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**Abstract**

We consider a hub network design model based on the Stackelberg hub location model, where two firms compete with each other to maximize their own profit. The firm as a leader first locates $p$ hubs and decides which OD pairs should be in services on the condition that the other firm as a follower locates $q$ hubs and decides its strategies in a similar way after that. To avoid the possibility of unprofitable services, we incorporate flow threshold constraints into the model. We formulate the leader's problem as a bilevel programming problem with the follower's problem as a lower level problem. We solve the problem with the complete enumeration method. The main objective is to make it clear how the network structure can be affected by the flow threshold constraints and the competitor's strategies.

## 1 Introduction

Since O'Kelly [4] formulated a discrete hub location problem as a quadratic integer programming problem, a variety of hub location models have been studied in the last two decades. However, studies on hub location problems in a competitive environment are scarce. Marianov, Serra and ReVelle [2] first addressed a competitive hub location model with the objective of maximizing the sum of captured flow and solved the problem using a tabu heuristic. Sasaki et al. [5] developed the Stackelberg hub location model, where two firms compete to maximize their own profit. A similar Stackelberg location-allocation model was presented by Serra and ReVelle [6] with the objective of minimize to maximum market capture by the follower firm.

Most hub location models studied so far assume that the firms provide their services for all OD pairs in a market. As a result, they also have to operate some routes with extremely low flows. To avoid the possibility of such unprofitable services, we incorporate flow threshold constraints into the model, which prohibit providing services not expecting enough captured flows. Campbell [1] first introduced threshold scheme on arcs as well as arc capacities into hub location models. We consider a hub network design model in a competitive environment, where hub locations and operating routes (services) are both determined.

The firm as a leader first locates $p$ hubs and decides which OD pairs should be in services on the condition that the other firm as a follower locates $q$ hubs and decide its strategies in a similar way after that. We formulate the leader's problem as a bilevel programming problem with follower's problem as the lower level problem. By introducing flow threshold constraints in to a competitive hub location model, we can enrich the model so as to develop a comprehensive hub network design model for more practical use.

This paper is organized as follows. In Section 2, we briefly review the Stackelberg hub location model which forms the basis of the new hub network design model. In Section 3, we explain the presented model and formulate it as a bilevel programming problem. In Section 4, we explain how to solve the problem by using a brute force procedure. In Section 5, we show computational results using real airlines' data, i.e. the CAB data. In Section 6, we give concluding remarks and mention some future work.

# 2 Brief Review of Stackelberg Hub Location Model

In this section, we briefly review the Stackelberg hub location model [5], which forms the basis of a new competitive hub network design model. In the Stackelberg hub location model, we assume the following conditions:

(i). There are one big firm and several medium firms exist in a market. They provide services for OD pairs by locating hubs with the objective of maximizing their own profit.

(ii). The trip demands among all OD pairs are assumed to be known and symmetric.

(iii). The level of captured passengers is determined by a logit function [3]. Specifically, we assume that there are $k$ services available for an OD pair and let $u_i, i = 1, \cdots, k$, be the disutility of the $i$-th service. Then the level of capture for the $i$-th service is determined by

$$L_i(u) = \frac{\exp[-\alpha u_i]}{\sum_{j=1}^{k} \exp[-\alpha u_j]}, \quad i = 1, \cdots, k, \tag{1}$$

where $\alpha > 0$ is a parameter.

(iv). The airfare for an OD pair is the same regardless of which firm provides the service, i.e., there is no price competition.

(v). The followers' service sets are subsets of the leader's service set and the followers' service sets are mutually disjoint, i.e., there is no competition among the followers.

(vi). The big firm is the leader and the other firms are the followers. After the leader locates its hub, the followers locate their hubs simultaneously. The leader firm knows that the follower firms are going to locate their new hubs after knowing the leader's decision. So the leader firm has to locate its new hub, given that the follower firms make optimal decisions.

(vii). Each hub can be located anywhere on the plane (continuous location model) and there is no capacity limit on the passengers who use it. Hubs are only for the use of a facility for transfer and they have no trip demand of their own.

(viii). Services between each OD pair are provided via one hub (one-stop service). Services through more than one hub and nonstop services are not allowed.

Under these assumptions, each firm locates its new hub one by one. Sasaki and Fukushima [5] reported interesting computational results of Stackelberg hub location model. Specifically, they make it clear how the optimal location and the market share are affected by the rival firms. On the other hand, the results bring new issues for further improvements of the model. As in the case with many hub location models addressed so far, the firms often have to provide services even if they capture few demand. Since the network structure is necessarily determined if the hub locations are fixed, the firms are forced to provide such unprofitable services. Moreover, the assumption that the service sets are predetermined seems to be unrealistic. To overcome these problems, it may be useful to consider a hub network design model, where the optimal location and the services to be provided are both determined. More precisely, we incorporate flow threshold constraints into the model to deal with the problems. We describe the threshold as a lower limit of the market share of each OD pair rather than the actual amount of captured demand. Namely, firms cannot provide any services whose captured market share does not reach to the predetermined level. By using the market share as a measure of threshold, it becomes easy to compare the results for problems of different size.

# 3 Formulation of Hub Network Design Model

Before we formulate the model, we provide a model description to make it clear the difference compared with the Stackelberg hub location model. Suppose that one leader and one follower exist in a market and they compete with each other to maximize their own profit as the same in the Stackelberg hub location model. The major difference is the network structure. Although the Stackelberg hub location model allows to locate hubs anywhere in a plane, we rather consider a discrete network model, where demand nodes and hub candidates are both given as a discrete node set. Let Firm A denote the leader firm and Firm B denote the follower firm. We employ the following notation:

- $N$: the set of demand nodes, $|N| = n$.
- $H$: the set of hub candidates, $|H| = h$.
- $\Pi$: the set of OD pairs, $\Pi \subseteq N \times N$.
- $d_\pi$: the direct distance between OD pair $\pi \in \Pi$.
- $c_{\pi k}$: the actual travel distance between OD pair $\pi \in \Pi$ via hub $k \in H$.
- $t_\pi$: the flow threshold of OD pair $\pi \in \Pi$, $0 \leq t_\pi \leq 0.5$.
- $W_\pi$: the trip demand (the number of passengers) for OD pair $\pi \in \Pi$.
- $F_\pi$: the airfare for OD pair $\pi \in \Pi$.
- $M$: a large number.

Note that the flow threshold $t_\pi$ is given by the market share. We introduce the design variables to describe which OD pairs should be in services as well as the location variables.

The decision variables of the firms are as follows:

- $x_k$: binary variable such that $x_k = 1$ if node $k \in H$ is selected as a Firm A's hub, and 0 otherwise.
- $y_k$: binary variable such that $y_k = 1$ if node $k \in H$ is selected as a Firm B's hub, and 0 otherwise.
- $u_\pi$: binary variable such that $u_\pi = 1$ if Firm A's service is provided on OD pair $\pi$, and $u_\pi = 0$ otherwise.
- $v_\pi$: binary variable such that $v_\pi = 1$ if Firm B's service is provided on OD pair $\pi$, and $v_\pi = 0$ otherwise.

As in the Stackelberg hub location model, we suppose that the captured demand level determined by the logit function given in (1), which is a function of those services' disutility. The disutility of Firm A's service $\eta_\pi^A(x_k)$ between OD pair $\pi = (i,j)$ using Firm A's hub $k \in H$ is defined as the ratio of the actual travel distance to the direct distance between the pair $(i,j)$, i.e., $\eta_\pi^A(x_k) = c_{\pi k}/d_\pi$. If a firm does not locate hub $k \in H$, no services through $k \in H$ are available. In such a case, the disutility of all service disutility through $k \in H$ is defined to be infinity. Therefore, the disutility of Firm A is given by

$$\eta_{\pi k}^A(x_k) = \begin{cases} c_{\pi k}/d_\pi, & \text{if } x_k = 1, \\ \infty, & \text{if } x_k = 0, \end{cases} \qquad \pi \in \Pi, k \in H.$$

In a similar manner, the disutility of Firm B's service $\eta_\pi^B(y_k)$ between OD pair $\pi = (i,j)$ using Firm B's hub $k \in H$ is given by

$$\eta_{\pi k}^B(y_k) = \begin{cases} c_{\pi k}/d_\pi, & \text{if } y_k = 1, \\ \infty, & \text{if } y_k = 0. \end{cases} \qquad \pi \in \Pi, k \in H.$$

Suppose that both Firm A and Firm B provide their services on an OD pair $\pi \in \Pi$. Then the market share of Firm A and Firm B on OD pair $\pi$ are given by

$$\phi_\pi(x,y) = \frac{\sum_{k\in H} \exp[-\alpha \eta_{\pi k}^A(x_k)]}{\sum_{k\in H} \exp[-\alpha \eta_{\pi k}^A(x_k)] + \sum_{k\in H} \exp[-\alpha \eta_{\pi k}^B(y_k)]}, \tag{2}$$

and

$$\psi_\pi(x,y) = \frac{\sum_{k\in H} \exp[-\alpha \eta_{\pi k}^B(y_k)]}{\sum_{k\in H} \exp[-\alpha \eta_{\pi k}^A(x_k)] + \sum_{k\in H} \exp[-\alpha \eta_{\pi k}^B(y_k)]} = 1 - \phi_\pi(x,y), \tag{3}$$

with a constant $\alpha > 0$, $x = (x_1, x_2, \cdots, x_h)^\top$, and $y = (y_1, y_2, \cdots, y_h)^\top$.

By taking design variables $u_\pi$ and $v_\pi$ into consideration, the actual market share captured by Firm A and Firm B are given by

$$\Phi_\pi(x,y,u,v) = \frac{\sum_{k\in H} \exp[-\alpha\eta^A_{\pi k}(x_k)]u_\pi}{\sum_{k\in H} \exp[-\alpha\eta^A_{\pi k}(x_k)]u_\pi + \sum_{k\in H} \exp[-\alpha\eta^B_{\pi k}(y_k)]v_\pi} \tag{4}$$

and

$$\Psi_\pi(x,y,u,v) = \frac{\sum_{k\in H} \exp[-\alpha\eta^B_{\pi k}(y_k)]v_\pi}{\sum_{k\in H} \exp[-\alpha\eta^A_{\pi k}(x_k)]u_\pi + \sum_{k\in H} \exp[-\alpha\eta^B_{\pi k}(y_k)]v_\pi}, \tag{5}$$

where $u = (u_1, u_2, \cdots, u_{|\Pi|})^\top$ and $v = (v_1, v_2, \cdots, v_{|\Pi|})^\top$. Consequently, the total revenues of Firm A and Firm B are given by

$$f(x,y,u,v) = \sum_{\pi\in\Pi} F_\pi W_\pi \Phi_\pi(x,y,u,v), \tag{6}$$

and

$$g(x,y,u,v) = \sum_{\pi\in\Pi} F_\pi W_\pi \Psi_\pi(x,y,u,v), \tag{7}$$

respectively. Now we proceed to formulating the problem. First we consider Firm B's problem. Given the Firm A's hub locations, Firm B will locate a hub so as to maximize its total revenue. So Firm B's problem, which is called HNDP-B, is written as follows:

[HNDP-B]

$$\begin{aligned}
\text{maximize}_{y,v} \quad & g(x,y,u,v) & & \\
\text{subject to} \quad & t_\pi - \psi_\pi(x,y) \le M(1-v_\pi), & \pi \in \Pi, & \quad (8)\\
& \sum_{k\in H} y_k = q, & & \quad (9)\\
& y_k \le 1 - x_k, & k \in H, & \quad (10)\\
& y_k \in \{0,1\}, & k \in H, & \\
& v_\pi \in \{0,1\}, & \pi \in \Pi. &
\end{aligned}$$

Constraints (8) prohibit providing services whose captured market share are less than flow threshold $t_\pi$. Constraint (9) ensures that Firm B locates $q$ hubs. Constraints (10) means that once Firm A locates hub $k \in H$, Firm B never locates hub $k \in H$. Firm A solves its own problem subject to the condition that Firm B finds the optimal solution of HNDP-B. More precisely, $[y,v] \in \arg\max\{g(x,y,u,v) | y \in Y, v \in V\}$ should be a constraint in Firm A's problem, where $Y$ and $V$ denote the feasible region of $y$ and $v$, respectively. Hence, Firm A's problem is stated as the following bilevel programming problem:

[HNDP]

$$
\begin{aligned}
\text{maximize} \quad & f(x,y,u,v) & & \\
\text{subject to} \quad & t_\pi - \phi_\pi(x,y) \le M(1-u_\pi), & \pi \in \Pi, & \quad (11)\\
& \sum_{k\in H} x_k = p, & & \quad (12)\\
& x_k \in \{0,1\} & k \in H, & \\
& u_\pi \in \{0,1\}, & \pi \in \Pi, & \\
& [y,v] \in \arg\max\{g(x,y,u,v) | y \in Y, v \in V\}. & &
\end{aligned}
$$

Constraints (11) prohibit providing services whose captured market share are less than flow threshold $t_\pi$. Constraint (12) ensures that Firm A locates $p$ hubs.

First, we establish that all demand is satisfied in HNDP. From (4) and (6), the value of function $f(x,y,u,v)$ increases as the value of $u_\pi$ increases. Also from (5) and (7), the value of function $g(x,y,u,v)$ increases as the value of $v_\pi$ increases. It follows that Firm A's service on OD pair $\pi$ that satisfies the threshold constraint $\phi_\pi(x,y) \ge t_\pi$ should be provided, i.e., $u_\pi = 1$ at the optimal solution. In a similar way, Firm B's service on OD pair $\pi$ that satisfies the threshold constraint $\psi_\pi(x,y) \ge t_\pi$ should be provided, i.e., $v_\pi = 1$ at the optimal solution. In addition, $\phi_\pi(x,y) + \psi_\pi(x,y) = 1$ is always satisfied for all $\pi$ by (2) and (3). Moreover, we define the value of $t_\pi$ ranges from 0 to 0.5 and hence at least $\phi_\pi(x,y) \ge t_\pi$ or $\psi_\pi(x,y) \ge t_\pi$ is always satisfied. Therefore, at least one of the two firms provide a service on each OD pair, implying that, all demand is satisfied, while passengers may not always take the most desired service.

# 4 Solution Method

We can obtain an optimal solution by the complete enumeration method. Assuming that $x$ and $y$ are fixed, we specify the following two sets: $H_A^1 = \{k \in H | x_k = 1\}$ and $H_B^1 = \{k \in H | y_k = 1\}$ . Then the market share of Firm A and Firm B on OD pair $\pi$ are given by

$$\tilde{\phi}_\pi = \frac{\displaystyle\sum_{k\in H_A^1} \exp[-\alpha c_{\pi k}/d_\pi]}{\displaystyle\sum_{k\in H_A^1} \exp[-\alpha c_{\pi k}/d_\pi] + \sum_{k\in H_B^1} \exp[-\alpha c_{\pi k}/d_\pi]}$$

and

$$\tilde{\psi}_\pi = \frac{\displaystyle\sum_{k\in H_B^1} \exp[-\alpha c_{\pi k}/d_\pi]}{\displaystyle\sum_{k\in H_A^1} \exp[-\alpha c_{\pi k}/d_\pi] + \sum_{k\in H_B^1} \exp[-\alpha c_{\pi k}/d_\pi]} = 1 - \tilde{\phi}_\pi,$$

respectively. Moreover, we define $\Pi_A^0 = \{\pi \in \Pi | \tilde{\phi}_\pi < t_\pi\}$ and $\Pi_B^0 = \{\pi \in \Pi | \tilde{\psi}_\pi < t_\pi\}$. It is necessary to be $u_\pi = 0$ for all $\pi \in \Pi_A^0$ and $v_\pi = 0$ for all $\pi \in \Pi_B^0$ to satisfy the constraints (8) and (11). As in the previous section, $f(x,y,u,v)$ and $g(x,y,u,v)$ are increasing functions of $u_\pi$ and $v_\pi$, respectively. Consequently, $u_\pi = 1$ for all $\pi \notin \Pi_A^0$ and $v_\pi = 1$ for all $\pi \notin \Pi_B^0$ to maximize the objective value under the condition of fixed $x$ and $y$. From the above observation, we see that to examine all possible combinations of $x$ and $y$ is sufficient to obtain the optimal solution of HNDP.

# 5 Computational Results

In this section, we report some computational results for the proposed model HNDP and examine how the optimal location and the total revenue affected by the flow threshold constraints and the passengers' preference (i.e., parameter $\alpha$). Computer programs were coded in MATLAB R13 (version 6.5.1). All programs were run on DELL DIMENSION 8300 computer with Intel Pentium 4 processor available in speeds of 3.0GHz operated under Windows XP professional with 2.0 GB DDR-SDRAM memory. We prepared the demand data based on the well-known U.S. 25 cities data evaluated in 1970 by CAB (Civil Aeronautics Board). For airfare data, we used the data supplied by http://www.airfare.com/. All figures presented in this section are prepared by using MATLAB and Mapping Toolbox (version 2.0.1).

For simplicity, we assume that the threshold is the same in all OD pairs and denote $t$ as the common flow threshold. We also assume that all demand nodes are hub candidates, that is, $H = N$. We solved 120 problems with $n = 25$ and various values of parameter $p$, $q$, $t$ and $\alpha$. More precisely, we solved the problem with $(p, q)$=(1,1), (1,2), (2,1), (2,2), (2,3) and (3,2), varying $\alpha$ from 1 to 4 by 1 and $t$ from 0.1 to 0.5 by 0.1. CPU time is about 3.5 seconds when $p = q = 1$, 35 seconds when $(p, q)$=(1,2) or (2,1), 390 seconds when $p = q = 2$, and about 2700 seconds when $(p, q)$=(2,3) or (3,2). In the results of many problems with $t = 0.5$, only one of the firms provides a service on each OD pair, a few exception is when the market is evenly shared by the two firms, that is when $\Phi_\pi(x, y) = \Psi_\pi(x, y)$. In this case, one of the firm capture all demand and the other capture nothing. Moreover, when $p = q = 1$, the results is the same as those brought by using all-or-nothing allocation rule. On the other hand, when $t = 0.1$, both firms provide their services on almost all OD pairs and demands are allocated by a logit function to each of them. Therefore, the results with $t = 0.1$ is approximate to those for problems with no flow threshold constraints. As we mention previously, we used a logit function so as to reflect passengers' various preferences. Note that the value of $\alpha$ becomes large, passenger preferences approach to all-or-nothing assignment.

We examine how the flow threshold constraints and the value of $\alpha$ affect the optimal objective value. Figure 1 shows the optimal objective values for the problems with $p = q = 1$, Figure 2 shows the optimal objective values for the problems with $p = q = 2$ and Figure 3 shows the optimal objective values for the problems with $p = 3$ and $q = 2$. In each figure, the solid line denotes the result for Firm A and the dotted line denotes the result for Firm B. The optimal objective values of Firm A are always larger than those of Firm B regardless of the value of $\alpha$ and $t$ when $(p, q)$=(2,2) and (3,2) (See Figure 2 and Figure 3). However, it is not necessarily the case, for example, Figure 1(a) and 1(b) indicate that Firm B's optimal objective values are larger. It follows that the leader does not always take advantage even on the condition that the follower is not allowed to locate hubs located by the leader. Moreover, in the result with $(p, q)$=(1,2) and (2,3), Firm B's optimal value is always larger. The reason is simply that the market share also depends on the number of located hubs. There is no clear relationship between the value of threshold and the leader's optimal value, however, the results indicate that it is advantageous to the leader in the case $t = 0.5$. Figure 4 displays optimal hub locations with $p = q = 2$ and $\alpha = 1$. Figure 5 displays optimal hub locations with $p = 3, q = 2$ and $\alpha = 3$. In these figure, "A" and "B" denote the optimal location of Firm A and Firm B, respectively. In both cases, optimal hub locations are very sensitive to the flow threshold, implying that, the threshold is one of the important factors. In the firms' point of view, the results may rather negative in the sense that they hard to find stable hub locations.

## 6 Conclusion and Future Work

We proposed a new hub network design problem in a competitive environment based on the Stackelberg hub location model. Specifically, we incorporated flow threshold constraints in to the model to determine which services should be provided. We formulated the problem as a bilevel programming problem, where the upper and lower problems are both 0-1 integer programming problems, and solved 120 instances by using the brute force procedure. Computational results showed that optimal location and the objective values are significantly affected by the value of threshold. We also observed that the leader cannot always take advantage even in the follower is prohibited to locate hubs at the same nodes of the leader.

In the presented model, we incorporated the flow threshold constraint on each OD pair. Another possibility is an arc flow oriented threshold, which is an interesting future work. It is also required to develop an effective solution method to solve larger problems.

## Acknowledgments

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## References

[1] J. F. Campbell: Integer programming formulations of discrete hub location problems, *European Journal of Operational Research*, **72** (1994) 387–405.

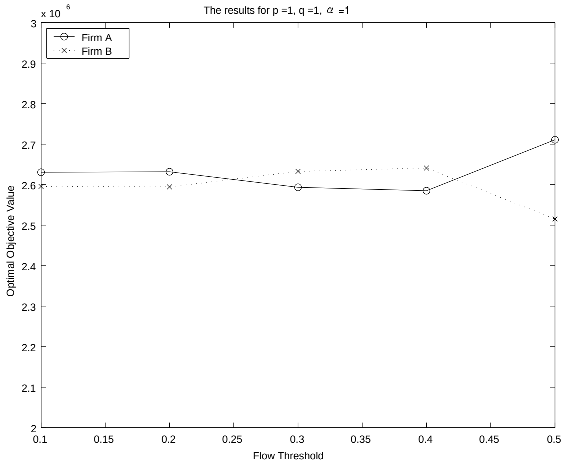
[2] V. Marianov, D. Serra and C. Revelle: Location of hubs in a competitive environment. *European Journal of Operational Research*, **114** (1999) 363–371.

[3] D. McFadden: Conditional logit analysis of qualitative choice behavior, In P. Zarembka (ed.): *Frontiers in Econometrics*, Academic Press, NY, 1974, pp. 105–142.

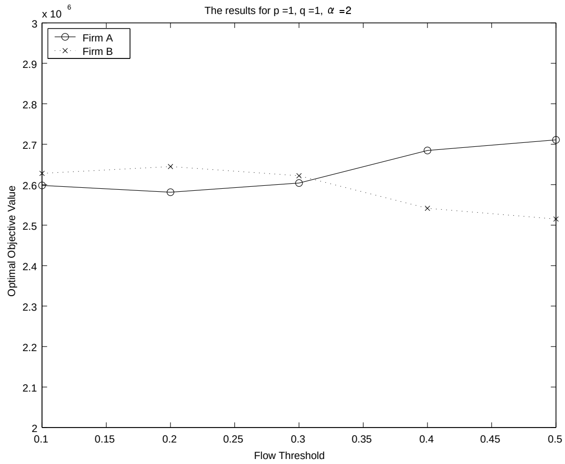
[4] M. E. O'Kelly: A quadratic integer program for the location of interacting hub facilities, *European Journal of Operational Research*, **32** (1987) 393–404.

[5] M. Sasaki and M. Fukushima: Stackelberg hub location problem, *Journal of the Operations Research Society of Japan*, **44** (2001) 390–402.

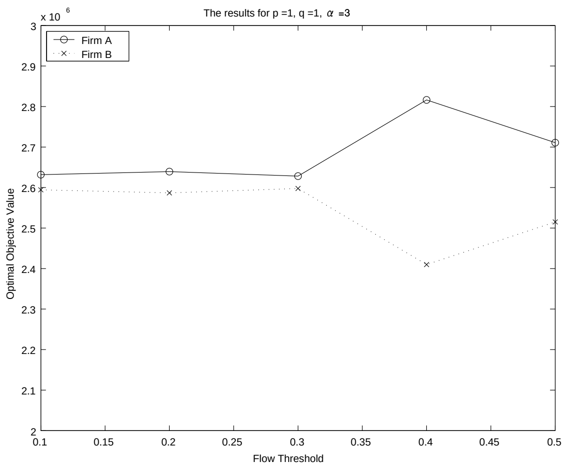
[6] D. Serra and C. ReVelle: Market capture by two competitors: the preemptive location problem, *Journal of Regional Science*, **34** (1994) 549–561.



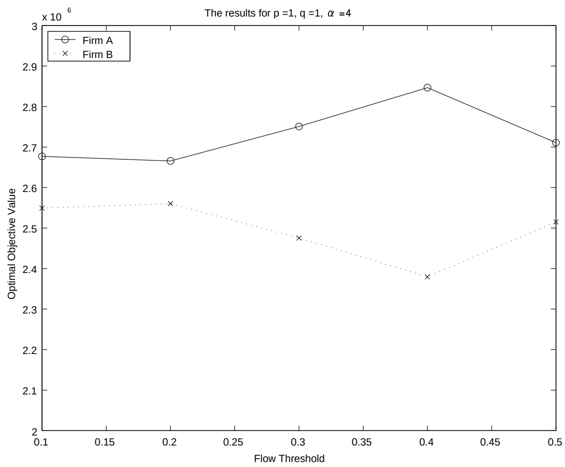

(a) $\alpha = 1$




(b) $\alpha = 2$

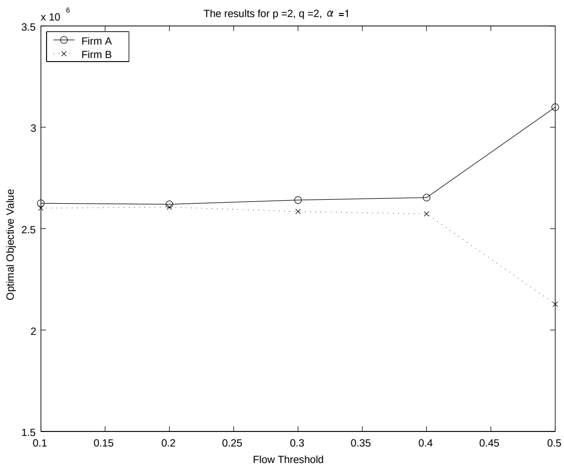



(c) $\alpha = 3$

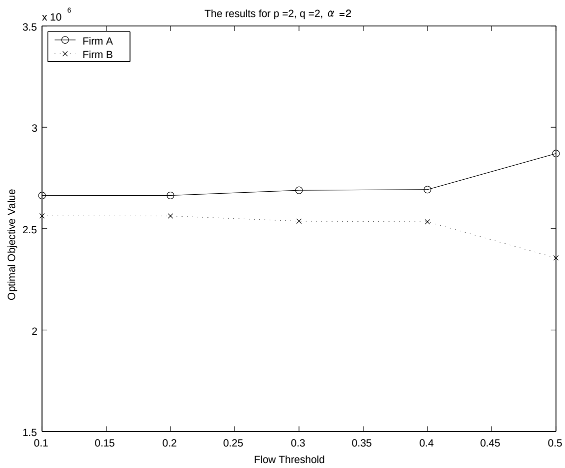



(d) $\alpha = 4$

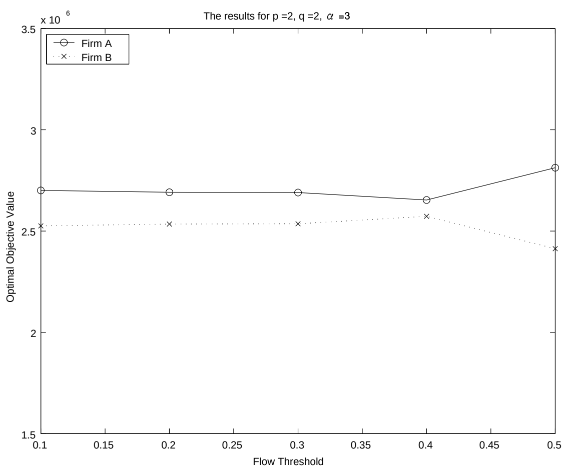
Figure 1: Optimal objective value for $n = 25, p = 1, q = 1$



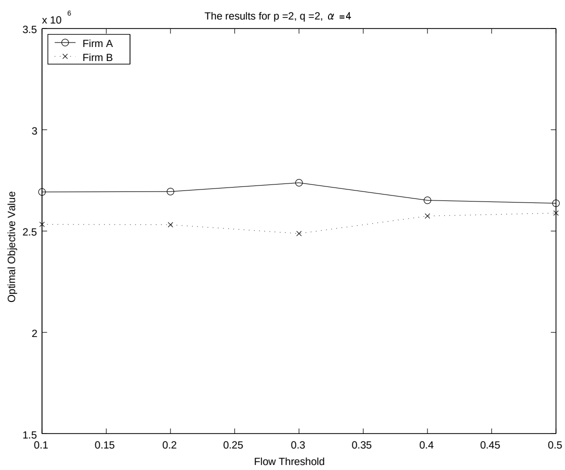

(a) $\alpha = 1$




(b) $\alpha = 2$

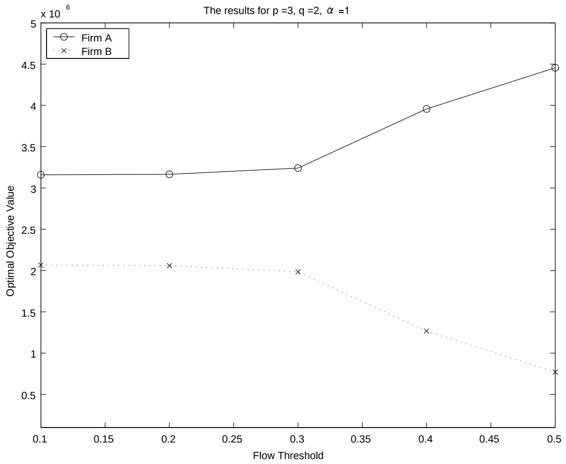



(c) $\alpha = 3$

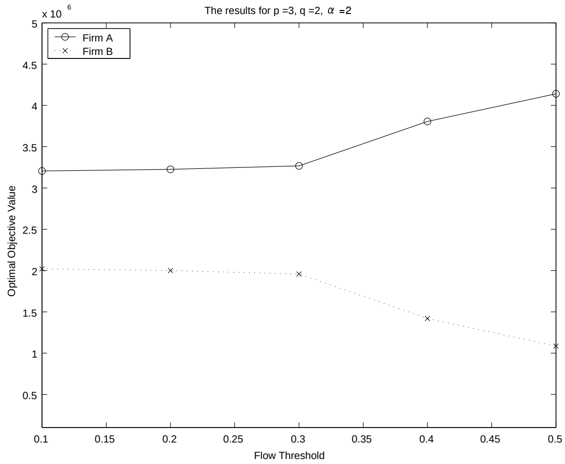



(d) $\alpha = 4$

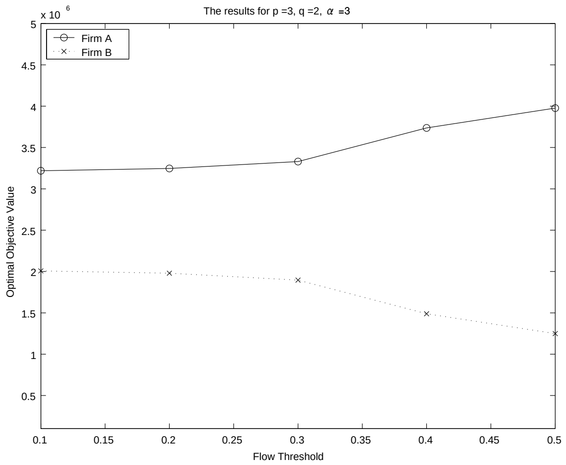
Figure 2: Optimal objective value for $n = 25, p = 2, q = 2$



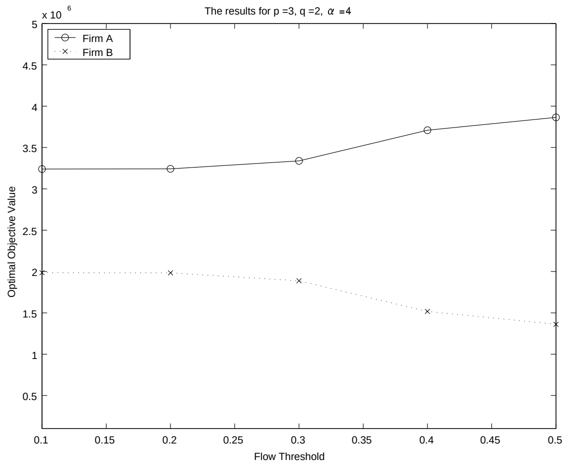

(a) $\alpha = 1$




(b) $\alpha = 2$

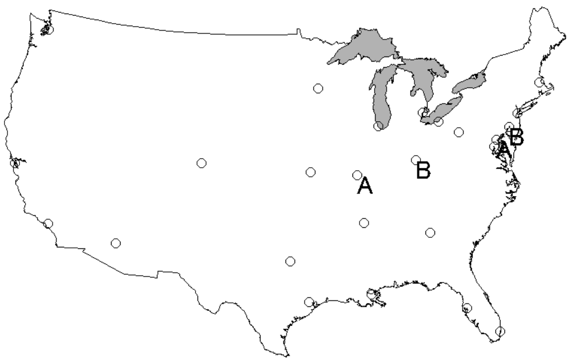



(c) $\alpha = 3$

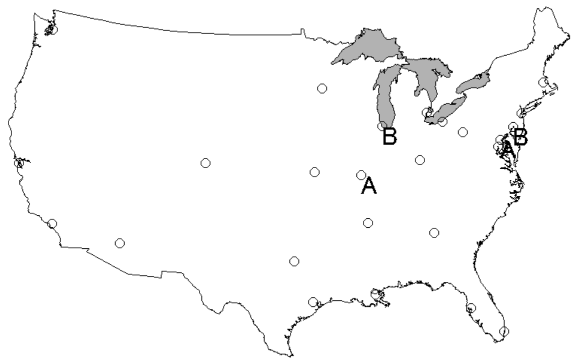



(d) $\alpha = 4$

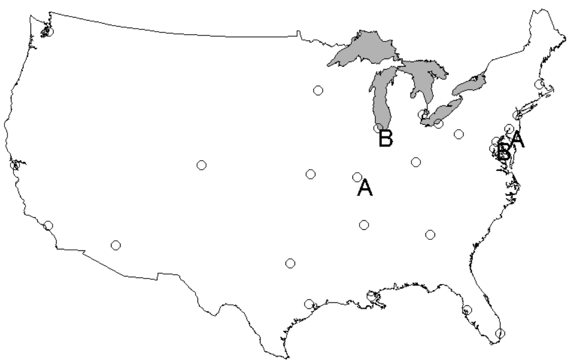
Figure 3: Optimal objective value for $n = 25, p = 3, q = 2$



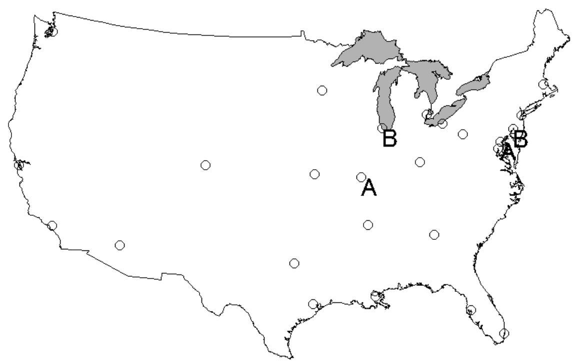

(a) Flow threshold=0.1



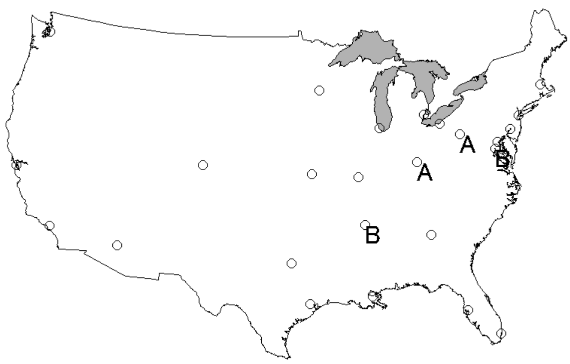

(b) Flow threshold=0.2




(c) Flow threshold=0.3

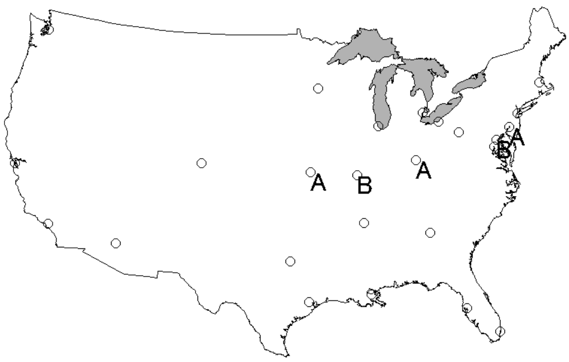



(d) Flow threshold=0.4

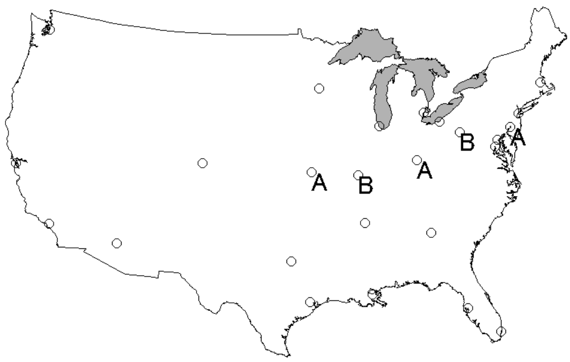



(e) Flow threshold=0.5

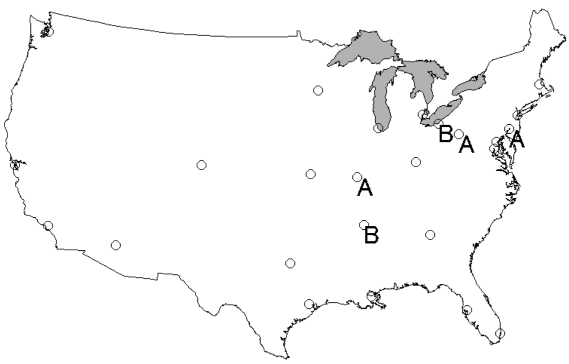
Figure 4: Results for $n = 25, p = 2, q = 2, \alpha = 1$



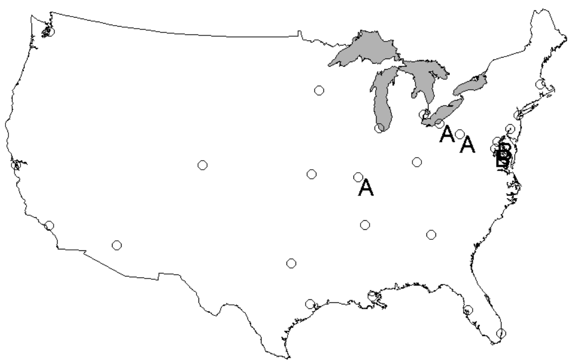

(a) Flow threshold=0.1



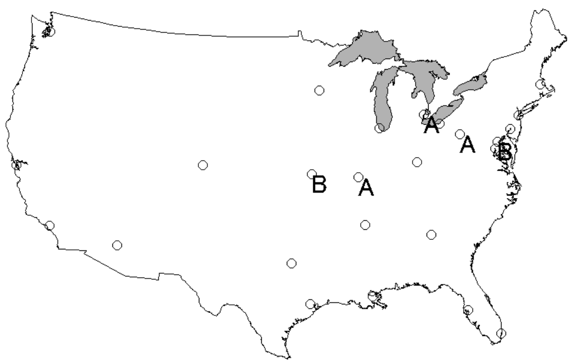

(b) Flow threshold=0.2




(c) Flow threshold=0.3




(d) Flow threshold=0.4




(e) Flow threshold=0.5

Figure 5: Results for $n = 25, p = 3, q = 2, \alpha = 3$