(X)\}$. We have $\Box A \in \mathbf{ant}(X)$, and using Lemma 3.6, $n > 0$. If $A \in \Delta_f$, then by Lemma 4.7(2), $\Box A \in \mathbf{suc}(X)$, which is in contradiction with Lemma 3.6. So, using (6), $A \in \mathbf{suc}(X) \cup \Delta_c \cup \Delta_d$. On the other hand, by $\Box A \in \mathbf{ant}(X)$ and Lemma 3.6, there exist $\ell \in \{0, \cdots, n-1\}$ and $Y \in \mathbf{G}(\ell)$ such that $A = \mathbf{for}(Y)$. By $A \in \mathbf{suc}(X) \cup \Delta_c \cup \Delta_d$, (1), (2) and (4), A is not a $\supset$-formula, and we have, $\mathbf{ant}(Y) = \emptyset$ and $A = \mathbf{for}(Y) = \bigvee \mathbf{suc}(Y)$.

If $\#(\mathbf{suc}(Y)) = 1$, then $\mathbf{BG}_\ell \supseteq \mathbf{suc}(Y) = \{\mathbf{for}(Y)\} = \{A\}$, and using (6), $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$. If $\#(\mathbf{suc}(Y)) > 1$, then $\mathbf{for}(Y)$ is a $\vee$-formula, and using (1) and (4), we have $A = \mathbf{for}(Y) = \bigvee \mathbf{suc}(Y) \in \Delta_d$, and so, $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$. Hence in any case, $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$. Also we note that $\mathbf{ant}(Y) = \emptyset \subseteq \mathbf{ant}(X)$. So, using Lemma 4.7(3), we have $\Box A = \Box\mathbf{for}(Y) \in \mathbf{suc}(X)$. This is in contradiction with $\Box A \in \mathbf{ant}(X)$ and Lemma 3.6. $\dashv$

LEMMA 4.11. *Let X be a sequent in $\mathbf{G}(n)$ and let be that $\Phi \subseteq \mathbf{ant}(\mathbf{sat}(X))$ and $\Psi \subseteq \mathbf{suc}(\mathbf{sat}(X))$. If I is an inference rule in* **S4** *except $(\rightarrow \Box)$ and (cut) whose lower sequent is $\Phi \rightarrow \Psi$, then $\Phi_1 \subseteq \mathbf{ant}(\mathbf{sat}(X))$ and $\Psi_1 \subseteq \mathbf{suc}(\mathbf{sat}(X))$, for some upper sequent $\Phi_1 \rightarrow \Psi_1$ of I.*

Proof. We use $\Gamma_c, \Gamma_d, \Delta_c, \Delta_d.\Delta_f$ as in Definition 4.8. If I is a weakening rule, then the lemma is clear, and so, we assume that I is not a weakening rule. Let A be the principal formula of I. We divide the cases.

The case that $A \in \Gamma_d$. There exist a set S and a formula B such that
(1.1) $A = (\bigvee S) \vee B$,
(1.2) $(S \cup \{B\}) \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1)) \neq \emptyset$,
(1.3) $S \cup \{B\} \subseteq \mathbf{BG}_{n-1}$,
(1.4) $\#(S) > 0$.
Also I is

$$\frac{\bigvee S, \Phi^* \rightarrow \Psi \qquad B, \Phi^* \rightarrow \Psi}{\Phi \rightarrow \Psi},$$

where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$. By (1.2), we have either $S \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$ or $\{B\} \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$. If $\{B\} \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$, then $B \in \mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$, and so, the left upper sequent satisfies the conditions. If $S \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$ and $\#(S) = 1$, then $\bigvee S \in \mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$, and so, the left upper sequent satisfies the conditions. If $S \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$ and $\#(S) > 1$, then using (1.3), $\bigvee S \in \Gamma_d \subseteq \mathbf{ant}(\mathbf{sat}(X))$, and so, the left upper sequent satisfies the conditions.

The case that $A \in \Delta_c$ can be shown similarly.

The case that $A \in \Gamma_c$. There exist a set S and a formula B such that
(2.1) $A = (\bigwedge S) \wedge B$,
(2.2) $S \subseteq \mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))$,
(2.3) $\{B\} \subseteq \mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))$,
(2.4) $\#(\mathbf{S}) > 0$.
Also I is either

$$\frac{\bigwedge S, \Phi^* \to \Psi}{\Phi \to \Psi} \quad \text{or} \quad \frac{B, \Phi^* \to \Psi}{\Phi \to \Psi},$$

where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$. By (2.3), $B \in \mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$, So, the upper sequent $B, \Phi^* \to \Psi$ satisfies the conditions. By (2.2), if $\#(S) = 1$, then $\bigwedge S \in \mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$; if not, $\bigwedge S \in \Gamma_c \subseteq \mathbf{ant}(\mathbf{sat}(X))$. So, the upper sequent $\bigwedge S, \Phi^* \to \Psi$ satisfies the conditions.

The case that $A \in \Delta_d$ can be shown similarly.

The case that $A \in \mathbf{ant}(X) \cup \mathbf{suc}(X)$. None of the member of $\mathbf{V}$ is principal formula. So, by Lemma 3.6, $A = \Box B \in \mathbf{ant}(X)$. Since I is not $(\to \Box)$, I is

$$\frac{B, \Phi^* \to \Psi}{\Phi \to \Psi},$$

where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$. By $A = \Box B \in \mathbf{ant}(X)$, we have $B \in \{C \mid \Box C \in \mathbf{ant}(X)\} \subseteq \mathbf{ant}(\mathbf{sat}(X))$. So, the upper sequent satisfies the conditions.

The case that $A \in \{C \mid \Box C \in \mathbf{ant}(X)\}$. We note that $n > 0$. By Lemma 3.6, there exist $i \in \{0, \cdots, n-1\}$ and $Y \in \mathbf{G}(i)$ such that $A = \mathbf{for}(Y)$. We note that $\Box A = \Box\mathbf{for}(Y) \in \mathbf{ant}(X)$. We define Z as $Z = (\mathbf{ant}(X) \cap \mathbf{BG}_i \to \mathbf{suc}(X) \cap \mathbf{BG}_i)$. Then by Lemma 4.7, $Z \in \mathbf{G}(i)$ and $\Box\mathbf{for}(Z) \in \mathbf{suc}(X)$. Using $\Box\mathbf{for}(Y) \in \mathbf{ant}(X)$ and Lemma 3.6, we have $Y \neq Z$. Using Lemma 3.6, we have $\mathbf{ant}(Y) \neq \mathbf{ant}(Z)$. In other words, $\mathbf{ant}(Y) \not\subseteq \mathbf{ant}(Z)$ or $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y)$. We divide the subcases.

The subcase that $\mathbf{ant}(Y) \not\subseteq \mathbf{ant}(Z)$. We note that $\mathbf{ant}(Y) \neq \emptyset$. So, I is

$$\frac{\Phi_1 \to \Psi_1, \bigwedge \mathbf{ant}(Y) \qquad \bigvee \mathbf{suc}(Y), \Phi_2 \to \Psi_2}{\Phi \to \Psi},$$

where $\Phi_1 \cup \Phi_2 \in \{\Phi, \Phi - \{A\}\}$ and $\Psi_1 \cup \Psi_2 = \Psi$. On the other hand, by $\mathbf{ant}(Y) \not\subseteq \mathbf{ant}(Z)$, there exists a formula $B \in \mathbf{ant}(Y) - \mathbf{ant}(Z)$. Using Lemma 3.6,

$$B \in \mathbf{ant}(Y) \cap \mathbf{suc}(Z) \subseteq \mathbf{ant}(Y) \cap (\mathbf{suc}(X) \cap \mathbf{BG}_i) \subseteq \mathbf{ant}(Y) \cap (\mathbf{suc}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))).$$

So, if $\#(\mathbf{ant}(Y)) = 1$, then $\bigwedge \mathbf{ant}(Y) = \{B\} \in \mathbf{suc}(X)$; if not, $\bigwedge \mathbf{ant}(Y) \in \Delta_c$. Hence the left upper sequent of I satisfies the conditions.

The subcase that $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y) \neq \emptyset$. By $\mathbf{ant}(Y) \neq \emptyset$, I is

$$\frac{\Phi_1 \to \Psi_1, \bigwedge \mathbf{ant}(Y) \qquad \bigvee \mathbf{suc}(Y), \Phi_2 \to \Psi_2}{\Phi \to \Psi},$$

where $\Phi_1 \cup \Phi_2 \in \{\Phi, \Phi - \{A\}\}$ and $\Psi_1 \cup \Psi_2 = \Psi$. On the other hand, by $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y)$, there exists a formula $B \in \mathbf{ant}(Z) - \mathbf{ant}(Y)$. Using Lemma 3.6,

$$B \in \mathbf{ant}(Z) \cap \mathbf{suc}(Y) \subseteq (\mathbf{ant}(X) \cap \mathbf{BG}_i) \cap \mathbf{suc}(Y) \subseteq (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \cap \mathbf{suc}(Y).$$

So, if $\#(\mathbf{suc}(Y)) = 1$, then $\bigvee \mathbf{suc}(Y) = \{B\} \in \mathbf{ant}(X)$; if not, $\bigvee \mathbf{suc}(Y) \in \Gamma_d$. Hence the right upper sequent satisfies the conditions.

The subcase that $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y) = \emptyset$. By $\mathbf{ant}(Y) = \emptyset$ and Lemma 3.6, we have $\mathbf{suc}(Y) = \mathbf{BG}_i$. If $\#(\mathbf{suc}(Y)) = \#(\mathbf{BG}_i) = 1$, then $\mathbf{suc}(Y) = \{A\} \subseteq \mathbf{V}$, and so, A is not a principal formula. So, we assume that $\#(\mathbf{suc}(Y)) > 1$. Then I is

$$\frac{\bigvee(\mathbf{suc}(Y) - \{C\}), \Phi^* \to \Psi \qquad C, \Phi^* \to \Psi}{\Phi \to \Psi},$$

where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$ and $\bigvee \mathbf{suc}(Y) = (\bigvee(\mathbf{suc}(Y) - \{C\})) \vee C$. On the other hand, we note by $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y)$, there exists a formula $B \in \mathbf{ant}(Z) - \mathbf{ant}(Y)$. Using Lemma 3.6,

$$B \in \mathbf{ant}(Z) \cap \mathbf{suc}(Y) \subseteq (\mathbf{ant}(X) \cap \mathbf{BG}_i) \cap \mathbf{suc}(Y) \subseteq (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \cap \mathbf{suc}(Y).$$

So, if $C = B$, then $C \in \mathbf{ant}(X)$, and so, the right upper sequent satisfies the conditions. If $C \neq B$, then $B \in \mathbf{suc}(Y) - \{C\}$ and $\bigvee(\mathbf{suc}(Y) - \{C\}) \in \Gamma_d$. So, the left upper sequent satisfies the conditions.

The case that $A \in \Delta_f$. There exists $\ell \leq n-1$ such that $A = \mathbf{for}(\mathbf{ant}(X) \cap \mathbf{S}_\ell \to \mathbf{suc}(X) \cap \mathbf{S}_\ell)$ and $\mathbf{ant}(X) \cap \mathbf{S}_\ell \neq \emptyset$. So, I is

$$\frac{\bigwedge(\mathbf{ant}(X) \cap \mathcal{S}_\ell), \Phi \to \Psi^*, \bigvee(\mathbf{suc}(X) \cap \mathbf{S}_\ell)}{\Phi \to \Psi},$$

where $\Psi^* \in \{\Psi, \Psi - \{A\}\}$. We note that $\bigwedge(\mathbf{ant}(X) \cap \mathcal{S}_\ell) \in \mathbf{ant}(X) \cup \Gamma_c$ and $\bigvee(\mathbf{suc}(X) \cap \mathbf{S}_\ell) \in \mathbf{suc}(X) \cup \Gamma_d$. So, the upper sequent of I satisfies the conditions. ⊣

Lemma 4.12. *Let X be a sequent in $\mathbf{G}(n+k)$ and let Y be a sequent in $\mathbf{G}^*(n)$. If $(\mathbf{ant}(Y))^\Box \neq (\mathbf{ant}(X))^\Box \cap \mathbf{BG}_n$. Then*

$$\to \mathbf{for}(Y), \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Proof. We use an induction on k.

Basis($k = 0$). By $X \in \mathbf{G}(n)$ and Lemma 3.6, $(\mathbf{ant}(Y))^\Box = (\mathbf{ant}(X))^\Box \cap \mathbf{BG}_n \neq (\mathbf{ant}(X))^\Box$. Also by $Y \in \mathbf{G}^*(n)$, we have

$$(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box \text{ implies } (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(Z))^\Box, \text{ for any } Z \in \mathbf{G}(n).$$

Hence we have $(\mathbf{ant}(Y))^\Box \not\subseteq (\mathbf{ant}(X))^\Box$. Using Lemma 3.7, we obtain the lemma.

Induction step($k > 0$). By $X \in \mathbf{G}(n+k)$, there exists $X_0 \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$ such that $X \in \mathbf{next}(X_0)$. By Lemma 4.6,

$$(\mathbf{ant}(X_0))^\Box \cap \mathbf{BG}_n = (\mathbf{ant}(X))^\Box \cap \mathbf{BG}_{n+k-1} \cap \mathbf{BG}_n = (\mathbf{ant}(X))^\Box \cap \mathbf{BG}_n \neq (\mathbf{ant}(Y))^\Box.$$

So, by the induction hypothesis, we have

$$\to \mathbf{for}(Y), \Box\mathbf{for}(X_0) \in \mathbf{S4}.$$

On the other hand, we note that $\Box\mathbf{for}(X_0) \to \Box\mathbf{for}(X) \in \mathbf{S4}$, and using (*cut*), we obtain the lemma. ⊣

Corollary 4.13. *Let X be a sequent in $\mathbf{G}(n+k)$ and let Y be a sequent in $\mathbf{G}^*(n)$. If $(\mathbf{ant}(Y))^\Box \neq (\mathbf{ant}(X))^\Box \cap \mathbf{BG}_n$. Then*

$$(\Box\mathbf{for}(X) \supset \mathbf{for}(Y)) \equiv \mathbf{for}(Y).$$

Proof. By Lemma 4.12 and (*cut*), we obtain the corollary. ⊣

Lemma 4.14. *Let X be a sequent in $\mathbf{G}(n)$ and let Y_1 be a sequent in $\mathbf{G}^*(k)$ ($k \in \{0, 1, \cdots, n-1\}$). If $\Box\mathbf{for}(Y_1) \in \mathbf{suc}(X)$, then*

$$\mathbf{for}(\mathbf{ant}(X)^\Box \to \mathbf{for}(Y_1)) \equiv \mathbf{for}(Y_1).$$

Proof. We define X_1 as follows:

$$X_1 = (\mathbf{ant}(X) \cap \mathbf{BG}_k \to \mathbf{suc}(X) \cap \mathbf{BG}_k).$$

Then

$$(\mathbf{ant}(X))^\Box = (\mathbf{ant}(X) \cap \mathbf{BG}_k)^\Box \cup (\mathbf{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box\mathbf{for}(\mathbf{G}(i)))$$

$$= \mathbf{ant}(X_1)^{\Box} \cup (\mathbf{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box\mathbf{for}(\mathbf{G}(i))).$$

So, it is sufficient to show the following two:

(1) for any $A \in \mathbf{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box\mathbf{for}(\mathbf{G}(i)))$, $A \supset \mathbf{for}(Y1) \equiv \mathbf{for}(Y_1)$,

(2) $\mathbf{for}(\mathbf{ant}(X_1)^{\Box} \to \mathbf{for}(Y_1)) \equiv \mathbf{for}(Y_1)$.

For (1). There exist a number $i \in \{k, k+1, \cdots, n-1\}$ and a sequent $Z \in \mathbf{G}(i)$ such that $A = \Box\mathbf{for}(Z)$. If $(\mathbf{ant}(Y_1))^{\Box} \neq (\mathbf{ant}(Z))^{\Box} \cap \mathbf{BG}_k$, then by Corollary 4.13, we obtain (1). So, we assume $(\mathbf{ant}(Y_1))^{\Box} = (\mathbf{ant}(Z))^{\Box} \cap \mathbf{BG}_k$. We divide the cases.

The case that $i = k$. Then by Lemma 3.8, we have

$$\Box\mathbf{for}(Z) \to \mathbf{for}(Y_1) \in \mathbf{S4}.$$

Using $(\to \Box)$, we have

$$\Box\mathbf{for}(Z) \to \Box\mathbf{for}(Y_1) \in \mathbf{S4}.$$

So, using $\Box\mathbf{for}(Y_1) \in \mathbf{suc}(X)$ and $\Box\mathbf{for}(Z) = A \in \mathbf{ant}(X)$, we have $X \in \mathbf{S4}$. Using Lemma 2.4(2), $X \notin \mathbf{G}(n)$, which is in contradiction with $X \in \mathbf{G}(n)$.

The case that $i > k$. We define Z_1 and Z_2 as follows:

$$Z_1 = (\mathbf{ant}(Z) \cap \mathbf{BG}_k \to \mathbf{suc}(Z) \cap \mathbf{BG}_k) \text{ and } Z_2 = (\mathbf{ant}(Z) \cap \mathbf{BG}_{k+1} \to \mathbf{suc}(Z) \cap \mathbf{BG}_{k+1}).$$

Then by Lemma 4.7, we have $Z_1 \in \mathbf{G}(k)$ and $Z_2 \in \mathbf{G}(k+1)$. Also by the assumption, we have $(\mathbf{ant}(Y_1))^{\Box} = (\mathbf{ant}(Z))^{\Box} \cap \mathbf{BG}_k = (\mathbf{ant}(Z_1))^{\Box}$, and using Lemma 3.9, $Z_1 \in \mathbf{G}^*(k)$. On the other hand, by $Z_2 \in \mathbf{G}(k+1)$, there exists a sequent $Z_1' \in \mathbf{G}(k) - \mathbf{G}^*(k)$ such that $Z_2 \in \mathbf{next}(Z_1')$. Using Lemma 4.6,

$$\begin{aligned} Z_1' &= (\mathbf{ant}(Z_2) \cap \mathbf{BG}_k \to \mathbf{suc}(Z_2) \cap \mathbf{BG}_k) \\ &= (\mathbf{ant}(Z) \cap \mathbf{BG}_{k+1} \cap \mathbf{BG}_k \to \mathbf{suc}(Z) \cap \mathbf{BG}_{k+1} \cap \mathbf{BG}_k) \\ &= (\mathbf{ant}(Z) \cap \mathbf{BG}_k \to \mathbf{suc}(Z) \cap \mathbf{BG}_k) = Z_1 \in \mathbf{G}^*(k), \end{aligned}$$

which is in contradiction with $Z_1' \in \mathbf{G}(k) - \mathbf{G}^*(k)$.

For (2). Suppose that $(\mathbf{ant}(X_1))^{\Box} \not\subseteq (\mathbf{ant}(Y_1))^{\Box}$. Then by Lemma 3.7, we have

$$\mathbf{ant}(X_1) \to \mathbf{suc}(X_1), \Box\mathbf{for}(Y_1) \in \mathbf{S4}.$$

So,

$$\mathbf{ant}(X) \cap \mathbf{BG}_k \to \mathbf{suc}(X) \cap \mathbf{BG}_k, \Box\mathbf{for}(Y_1) \in \mathbf{S4}.$$

Hence $X \in \mathbf{S4}$, which is in contradiction with Lemma 2.4(2) and $X \in \mathbf{G}(n)$. So, we have $(\mathbf{ant}(X_1))^{\Box} \subseteq (\mathbf{ant}(Y_1))^{\Box}$, and so,

$$(\bigwedge(\mathbf{ant}(X_1))^{\Box}) \supset \mathbf{for}(Y_1) \equiv (\bigwedge(\mathbf{ant}(X_1))^{\Box} \cup \mathbf{ant}(Y_1)) \supset \bigvee \mathbf{suc}(Y_1) \equiv \bigwedge \mathbf{ant}(Y_1) \supset \bigvee \mathbf{suc}(Y_1).$$

Hence we obtain (2). ⊣

Lemma 4.15. *Let X be a sequent in $\mathbf{G}(n+k)$ and let Y_0 be a sequent in $\mathbf{G}(n)-\mathbf{G}^*(n)$. Let X_1 be a sequent in $\mathbf{next}(X)$. If $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1)$, then there exists a sequent $Y \in \mathbf{G}^{n+k}$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$, $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y)$.*

Proof. We use an induction on k.

Basis($k = 0$). The lemma is clear from $Y_0 \in \mathbf{G}(n)$ and $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1)$.

Induction step($k > 0$). By $X \in \mathbf{G}(n+k)$, there exists a sequent $X_0 \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$ such that $X \in \mathbf{next}(X_0)$. Also by $k > 0$ and Lemma 3.6, $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)) = \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n))$. Using the induction hypothesis, there exists a sequent $Y_2 \in \mathbf{G}^{n+k-1}$

such that $\Box\mathbf{for}(Y_2) \in \mathbf{suc}(X)$, $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y_2)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y_2)$. If $Y_2 \in \bigcup_{i=0}^{n+k-1} \mathbf{G}^*(i)$, then $Y_2 \in \mathbf{G}^{n+k}$, and we obtain the lemma. So, we assume that $Y_2 \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$. On the other hand, by Lemma 2.4 and Lemma 4.4, we have $X_1 \notin \mathbf{prov}_2(X)$. Using the four conditions

$\Box\mathbf{for}(X) \in \mathbf{suc}(X_1)$,
$\Box\mathbf{for}(Y_2) \in \mathbf{suc}(X)$,
$Y_2 \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$ and
$X \in \mathbf{G}(n+k)$,

we have

$$\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_2) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Z))^\Box\}) \not\subseteq \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n+k)).$$

So, there exists a sequent $Y \in \mathbf{next}(Y_2)$ such that $(\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y))^\Box$ and $\Box\mathbf{for}(Y) \notin \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n+k))$. By $Y \in \mathbf{next}(Y_2)$, we have $Y \in \mathbf{G}(n+k) \subseteq \mathbf{G}^{n+k}$. Using $\Box\mathbf{for}(Y) \notin \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n+k))$ and Lemma 3.6, we have $\Box\mathbf{for}(Y) \notin \mathbf{ant}(X_1)$ and $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$. Also by $Y \in \mathbf{next}(Y_2)$, we have $\mathbf{ant}(Y_2) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_2) \subseteq \mathbf{suc}(Y)$. Hence we have $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y)$. ⊣

Lemma 4.16. *Let X and Y be sequents in $\mathbf{G}(n) - \mathbf{G}^*(n)$ and let X_1 be a sequent in $\mathbf{next}^+(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X))$. If $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$, then*

$$(\Gamma_Y, \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Delta_Y) \in \mathbf{next}^+(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X)),$$

where

$\Delta_Y = \{\Box\mathbf{for}(Z) \in \mathbf{suc}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid \mathbf{ant}(Y)^\Box \subseteq \mathbf{ant}(Z)^\Box\}$ *and*
$\Gamma_Y = \{\Box\mathbf{for}(Z) \in \mathbf{suc}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid \mathbf{ant}(Y)^\Box \not\subseteq \mathbf{ant}(Z)^\Box\}$.

Proof. we define the sequent Y_1 as follows:

$$Y_1 = (\Gamma_Y, \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Delta_Y).$$

It is not hard to see that $Y_1 \in \mathbf{next}^+(Y)$. So, it is sufficient to show the following three:

(1) $Y_1 \notin \mathbf{prov}_1(Y)$,
(2) $Y_1 \notin \mathbf{prov}_2(Y)$,
(3) $Y_1 \notin \mathbf{prov}_3(Y)$.

For (1). Suppose that $Y_1 \in \mathbf{prov}_1(Y)$. Then there exists a sequent $Z \in \mathbf{G}(n)$ such that $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_1)$, $(\mathbf{ant}(Y))^\Box \neq\subseteq (\mathbf{ant}(Z))^\Box$. By Lemma 3.6, we have $\Box\mathbf{for}(Z) \notin \mathbf{BG}_n \supseteq \mathbf{suc}(Y)$, and using $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_1) = \mathbf{suc}(Y) \cup \Delta_Y$, we have $\Box\mathbf{for}(Z) \in \Delta_Y$. So, $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box$. This is in contradiction with $(\mathbf{ant}(Y))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box$.

For(2). Suppose that $Y_1 \in \mathbf{prov}_2(Y)$. Then there exist sequents $Z \in \mathbf{G}(n)$ and $Z_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that

(2.1) $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_1)$,
(2.2) $\Box\mathbf{for}(Z_0) \in \mathbf{suc}(Z)$,
(2.3) $\Box\mathbf{for}(\{Z' \in \mathbf{next}(Z_0) \mid (\mathbf{ant}(Z))^\Box \subseteq (\mathbf{ant}(Z'))^\Box\}) \subseteq \mathbf{suc}(Y_1) \cap \Box\mathbf{for}(\mathbf{G}(n))$.

Similarly to (1), by (2.1), we have

(2.4) $\Box\mathbf{for}(Z) \in \mathbf{suc}(X_1)$,

Also by Lemma 3.6, we have $\mathbf{suc}(Y_1) \cap \Box\mathbf{for}(\mathbf{G}(n)) = \Delta_Y$, and using (2.3), we have

(2.5) $\Box\mathbf{for}(\{Z' \in \mathbf{next}(Z_0) \mid (\mathbf{ant}(Z))^\Box \subseteq (\mathbf{ant}(Z'))^\Box\}) \subseteq \Delta_Y \subseteq \mathbf{suc}(X_1) \cap \mathbf{G}(n)$.

By (2.4), (2.2), (2.5) and $X_1 \in \mathbf{next}^+(X)$, we obtain $X_1 \in \mathbf{prov}_2(X)$, which is in contradiction with $X_1 \notin \mathbf{prov}_2(X)$.

For (3). Suppose that $Y_1 \in \mathbf{prov}_3(Y)$. Then there exist sequents $Z, Z' \in \mathbf{G}^*(n)$ such that

(3.1) $\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_1)$, (3.2) $\Box\mathbf{for}(Z') \in \mathbf{suc}(Y_1)$

(3.3) $(\mathbf{ant}(Z))^{\Box} = (\mathbf{ant}(Z'))^{\Box}$.

Similarly to (1), by (3.2), we have

(3.4) $\Box\mathbf{for}(Z') \in \Delta_Y \subseteq \mathbf{suc}(X_1)$.

By $\Box\mathbf{for}(Z') \in \Delta_Y$, we have $(\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(Z'))^{\Box}$. Using (3.3), $(\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box}$. So, we have $\Box\mathbf{for}(Z') \notin \Gamma_Y$. Using (3.1), we have $\Box\mathbf{for}(Z') \in \mathbf{ant}(X_1) \cup \mathbf{ant}(Y)$. Similarly to (1), we have

(3.5) $\Box\mathbf{for}(Z') \in \mathbf{ant}(X_1)$.

By (3.4), (3.5), (3.3) and $X_1 \in \mathbf{next}^{+}(X)$, we obtain $X_1 \in \mathbf{prov}_3(X)$, which is in contradiction with $X_1 \notin \mathbf{prov}_3(X)$. ⊣

Lemma 4.17. *Let $\mathcal{P}$ be a cut-free proof figure in* **S4** *whose end sequent is $\Phi \to \Psi$. Then for any $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and for any $X_1 \in \mathbf{next}^{+}(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X))$,*

$$(\Phi \to \Psi) \notin \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_1))\}.$$

Proof. We use an induction on $\mathcal{P}$.

Basis($\mathcal{P}$ consists of an axiom). Suppose that

$$(\Phi \to \Psi) \in \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_1))\}.$$

Then by Lemma 4.10, $\Phi \cap \Psi = \emptyset$, which is not an axiom.

Induction step ($\mathcal{P}$ has the inference rule introducing the end sequent). Suppose that

$$(\Phi \to \Psi) \in \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_1))\}.$$

and let I be the inference rule introducing the end sequent in $\mathcal{P}$.

If I is not $(\to \Box)$, then by Lemma 4.11, an upper sequent I belongs to

$$\{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_1))\}.$$

This is in contradiction with the induction hypothesis.

So, we assume that I is $(\to \Box)$. Then there exist a set Γ and a sequent Y_0 such that

(1) $\Gamma \subseteq \mathbf{ant}(X_1)^{\Box}$,

(2) $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1)^{\Box}$,

(3) $(\Phi \to \Psi) = (\Gamma \to \Box\mathbf{for}(Y_0))$,

(4) I is $\dfrac{\Gamma \to \mathbf{for}(Y_0)}{\Gamma \to \Box\mathbf{for}(Y_0)}$.

We divide the cases.

The case that $Y_0 \in \mathbf{G}^*(k)$ for some $k \leq n$. By Lemma 4.14, (1) and (2),

$$\mathbf{for}(\Gamma \to \mathbf{for}(Y_0)) \equiv \mathbf{for}(\mathbf{ant}(X_1)^{\Box} \to \mathbf{for}(Y_0)) \equiv \mathbf{for}(Y_0).$$

Using (4), we have $Y_0 \in \mathbf{S4}$, which is in contradiction with $Y_0 \in \mathbf{G}^*(k)$ and Lemma 2.4(2).

The case that $Y_0 \notin \mathbf{G}^*(k)$ for any $k \leq n$. Then by Lemma 3.6, $Y_0 \in \mathbf{G}(k) - \mathbf{G}^*(k)$ for some $k \leq n$. Using (2) and Lemma 4.15, there exists a sequent $Y \in \mathbf{G}^n$ such that

(5) $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$,

(6) $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y)$.

By (6), we have $\mathbf{for}(Y_0) \to \mathbf{for}(Y) \in \mathbf{S4}$, and using (4), we have $\Gamma \to \mathbf{for}(Y) \in \mathbf{S4}$. If $Y \in \mathbf{G}^*(i)$ for some $i \leq n$, then using (1), (5) and Lemma 4.14, we obtain a contradiction similarly to the above case. So, by $Y \in \mathbf{G}^n$ we can assume that $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$. Then by (5) and Lemma 4.16,

$Y_1 = (\Gamma_Y, \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Delta_Y) \in \mathbf{next}^{+}(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X))$, where Δ_Y and Γ_Y are as in Lemma 4.16. By (6), we have $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y_1)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y_1)$. Using Lemma 4.7(3),

$$\mathbf{for}(Y_0) = \mathbf{for}(\mathbf{ant}(Y_0) \to \mathbf{suc}(Y_0)) = \mathbf{for}(\mathbf{ant}(Y_1) \cap \mathbf{BG}_k \to \mathbf{suc}(Y_1) \cap \mathbf{BG}_k) \in \mathbf{suc}(\mathbf{sat}(Y_1)).$$

On the other hand, by $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$ and $Y \in \mathbf{G}(n)$, we have

$$(\mathbf{ant}(X))^{\Box} \not\subseteq (\mathbf{ant}(Y))^{\Box} \text{ implies } X_1 \in \mathbf{prov}_1(X).$$

So, using $X_1 \notin \mathbf{prov}_1(X)$, we have

$$\Gamma \subseteq (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box} \subseteq \mathbf{ant}(\mathbf{sat}(Y_1)).$$

So, the upper sequent of I belongs to

$$\{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(Y_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(Y_1))\}$$

for $Y_1 \in \mathbf{next}^+(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X))$. This is in contradiction with the induction hypothesis. ⊣

By the above lemma and Lemma 1.1(2), we obtain

COROLLARY 4.18. *Let X be a sequent in $\mathbf{G}(n) - \mathbf{G}^*(n)$. Then*

$$\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X) \supseteq \mathbf{prov}(X).$$

From Lemma 4.3, Lemma 4.4, Lemma 4.5 and Corollary 4.18, we obtain Theorem 4.2.

5. m-universal model

Here we construct n-universal model $\langle W, R, P\rangle$ and clarify the image of P, $\{P(A) \mid A \in \mathbf{S}(V)\}$.

DEFINITION 5.1. The Kripke model $\mathbf{UM}$ is defined as

$$\mathbf{UM} = \langle W_u, R_u, P_u\rangle,$$

where

$W_u = \bigcup_{n=0}^{\infty} \mathbf{G}^*(n)$,
R_u is the transitive closure of $\{(X, Y) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y) \text{ or } (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}\}$,
$P_u(p_i) = \{X \mid p_i \in \mathbf{ant}(X)\}$.

DEFINITION 5.2. For a sequent $X \in \mathbf{G}^n$, we define the sets $\overrightarrow{X}$ and $\overleftarrow{X}$ inductively as follows:
(1.1) $X \in \overrightarrow{X}$,
(1.2) $Y \in \overrightarrow{X}$ and $Y \notin W_u$ imply $\mathbf{next}(Y) \subseteq \overrightarrow{X}$,
(2.1) $X \in \overleftarrow{X}$,
(2.2) $X \in \mathbf{next}(Y)$ implies $Y \in \overleftarrow{X}$.

We note that

$$\overleftarrow{X} = \{\mathbf{ant}(X) \cap \mathbf{BG}_\ell \to \mathbf{suc}(X) \cap \mathbf{BG}_\ell \mid 0 \le \ell \le n\}$$

and

$$Y \in \overrightarrow{X} \text{ if and only if } X \in \overleftarrow{Y}.$$

THEOREM 5.3.
(1) $\mathbf{UM}$ *is the n-universal model.*
(2) $\{W_u - P(A) \mid A \in \mathbf{S}(V)\}$ *is the union of finite number of sets in* $\{\overrightarrow{X} \cap W_u \mid X \in \mathbf{G}(n), n \ge 0\}$.
(3) $\{P(A) \mid A \in \mathbf{S}(V)\}$ *is the intersection of finite number of sets in* $\{W_u - \overrightarrow{X} \mid X \in \mathbf{G}(n), n \ge 0\}$.

To prove the above theorem, we need some lemmas.

LEMMA 5.4. *Let X and Y be sequents in $\mathbf{G}(n)$ and let X_1 be a sequent in $\mathbf{next}(X) \cap \mathbf{G}^*(n+1)$. If $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$, then either $Y \in \mathbf{G}^*(n)$ or $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$.*

PROOF. Suppose that
(1) $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$,
(2) $Y \notin \mathbf{G}^*(n)$,
(3) $(\mathbf{ant}(X))^\Box \neq (\mathbf{ant}(Y))^\Box$.
By (1), $X_1 \in \mathbf{next}(X)$ and Theorem 4.2, we have $X_1 \notin \mathbf{prov}(X) \supseteq \mathbf{prov}_1(X)$, and so,
(4) $(\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y))^\Box$.
Let be that

$$\mathbf{S}_Y = \Box\mathbf{for}(\{Z \in \mathbf{G}(n) \mid (\mathbf{ant}(Y))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\})$$

and

$$Y_1 = (\mathbf{ant}(X_1) \cap \mathbf{G}(n), \mathbf{S}_Y, \mathbf{ant}(Y) \to \mathbf{suc}(Y), (\mathbf{suc}(X_1) \cap \mathbf{G}(n)) - \mathbf{S}_Y).$$

By (4), we note $\Box\mathbf{for}(X) \in \mathbf{S}_Y$, $(\mathbf{ant}(X_1))^\Box \subseteq (\mathbf{ant}(Y_1))^\Box$ and $(\mathbf{suc}(Y_1))^\Box \subseteq (\mathbf{suc}(X_1))^\Box$. So, if $Y_1 \in \mathbf{next}(Y)$, then it is in contradiction with $X_1 \in \mathbf{G}^*(n+1)$. Using Theorem 4.2, we have only to show $Y_1 \in \mathbf{next}(Y)$, the following four:
(5) $Y_1 \in \mathbf{next}^+(Y)$,
(6) $Y_1 \notin \mathbf{prov}_1(Y)$,
(7) $Y_1 \notin \mathbf{prov}_2(Y)$,
(8) $Y_1 \notin \mathbf{prov}_3(Y)$.
For (5). By (4), we have $Y \notin \mathbf{S}(Y)$, and using (1), $\Box\mathbf{for}(Y) \in \mathbf{suc}(Y_1)$. Clearly $(\mathbf{ant}(Y_1) \cup \mathbf{suc}(Y_1)) \cap \mathbf{G}(n) = \mathbf{G}(n)$ and $(\mathbf{ant}(Y_1) \cap \mathbf{suc}(Y_1)) \cap \mathbf{G}(n) = \emptyset$.

For (6). By $\mathbf{S}_Y \in \mathbf{ant}(Y_1)$, we have $Y_1 \notin \mathbf{prov}_1(Y)$.

For (7). Suppose that $Y_1 \in \mathbf{prov}_2(Y)$. Then there exist sets Γ_0 and Δ_0 and a sequent $Z_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that
(7.1) $\Box\mathbf{for}(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Z_0)) \in \mathbf{suc}(Y_1)$,
(7.2) $(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Z_0)) \in \mathbf{G}(n)$,
(7.3) for any $Z \in \mathbf{next}(Z_0)$ $\Gamma_0 \subseteq (\mathbf{ant}(Z))^\Box$ implies $\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_1) \cap \Box\mathbf{for}(\mathbf{G}(n))$.
By (7.1), we have
(7.4) $\Box\mathbf{for}(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Z_0)) \in \mathbf{suc}(X_1)$.
Also by (5) and (7.1), we have $(\mathbf{ant}(Y))^\Box \subseteq \Gamma_0^\Box$. Using (7.3), for any $Z \in \mathbf{next}(Z_0)$, if $\Gamma_0 \subseteq (\mathbf{ant}(Z))^\Box$, then $\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_1) \cap \Box\mathbf{for}(\mathbf{G}(n))$ and $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box$, and so,
(7.5) $\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_1) \cap \Box\mathbf{for}(\mathbf{G}(n)) - \mathbf{S}_Y \subseteq \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n))$.
By (7.4), (7.2), (7.5) and $X_1 \in \mathbf{next}^+(X)$, we obtain $X_1 \in \mathbf{prov}_2(X)$, which is in contradiction with Theorem 4.2 and $X_1 \in \mathbf{next}(X)$.

For (8). Suppose that $Y_1 \in \mathbf{prov}_2(Y)$. Then there exist two sequents $Z, Z' \in \mathbf{G}(n)$ such that $\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_1)$, $\Box\mathbf{for}(Z') \in \mathbf{suc}(Y_1)$ and $(\mathbf{ant}(Z))^\Box = (\mathbf{ant}(Z'))^\Box$. By $\Box\mathbf{for}(Z') \in \mathbf{suc}(Y_1)$, we have $\Box\mathbf{for}(Z') \notin \mathbf{S}_Y$. Using $(\mathbf{ant}(Z))^\Box = (\mathbf{ant}(Z'))^\Box$, we have $\Box\mathbf{for}(Z) \notin \mathbf{S}_Y$. Using $\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_1)$, we have $\Box\mathbf{for}(Z) \in \mathbf{ant}(X_1)$. Also by $\Box\mathbf{for}(Z') \in \mathbf{suc}(Y_1)$, we have we have $\Box\mathbf{for}(Z') \in \mathbf{suc}(X_1)$. Using $X_1 \in \mathbf{next}^+(X)$, we obtain $X_1 \in \mathbf{prov}_3(X)$, which is in contradiction with Theorem 4.2 and $X_1 \in \mathbf{next}(X)$. ■

LEMMA 5.5. *Let X_1 be a sequent in $\mathbf{G}^*(n+1)$ and let Y_0 be a sequent in $\mathbf{G}(k)$ $(k = 0, 1, \cdots, n)$. If $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1)$ and $(\mathbf{ant}(X_1))^\Box \cap \mathbf{BG}_k \subsetneq (\mathbf{ant}(Y_0))^\Box$, then there exists a sequent $Y \in W_u$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$ and $\Box\mathbf{for}(Y_0) \in \{\Box\mathbf{for}(Y)\} \cup \mathbf{suc}(Y)$.*

PROOF. We use an induction on $n - k$.

Basis($k = n$). We note that $(\mathbf{ant}(X))^\Box \cap \mathbf{BG}_n \subsetneq (\mathbf{ant}(Y_0))^\Box$. Using Lemma 5.4, we have $Y_0 \in \mathbf{G}^*(n) \subseteq W_u$.

Induction step($k < n$). If $Y_0 \in \mathbf{G}^*(k)$, then the lemma is clear. We assume that $Y_0 \in \mathbf{G}(k) - \mathbf{G}^*(k)$. By Lemma 4.7(1), we have

$$Y = (\mathbf{ant}(X) \cap \mathbf{BG}_{k+1} \to \mathbf{suc}(X) \cap \mathbf{BG}_{k+1}) \in \mathbf{G}(k+1)$$

and $\Box\mathbf{for}(()Y_0)$ belongs to the succedent of the above sequnet. Also by Lemma 4.6

$$Y_1 = (\mathbf{ant}(X) \cap \mathbf{BG}_{k+2} \to \mathbf{suc}(X) \cap \mathbf{BG}_{k+2}) \in \mathbf{next}(Y)$$

Using Theorem 4.2, $Y_1 \notin \mathbf{prov}_2(Y)$. So, there exists a sequent $Z \in \mathbf{next}(Y_0)$ such that $(\mathbf{ant}(Y)^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box}$ and $\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_1) \subseteq \mathbf{ant}(X_1)$. We note that

$$(\mathbf{ant}(X_1))^{\Box} \cap \mathbf{BG}_{k+1} \subsetneq ((\mathbf{ant}(X_1))^{\Box} \cap \mathbf{G}(k+1)) \cup (\mathbf{ant}(Y_0))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box}.$$

So, using the induction hypothesis, then there exists a sequent $Z_1 \in W_u$ such that $\Box\mathbf{for}(Z_1) \in \mathbf{suc}(X_1)$ and $\Box\mathbf{for}(Z) \in \{\Box\mathbf{for}(Z_1)\} \cup \mathbf{suc}(Z_1)$. Using $\Box\mathbf{for}(Y_0) \to \Box\mathbf{for}(Z) \in \mathbf{S4}$, we have $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(Z_1)$. Hence we obtain the lemma. ■

Lemma 5.6. *Let X_1 be a sequent in $\mathbf{G}(n)$ and let Y_0 be a sequnet in $\mathbf{G}(k) - \mathbf{G}^*(k)$ $(k = 0, 1, \cdots, n-1)$. Then $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1)$ and $(\mathbf{ant}(X_1))^{\Box} \cap \mathbf{BG}_k = (\mathbf{ant}(Y_0))^{\Box}$ imply*

$$((\mathbf{ant}(X_1))^{\Box}, \mathbf{ant}(Y_0) \cap \mathbf{V} \to \mathbf{suc}(Y_0) \cap \mathbf{V}, (\mathbf{suc}(X_1))^{\Box}) \in \mathbf{G}(n).$$

Proof. We use an induction on n. Basis($n = 0$) is clear. We show Induction Step($n > 0$). By $n > 0$, there exists a sequent $X \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X_1 \in \mathbf{next}(X)$. Using $(\mathbf{ant}(Y_0))^{\Box} = (\mathbf{ant}(X_1))^{\Box} \cap \mathbf{BG}_k$, we have $(\mathbf{ant}(Y_0))^{\Box} = (\mathbf{ant}(X))^{\Box} \cap \mathbf{BG}_k$. We define Y as

$$Y = ((\mathbf{ant}(X))^{\Box}, \mathbf{ant}(Y_0) \cap \mathbf{V} \to \mathbf{suc}(Y_0) \cap \mathbf{V}, (\mathbf{suc}(X))^{\Box}) \in \mathbf{G}(n).$$

So, if $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X)$, then by the induction hypothesis, we have $Y \in \mathbf{G}(n)$; if not, we have $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1) \cap \mathbf{G}(n-1)$, and using $(\mathbf{ant}(Y_0))^{\Box} = (\mathbf{ant}(X))^{\Box} \cap \mathbf{BG}_k = (\mathbf{ant}(X))^{\Box} =$, we also have $Y = Y_0 \in \mathbf{G}(n)$. So, in any case, we have $Y =\in \mathbf{G}(n)$. Using $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and Lemma 3.9, we have $Y =\notin \mathbf{G}^*(n)$. So,

$$((\mathbf{ant}(X_1))^{\Box}, \mathbf{ant}(Y_0) \cap \mathbf{V} \to \mathbf{suc}(Y_0) \cap \mathbf{V}, (\mathbf{suc}(X_1))^{\Box}) \in \mathbf{next}^+(Y).$$

Also it is not hard to see that the above sequent belongs to none of $\mathbf{prov}_1(Y)$, $\mathbf{prov}_2(Y)$ and $\mathbf{prov}_3(Y)$, and so, it belongs to $\mathbf{next}(Y) \subseteq \mathbf{G}(n)$. ■

Lemma 5.7. *Let X and Y be sequents in W_u. If $R_u(X, Y)$, then either one of the following two holds:*
(1) $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$,
(2) $\Box\mathbf{for}(\{Z \in W_u \mid (\mathbf{suc}(Y))^{\Box} = (\mathbf{suc}(Z))^{\Box}\}) \cup (\mathbf{suc}(Y))^{\Box} \subseteq (\mathbf{suc}(X))^{\Box}$.

Proof. By $R_u(X, Y)$, there exist sequent $X_1, \cdots, X_\ell \in W_u$ such that
$X = X_1$,
$\Box\mathbf{for}(X_{i+1}) \in \mathbf{suc}(X_i)$ or $(\mathbf{ant}(X_i))^{\Box} = (\mathbf{ant}(X_{i+1}))^{\Box}$,
$X_\ell = Y$.
We use an induction on ℓ. If $\ell = 1$, then $X = X_1 = X_\ell = Y$, and so, (1) holds. Suppose that $\ell > 1$. By the induction hypothesis, we have either one of the following two:
(3) $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(X_{\ell-1}))^{\Box}$,
(4) $\Box\mathbf{for}(\{Z \in W_u \mid (\mathbf{suc}(X_\ell))^{\Box} = (\mathbf{suc}(Z))^{\Box}\}) \cup (\mathbf{suc}(X_\ell))^{\Box} \subseteq (\mathbf{suc}(X))^{\Box}$.
Also either one of the following two holds:
(5) $(\mathbf{ant}(X_{\ell-1}))^{\Box} = (\mathbf{ant}(Y))^{\Box}$,
(6) $\Box\mathbf{for}(\{Z \in W_u \mid (\mathbf{suc}(Y))^{\Box} = (\mathbf{suc}(Z))^{\Box}\}) \cup (\mathbf{suc}(Y))^{\Box} \subseteq (\mathbf{suc}(X_{\ell-1}))^{\Box}$.
Using Lemma 3.6, we have
(3) and (5) imply (1),
(3) and (6) imply (2),
(4) and (5) imply (2),
(4) and (6) imply (2). ■

Lemma 5.8. *Let X be a sequent in $\mathbf{G}^*(n)$. If $(\mathbf{UM}, X) \not\models \mathbf{for}(X)$, then for any $Y \in \mathbf{G}(n+k)$ $(k \geq 0)$.*

$$(\mathbf{UM}, X) \not\models Y \textit{ if and only if } X = Y.$$

PROOF. If $k = 0$, then by Lemma 3.6, we obtain the lemma. Suppose that $k > 0$. We define Z_0 as

$$Z_0 = (\mathbf{ant}(Y) \cap \mathbf{BG}_n \to \mathbf{suc}(Y) \cap \mathbf{BG}_n).$$

By sketching the proof of Lemma 4.7(1), we have

$$Y \in \mathbf{G}(n+k) - \mathbf{G}^*(n+k) \text{ implies } Z_0 \in \mathbf{G}(n) - \mathbf{G}^*(n).$$

Also by $k > 0$ considering the fact that $Z \in \mathbf{next}(Z_0')$ for some $Z_0' \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$, we have

$$Z_0 \in \mathbf{G}(n) - \mathbf{G}^*(n).$$

By $X \in \mathbf{G}^*(n)$, we have $X \neq (\mathbf{ant}(Y) \cap \mathbf{BG}_n \to \mathbf{suc}(Y) \cap \mathbf{BG}_n)$. So, smilarly to the case that $k = 0$, $(\mathbf{UM}, X) \models \mathbf{ant}(Y) \cap \mathbf{BG}_n \to \mathbf{suc}(Y) \cap \mathbf{BG}_n$, and so $(\mathbf{UM}, X) \models Y$. ■

LEMMA 5.9. *Let X be a sequent in $\mathbf{G}^*(n)$ and let A be a formula in $\mathbf{BG}_k$. Then $(\mathbf{UM}, X) \not\models A$ if and only if either one of the following two holds:*
(1) $A \in \mathbf{suc}(X)$,
(2) $A \in \{\Box\mathbf{for}(Y) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box\}$.

PROOF. We use an induction on $n\omega + k$.
Basis($n = 0$). Clear from $\mathbf{G}_0^* = \emptyset$.
Induction step($n > 0$).
If $A \in \mathbf{V}$ ($k = 0$), then from the definition of P_u, we obtain

$$(\mathbf{UM}, X) \not\models A \text{ if and only if } A \in \mathbf{suc}(X).$$

So, we assume that $A \notin \mathbf{V}$. Then there exists a sequent $Y_0 \in \mathbf{G}(k')$ such that $k' < k$ and $A = \Box\mathbf{for}(Y_0)$.

We show the "only if" part. Suppose that $(\mathbf{UM}, X) \not\models \Box\mathbf{for}(Y_0)$. Then there exists a sequent $X_0 \in W_u$ such that $R_u(X, X_0)$ and $(\mathbf{UM}, X_0) \not\models \mathbf{for}(Y_0)$. Using Lemma 5.7, either one of the following two holds:
(3) $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(X_0))^\Box$,
(4) $\Box\mathbf{for}(\{Z \in W_u \mid (\mathbf{suc}(X_0))^\Box = (\mathbf{suc}(Z))^\Box\}) \cup (\mathbf{suc}(X_0))^\Box \subseteq (\mathbf{suc}(X))^\Box$.
We divide the cases.

The case that (3) holds. We note that $X_0 \in \mathbf{G}^*(n)$. If $k' \geq n$, then by Lemma 5.8 and $(\mathbf{UM}, X_0) \not\models \mathbf{for}(Y_0)$, we have $X_0 = Y_0$, and using (3), we obtain (2). So, we assume that $k' < n$. Then by Lemma 3.6(1), $\Box\mathbf{for}(Y_0) \in \mathbf{ant}(X_0) \cup \mathbf{suc}(X_0)$. Using Lemma 2.4, $X_0 \notin \mathbf{S4}$, and so, $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_0)$. Using (3) and Lemma 3.6(1), we obtain (1).

The case that (4) holds. By $\Box\mathbf{for}(X_0) \in (\mathbf{suc}(X))^\Box$, we have $X_0 \in \mathbf{G}^*(k)$ for some $k < n$. Also by $(\mathbf{UM}, X_0) \not\models \mathbf{for}(Y_0)$, we have $(\mathbf{UM}, X_0) \not\models \Box\mathbf{for}(Y_0)$. Using the induction hypothesis, either one of the following two holds:
(5) $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_0)$,
(6) $\Box\mathbf{for}(Y_0) \in \Box\mathbf{for}(\mathbf{G}(k))$ and $(\mathbf{ant}(X_0))^\Box = (\mathbf{ant}(Y_0))^\Box$.
Using (4), we obtain (1).

We show the "if" part.

Suppose that (1) holds. Then $A = \Box\mathbf{for}(Y_0) \in \mathbf{suc}(X)$. If $(\mathbf{ant}(X))^\Box \cap \mathbf{BG}_k \not\subseteq (\mathbf{ant}(Y_0))^\Box$, then $X \in \mathbf{S4}$, which is in contardiction with $X \in \mathbf{G}(n)$. So, we assume that $(\mathbf{ant}(X))^\Box \cap \mathbf{BG}_k \subseteq (\mathbf{ant}(Y_0))^\Box$ and divide the cases.

The case that $(\mathbf{ant}(X))^\Box \cap \mathbf{BG}_k = (\mathbf{ant}(Y_0))^\Box$. By Lemma 5.6,

$$Y = ((\mathbf{ant}(X))^\Box, \mathbf{ant}(Y_0) \cap \mathbf{V} \to \mathbf{suc}(Y_0) \cap \mathbf{V}, (\mathbf{suc}(X))^\Box) \in \mathbf{G}(n).$$

Using Lemma 3.9, $Y \in \mathbf{G}^*(n)$. Also we have $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y)$. Using the induction hypothesis, for any $B \in \mathbf{BG}_{k'}$,
$B \in \mathbf{ant}(Y_0)$ implies $(\mathbf{UM}, Y) \models B$,
$B \in \mathbf{suc}(Y_0)$ implies $(\mathbf{UM}, Y) \not\models B$.
Hence $(\mathbf{UM}, Y) \not\models \mathbf{for}(Y_0)$. By the definition of R_u, we have $R_u(X, Y)$. Hence $(\mathbf{UM}, X) \not\models \Box\mathbf{for}(Y_0)$.

The case that $(\mathbf{ant}(X))^{\Box} \cap \mathbf{BG}_k \subsetneq (\mathbf{ant}(Y_0))^{\Box}$. By Lemma 5.5, there exists a sequent $Y \in W_u$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ and $\Box\mathbf{for}(Y_0) \in \{\Box\mathbf{for}(Y)\} \cup \mathbf{suc}(Y)$. By $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$, we have $R_u(X, Y)$ and $Y \in \mathbf{G}^*(\ell)$ for some $\ell < n$. Using the induction hypothesis, $(\mathbf{UM}, Y) \not\models \Box\mathbf{for}(Y_0)$. By $R_u(X, Y)$ and the transitivity of R_u, we obtain $(\mathbf{UM}, X) \not\models \Box\mathbf{for}(Y_0)$.

Suppose that (2) holds. Then by Lemma 3.9, we have $Y_0 \in \mathbf{G}^*(n)$ and $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y_0))^{\Box}$, and so, $Y_0 \in W_u$ and $R_u(X, Y_0)$. By the induction hypothesis, for any $B \in \mathbf{BG}_{k'}$,

$B \in \mathbf{ant}(Y_0)$ implies $(\mathbf{UM}, Y_0) \models B$,

$B \in \mathbf{suc}(Y_0)$ implies $(\mathbf{UM}, Y_0) \not\models B$.

Hence $(\mathbf{UM}, Y_0) \not\models \mathbf{for}(Y_0)$. Using $R_u(X, Y_0)$, $(\mathbf{UM}, X) \not\models \Box\mathbf{for}(Y_0)$. ■

COROLLARY 5.10. *Let X be a sequent in W_u. Then*

$$(\mathbf{UM}, X) \not\models \mathbf{for}(X).$$

PROOF. By Lemma 5.9 and Lemma 3.6, we obtain the corollary. ■

LEMMA 5.11. *Let X be a sequent in $\mathbf{G}(n)$. Then for any $Y \in W_u$,*

$$(\mathbf{UM}, Y) \not\models \mathbf{for}(X) \text{ if and only if } Y \in \overrightarrow{X}.$$

PROOF. By Corollary 5.10, we have

$$(\mathbf{UM}, Y) \not\models \mathbf{for}(Y).$$

If $Y \in \mathbf{G}^*(k)$ for some $k \leq n$, then by Lemma 5.8,

$$(\mathbf{UM}, Y) \not\models \mathbf{for}(X) \text{ if and only if } Y = X,$$

and so,

$$(\mathbf{UM}, Y) \not\models \mathbf{for}(X) \text{ if and only if } Y \in \overrightarrow{X}.$$

So, we assume that $Y \in \mathbf{G}^*(k)$ for some $k > n$. Also we define the sequent Y_0 as

$$Y_0 = (\mathbf{ant}(Y) \cap \mathbf{BG}_n \to \mathbf{suc}(Y) \cap \mathbf{BG}_n).$$

We note that

$$Y \in \overrightarrow{X} \text{ if and only if } X = Y_0,$$

and

$$(\mathbf{UM}, Y) \not\models \mathbf{for}(Y_0).$$

Using $X, Y_0 \in \mathbf{G}(n)$,

$$(\mathbf{UM}, Y) \not\models \mathbf{for}(X) \text{ if and only if } X = Y_0.$$

Hence we obtain the lemma. ■

LEMMA 5.12. *Let X be a sequent in $\mathbf{G}(n)$. Then there exists a sequent $Y \in W_u \cap \overrightarrow{X}$.*

PROOF. If $X \in \mathbf{G}^*(n)$, then $X \in W_u \cap \overrightarrow{X}$, and so, we obtain the lemma. If $X \in \mathbf{G}^*(0)$, then using Theorem 2.3, the sequent

$$\Box\mathbf{for}(\mathbf{G}(0) - \{X\}), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(X)$$

belongs to $\mathbf{next}(X) \cap W_u$, and so, we obtain the lemma.

So, we assume that $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and $n > 0$. Let X_1 be the sequent in $\mathbf{next}^+(X)$ defined as

$$X_1 = (\mathbf{G}(n) - \mathbf{C}(X), \mathbf{ant}(X) \to \mathbf{suc}(X), \mathbf{C}(X),)$$

where $\mathbf{C}(X) = \{\Box\mathbf{for}(Z) \mid Z \in \mathbf{G}(n), (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Z))^{\Box}\}$. It is not hard to see that

$$X_1 \notin \mathbf{prov}_1(X) \cup \mathbf{prov}_2(X) \cup \mathbf{prov}_3(X).$$

Hence using Theorem 4.2, we obtain $X_1 \in \mathbf{next}(X) \subseteq \mathbf{G}(n+1)$.

We show $X_1 \in \mathbf{G}^*(n+1)$. Suppose that $X_1 \notin \mathbf{G}^*(n+1)$. Then there exist sequents $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and $Y_1 \in \mathbf{next}(Y)$ such that $(\mathbf{ant}(X_1))^\Box \subsetneq (\mathbf{ant}(Y_1))^\Box$. Using Lemma 3.6, there exists a sequent $Z \in \mathbf{G}(n)$ such that $\Box\mathbf{for}(Z) \in \mathbf{C}(X) \cap (\mathbf{ant}(Y_1))^\Box$. Also by $n > 0$, there exists a sequent $Z_0 \in \mathbf{G}(n-1)$ such that $Z \in \mathbf{next}(Z_0)$. Using Lemma 3.6, we have $\Box\mathbf{for}(\mathbf{next}(Z_0)) \cap \mathbf{C}(X) = \{\Box\mathbf{for}(Z)\}$. Hence

$$\Box\mathbf{for}(\mathbf{next}(Z_0)) \subseteq (\mathbf{ant}(X_1))^\Box \cup \{\Box\mathbf{for}(Z)\} \subseteq (\mathbf{ant}(Y_1))^\Box$$

Also by $Z \in \mathbf{next}(Z_0)$, we have $\Box\mathbf{for}(Z_0) \notin \mathbf{ant}(Z)$, using $Z \in \mathbf{C}(X)$, $\Box\mathbf{for}(Z_0) \notin \mathbf{ant}(X)$, and so, $\Box\mathbf{for}(Z_0) \in \mathbf{suc}(X) = \mathbf{suc}(Y) \subseteq \mathbf{suc}(Y_1)$. Using Corollary 3.4(1), we have $Y_1 \in \mathbf{S4}$, which is in contradiction with $Y_1 \in \mathbf{next}(Y)$.

■

LEMMA 5.13. *Let A be a formula. Then*

$$A \in \mathbf{S4} \textit{ if and only if } \mathbf{UM} \models A.$$

PROOF. It is easily seen that **UM** is a reflexive and transitive Kripke model. So, we have the "only if" part.

Suppose that $A \in [\bigwedge \mathbf{for}(\mathcal{S})]$ for some $\mathcal{S} \in 2^{\mathbf{G}^n} - \{\emptyset\}$. Let be that $X \in \mathcal{S}$. By Lemma 5.12 and Lemma 5.11, there exists $Y \in W_u$ such that $(\mathbf{UM}, Y) \models \Box\mathbf{for}(X)$. Hence $(\mathbf{UM}, Y) \models A$. ■

From Theorem 2.3 and Lemma 5.11, we obtain Theorem 5.3(2), and so, we have Theorem 5.3 (3). Using Theorem 5.13, we obtain Theorem 5.3(1).

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