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Sequent Systems and Provability Logic R^{-} for Rosser Sentences

Katsumi Sasaki

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Sequent Systems and Provability Logic $\mathbf{R}^-$ for Rosser Sentences

Katsumi Sasaki
Department of Mathematical Sciences, Nanzan University,
27 Seirei-Cho, Seto 489-0863, Japan
e-mail: sasaki@ms.nanzan-u.ac.jp

Abstract

To discuss Rosser sentences, Guaspari and Solovay [GS79] enriched the modal language by adding, for each $\Box A$ and $\Box B$, the formulas $\Box A \prec \Box B$ and $\Box A \preceq \Box B$, with arithmetic realizations. They introduced provability logics $\mathbf{R}^-$, $\mathbf{R}$ and $\mathbf{R}^\omega$ with enriched language by extending the unimodal provability logic $\mathbf{GL}$ and proved kinds of arithmetic completeness for them.

A sequent system for $\mathbf{R}^-$, the most preliminary logic among the logics they introduced, was given in Sasaki and Ohama [SO03]. They proved a cut-elimination theorem in weakened form, and as a result, a kind of subformula property was shown. However, considering a cut-free system for $\mathbf{GL}$, their system has a cut, which seems to be removable. Here we introduce another system with a kind of subformula property and discuss what kinds of cuts are removable from the system in [SO03]. Also we give a proof of completeness theorem without the extension lemma in [GS79], which was used in [SO03].

1 The logic $\mathbf{R}^-$

In this section, we introduce the logic $\mathbf{R}^-$. We use logical constant $\bot$ (contradiction), and logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\Box$ (provability), $\preceq$ (witness comparison), and $\prec$ (witness comparison). Formulas are defined inductively as follows:

(1) propositional variables and $\bot$ are formulas,

(2) if A and B are formulas, then so are $(A \wedge B), (A \vee B), (A \supset B), (\Box A), (\Box A \prec \Box B)$ and $(\Box A \preceq \Box B)$.
A formula of the form $\Box A$ is said to be a $\Box$-formula. Also a formula of the form $\Box A \preceq \Box B$ ($\Box A \prec \Box B$) is said to be a $\preceq$-formula ($\prec$-formula). By Σ, we mean the set of all $\Box$-formulas, all $\prec$-formulas and all $\preceq$-formulas.

The modal system $\mathbf{R}^-$ is defined by the following axioms and inference rules.

Axioms of $\mathbf{R}^-$

$A1$: all tautologies,
$A2 : \Box(A \supset B) \supset (\Box A \supset \Box B)$,
$A3 : \Box(\Box A \supset A) \supset \Box A$,
$A4 : A \supset \Box A$, where $A \in \Sigma$,
$A5 : (\Box A \preceq \Box B) \supset \Box A$,
$A6 : ((\Box A \preceq \Box B) \wedge (\Box B \preceq \Box C)) \supset (\Box A \preceq \Box C)$,
$A7 : (\Box A \vee \Box B) \supset ((\Box A \preceq \Box B) \vee (\Box B \prec \Box A))$,
$A8 : (\Box A \prec \Box B) \supset (\Box A \preceq \Box B)$,
$A9 : ((\Box A \preceq \Box B) \wedge (\Box B \prec \Box A)) \supset \bot$,

Inference rules of $\mathbf{R}^-$

$MP : A, A \supset B \in \mathbf{R}^-$ implies $B \in \mathbf{R}^-$,
$N : A \in \mathbf{R}^-$ implies $\Box A \in \mathbf{R}^-$.

In [GS79] and Smoriński [Smo85], the following two formulas are also axioms of $\mathbf{R}^-$, but they are redundant (cf. De Jongh [Jon87] and Voorbraak [Voo90]).

$A10 : \Box A \supset (\Box A \preceq \Box A)$,
$A11 : (\Box A \wedge (\Box B \supset \bot)) \supset (\Box A \prec \Box B)$.

We introduce Kripke semantics for $\mathbf{R}^-$, following [Smo85].[1]

[1] [Smo85] uses a rooted frame, but our frame does not necessarily be rooted.

Definition 1.1. A Kripke pseudo-model for $\mathbf{R}^-$ is a triple $\langle \mathbf{W}, <, \models \rangle$ where
(1) $\mathbf{W}$ is a non-empty finite set,
(2) $<$ is an irreflexive and transitive binary relation on $\mathbf{W}$ satisfying

$$\alpha < \gamma \text{ and } \beta < \gamma \text{ imply either one of } \alpha = \beta,\ \alpha < \beta \text{ or } \beta < \alpha,$$

(3) $\models$ is a valuation satisfying, in addition to the usual boolean laws,

$$\alpha \models \Box A \text{ if and only if for any } \beta \in \alpha\!\uparrow (= \{\gamma \mid \alpha < \gamma\}),\ \beta \models A.$$

Definition 1.2. A Kripke pseudo-model $\langle \mathbf{W}, <, \models \rangle$ for $\mathbf{R}^-$ is said to be a Kripke model for $\mathbf{R}^-$ if the following conditions hold, for any formula D,
(1) if $D \in \Sigma$ and $\alpha \models D$, then for any $\beta \in \alpha\!\uparrow, \beta \models D$,
(2) if D is either one of the axioms $A5, A6, A7, A8$ and $A9$, then $\alpha \models D$.

Lemma 1.3([GS79]). *$A \in \mathbf{R}^-$ if and only if A is valid in any Kripke model for $\mathbf{R}^-$.*

2 A sequent system $\mathbf{GR}^-$

In this section we introduce a sequent system $\mathbf{GR}^-$ for $\mathbf{R}^-$. We use Greek letters, Γ and Δ, possibly with suffixes, for finite sets of formulas. The expression $\Box\Gamma$ denotes the set $\{\Box A \mid A \in \Gamma\}$. By a sequent, we mean the expression $\Gamma \to \Delta$. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

By $\mathsf{Sub}(A)$, we mean the set of subformulas of A. We put

$$\mathsf{Sub}^+(A) = \mathsf{Sub}(A) \cup \{\Box B \preceq \Box C | \Box B, \Box C \in \mathsf{Sub}(A)\} \cup \{\Box B \prec \Box C | \Box B, \Box C \in \mathsf{Sub}(A)\},$$

$$\mathsf{Sub}(\Gamma \to \Delta) = \bigcup_{B \in \Gamma \cup \Delta} \mathsf{Sub}(B), \qquad \mathsf{Sub}^+(\Gamma \to \Delta) = \bigcup_{B \in \Gamma \cup \Delta} \mathsf{Sub}^+(B).$$

By the sequent system **LK** for the classical propositional logic, we mean the system defined by the following axioms and inference rules in the usual way.

Axioms of LK:

$$A \to A$$

$$\bot \to$$

Inference rules of LK:

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}(w \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}(\to w)$$

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}(cut)$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta}(\wedge \to_i) \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2}(\to \vee_i)$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}(\supset\to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to\supset)$$

The system $\mathbf{GR}^-$ is obtained from the sequent system $\mathbf{LK}$ by adding the following axioms and inference rules in the usual way.

Additional axioms of $\mathbf{GR}^-$

$GA1$: $\Box A \preceq \Box B, \Box B \preceq \Box C \rightarrow \Box A \preceq \Box C$
$GA2$: $\Box A \rightarrow \Box A \preceq \Box B, \Box B \prec \Box A$
$GA3$: $\Box B \rightarrow \Box A \preceq \Box B, \Box B \prec \Box A$
$GA4$: $\Box A \prec \Box B \rightarrow \Box A \preceq \Box B$
$GA5$: $\Box A \preceq \Box B, \Box B \prec \Box A \rightarrow$

Additional inference rules of $\mathbf{GR}^-$

$$\frac{\Box A, \Sigma^f, \Gamma \rightarrow A}{\Sigma^f, \Box\Gamma \rightarrow \Box A}(\rightarrow \Box) \qquad \frac{\Box A, \Gamma \rightarrow \Delta}{\Box A \preceq \Box B, \Gamma \rightarrow \Delta}(\preceq\rightarrow) \qquad \frac{\Gamma \rightarrow \Delta, \Box A}{\Gamma \rightarrow \Delta, \Box A \preceq \Box A}(\rightarrow\preceq)$$

where Σ^f is a finite subset of Σ.

The system $\mathbf{GR}_1^-$ is the system obtained from $\mathbf{GR}^-$ by restricting a cut to the following form:

$$\frac{\Gamma \rightarrow \Delta, \Box A \odot \Box B \qquad \Box A \odot \Box B, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}$$

where $\odot \in \{\prec, \preceq\}$, and $\Box A$ and $\Box B$ are subformulas of a formula occurring in the lower sequent.

Theorem 2.1([SO03]). *The following conditions are equivalent:*
(1) $A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n \in \mathbf{GR}_1^-$,
(2) $A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n \in \mathbf{GR}^-$,
(3) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n \in \mathbf{R}^-$,
(4) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ *is valid in any Kripke model for* $\mathbf{R}^-$.

Corollary 2.2. *If a sequent S is provable in $\mathbf{GR}^-$, then there exists a proof figure $\mathcal{P}$ for S such that each formula occurring in $\mathcal{P}$ belongs to $\mathsf{Sub}^+(S)$.*

3 A sequent system for $\mathbf{R}^-$ without additional axioms

Theorem 2.1 provides the decision procedure for the provability of $\mathbf{R}^-$, but does not say that every cut in $\mathbf{GR}_1^-$ is necessary. For instance, the following cut seems to be removable if $\Gamma \rightarrow \Delta$ does not have any $\prec$-formula and $\preceq$-formula. Because we can easily see the $\prec$-free and $\preceq$-free fragment of $\mathbf{GR}^-$ is the system for the provability logic $\mathbf{GL}$, the $\prec$-free and $\preceq$-free fragment of $\mathbf{R}^-$, described in Valentini [Val83] and Avron [Avr84] and enjoying a cut-elimination theorem.

$$\frac{\dfrac{\Gamma \rightarrow \Delta, \Box A}{\Gamma \rightarrow \Delta, \Box A \preceq \Box A}(\rightarrow\preceq) \quad \dfrac{\Box A, \Gamma \rightarrow \Delta}{\Box A \preceq \Box A, \Gamma \rightarrow \Delta}(\preceq\rightarrow)}{\Gamma \rightarrow \Delta}(cut)$$

Here we introduce another system for $\mathbf{R}^-$ by adding only inference rules to $\mathbf{LK}$ and prove cut-elimination theorem. Also we consider what kind of cuts are removable from $\mathbf{GR}_1^-$.

The system $\mathbf{GR}_2^-$ is obtained from $\mathbf{LK}$ by adding the following inference rules in the usual way.

Additional inference rules of $\mathbf{GR}_2^-$

$(\rightarrow \Box), (\rightarrow\preceq), (\preceq\rightarrow)$ are as in $\mathbf{GR}^-$,

$$\frac{\Box A \preceq \Box B, \Gamma \to \Delta, \Box B \preceq \Box A}{\Box A \prec \Box B, \Gamma \to \Delta}(\prec\to)$$

$$\frac{\Gamma \to \Delta, \Box C \preceq \Box D \quad \Gamma \to \Delta, \Box D \preceq \Box E \quad \Box C \preceq \Box E, \Gamma \to \Delta}{\Gamma \to \Delta}(tran)$$

$$\frac{\Gamma \to \Delta, \Box C, \Box D \quad \Box C \prec \Box D, \Gamma \to \Delta \quad \Box D \prec \Box C, \Gamma \to \Delta \quad \Box C \preceq \Box D, \Box D \preceq \Box C, \Gamma \to \Delta}{\Gamma \to \Delta}(lin)$$

where $\Box C, \Box D$ and $\Box E$ are different subformulas occurring in the lower sequent.

The system $\mathbf{GR}_3^-$ is the system obtained from $\mathbf{GR}_2^-$ by removing cuts.

Lemma 3.1. *The following conditions are equivalent:*
(1) $\Gamma \to \Delta \in \mathbf{GR}^-$,
(2) $\Gamma \to \Delta \in \mathbf{GR}_2^-$.

Proof. For "(1) implies (2)". Additional inference rules of $\mathbf{GR}^-$ are also inference rule in $\mathbf{GR}_2^-$. So, it is sufficient to show the provability of the additional axioms of $\mathbf{GR}^-$ in $\mathbf{GR}_2^-$. Axioms $GA4(\Box A \prec \Box B \to \Box A \preceq \Box B)$ and $GA5(\Box A \preceq \Box B, \Box B \prec \Box A \to)$ are shown by the following figures.

$$\cfrac{\cfrac{\Box A \preceq \Box B \to \Box A \preceq \Box B}{\Box A \preceq \Box B \to \Box A \preceq \Box B, \Box B \preceq \Box A}\ (\to w)}{\Box A \prec \Box B \to \Box A \preceq \Box B}\ (\prec\to)$$

$$\cfrac{\cfrac{\Box A \preceq \Box B \to \Box A \preceq \Box B}{\Box B \preceq \Box A, \Box A \preceq \Box B \to \Box A \preceq \Box B}\ (\to w)}{\Box A \preceq \Box B, \Box B \prec \Box A \to}\ (\prec\to)$$

For $GA1(\Box A \preceq \Box B, \Box B \preceq \Box C \to \Box A \preceq \Box C)$. If A, B and C are different, then the provability can be shown by $(tran)$ and weakening rules. If $A = B$ or $B = C$, then it can be shown by weakening rules. If $A = C$, by the following figure.

$$\cfrac{\cfrac{\cfrac{\Box A \to \Box A}{\Box A \to \Box A \preceq \Box A}\ (\to\preceq)}{\Box A \preceq \Box B \to \Box A \preceq \Box A}\ (\preceq\to)}{\Box A \preceq \Box B, \Box B \preceq \Box A \to \Box A \preceq \Box A}\ (w\to)$$

For $GA2(\Box A \to \Box A \preceq \Box B, \Box B \prec \Box A)$. If $A \neq B$, then the provability can be shown by (lin), the provability of $GA4$ and weakening rules. If $A = B$, then it can be shown by $(\to\preceq)$ and weakening rules. The provability of $GA3$ can be shown similarly to $GA2$.

For "(2) implies (1)". By the figures in the next page, each inference rule in $\mathbf{GR}_2^-$ preserves the provability of $\mathbf{GR}^-$. ⊣

Theorem 3.2. *The following conditions are equivalent:*
(1) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_3^-$,
(2) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_2^-$.

"(1) implies (2)" is clear. To prove "(2) implies (1)", we need some preparations.

$(\prec\to)$:

$$\dfrac{\dfrac{\Box A \prec \Box B \to \Box A \preceq \Box B \qquad \dfrac{\Box A \preceq \Box B, \Gamma \to \Delta, \Box B \preceq \Box A}{\Box A \preceq \Box B, \Box A \prec \Box B, \Gamma \to \Delta, \Box B \preceq \Box A}\,(w\to)}{\Box A \prec \Box B, \Gamma \to \Delta, \Box B \preceq \Box A}\,(cut) \qquad \Box B \preceq \Box A, \Box A \prec \Box B \to}{\Box A \prec \Box B, \Gamma \to \Delta}\,(cut)$$

$(tran)$:

$$\dfrac{\dfrac{\Gamma \to \Delta, \Box C \preceq \Box D \qquad \dfrac{\Gamma \to \Delta, \Box D \preceq \Box E \quad \Box D \preceq \Box E, \Box C \preceq \Box D \to \Box C \preceq \Box E}{\Box C \preceq \Box D, \Gamma \to \Delta, \Box C \preceq \Box E}\,(cut)}{\Gamma \to \Delta, \Box C \preceq \Box E}\,(cut) \qquad \Box C \preceq \Box E, \Gamma \to \Delta}{\Gamma \to \Delta}\,(cut)$$

(lin):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, \Box C, \Box D}{\Gamma \to \Delta, \Box C, \Box D \preceq \Box D}\,(\to\preceq) \qquad \mathcal{P}(D,C)}{\Gamma \to \Delta, \Box C \preceq \Box D, \Box D \preceq \Box C, \Box C}\,(cut)}{\Gamma \to \Delta, \Box C \preceq \Box D, \Box D \preceq \Box C, \Box C \preceq \Box C}\,(\to\preceq) \qquad \mathcal{P}(C,D)}{\Gamma \to \Delta, \Box C \preceq \Box D, \Box D \preceq \Box C}\,(cut) \qquad \mathcal{Q}(C,D)}{\Gamma \to \Delta, \Box C \preceq \Box D}\,(cut) \qquad \dfrac{\mathcal{Q}(D,C) \quad \Box D \preceq \Box C, \Box C \preceq \Box D, \Gamma \to \Delta}{\Box C \preceq \Box D, \Gamma \to \Delta}\,(cut)}{\Gamma \to \Delta}\,(cut)$$

$\mathcal{P}(C,D)$:

$$\dfrac{\dfrac{\Box C \to \Box C \preceq \Box D, \Box D \prec \Box C \quad \Box D \prec \Box C \to \Box D \preceq \Box C}{\Box C \to \Box C \preceq \Box D, \Box D \preceq \Box C}\,(cut)}{\Box C \preceq \Box C \to \Box C \preceq \Box D, \Box D \preceq \Box C}\,(\preceq\to)$$

$\mathcal{Q}(C,D)$:

$$\dfrac{\dfrac{\Box D \to \Box C \preceq \Box D, \Box D \prec \Box C \quad \Box D \prec \Box C, \Gamma \to \Delta}{\Box D, \Gamma \to \Delta, \Box C \preceq \Box D}\,(cut)}{\Box D \preceq \Box C, \Gamma \to \Delta, \Box C \preceq \Box D}\,(\preceq\to)$$

Definition 3.3. A sequent $\Gamma \to \Delta$ is said to be saturated if the following conditions hold:

(1) if $A \wedge B \in \Gamma$, then $A, B \in \Gamma$, (2) if $A \wedge B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$,
(3) if $A \vee B \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$, (4) if $A \vee B \in \Delta$, then $A, B \in \Delta$,
(5) if $A \supset B \in \Gamma$, then $A \in \Delta$ or $B \in \Gamma$, (6) if $A \supset B \in \Delta$, then $A \in \Gamma$ and $B \in \Delta$,
(7) if $\Box A \preceq \Box B \in \Gamma$, then $\Box A \in \Gamma$, (8) if $\Box A \preceq \Box A \in \Delta$, then $\Box A \in \Delta$,
(9) if $\Box A \prec \Box B \in \Gamma$, then $\Box A \preceq \Box B \in \Gamma$ and $\Box B \preceq \Box A \in \Delta$,
(10) if $\Box C, \Box D$ and $\Box E$ is distinct subformulas in $\mathsf{Sub}(\Gamma \to \Delta)$, then either one of $\Box C \preceq \Box E \in \Gamma$, $\Box C \preceq \Box D \in \Delta$, or $\Box D \preceq \Box E \in \Delta$ holds,
(11) if $\Box C$ and $\Box D$ is distinct subformulas in $\mathsf{Sub}(\Gamma \to \Delta)$, then either one of $\Box C, \Box D \in \Delta$, $\Box C \prec \Box D \in \Gamma$, $\Box D \prec \Box C \in \Gamma$ or $\Box C \preceq \Box D, \Box D \preceq \Box C \in \Gamma$ holds.

Lemma 3.4. *If $\Gamma \to \Delta \notin \mathbf{GR}_3^-$, then there exists a sequent $\Gamma' \to \Delta'$ satisfying the following three conditions:*

(1) $\Gamma' \to \Delta' \notin \mathbf{GR}_3^-$,
(2) $\Gamma' \to \Delta'$ *is saturated,*
(3) $\Gamma \subseteq \Gamma' \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$ *and* $\Delta \subseteq \Delta' \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$.

Proof. Let it be that $p \notin \mathsf{Sub}(\Gamma \to \Delta)$. Since $\mathsf{Sub}^+(\Gamma \to \Delta)$ is finite, there exist formulas

$$A_0, A_1 \cdots, A_{n-1}$$

such that

$$\mathsf{Sub}^+(\Gamma \to \Delta) \cup \{\Box B \wedge \Box C \wedge \Box D \wedge p \mid \Box B, \Box C, \Box D \in \mathsf{Sub}(\Gamma \to \Delta), B \neq C, C \neq D, D \neq B\}$$
$$= \{A_0, A_1 \cdots, A_{n-1}\}.$$

We define a sequence of sequents

$$(\Gamma_0 \to \Delta_0), (\Gamma_1 \to \Delta_1), \cdots, (\Gamma_k \to \Delta_k), \cdots$$

inductively as follows.

Step 0: $(\Gamma_0 \to \Delta_0) = (\Gamma \to \Delta)$.

Step $k+1$: If $A_{k \bmod n} = \Box B \preceq \Box C$, then

$$(\Gamma_{k+1} \to \Delta_{k+1}) = \begin{cases} (\Box B, \Gamma_k \to \Delta_k) & \text{if } \Box B \preceq \Box C \in \Gamma \\ (\Gamma_k \to \Delta_k, \Box B) & \text{if } \Box B \preceq \Box C \in \Delta \text{ and } B = C \\ (\Gamma_k \to \Delta_k) & \text{otherwise} \end{cases}$$

If $A_{k \bmod n} = \Box B \prec \Box C$ and $B \neq C$, then

$$(\Gamma_{k+1} \to \Delta_{k+1}) = \begin{cases} S_1 & \text{if } S_1 \notin \mathbf{GR}_3^- \\ (\Box B \preceq \Box C, \Box B \prec \Box C, \Gamma_k & \\ \qquad \to \Delta_k, \Box C \preceq \Box B) & \text{if } S_1 \in \mathbf{GR}_3^- \text{ and } S_2 \notin \mathbf{GR}_3^- \\ S_3 & \text{if } S_1, S_2 \in \mathbf{GR}_3^- \text{ and } S_3 \notin \mathbf{GR}_3^- \\ S_4 & \text{if } S_1, S_2, S_3 \in \mathbf{GR}_3^- \text{ and } S_4 \notin \mathbf{GR}_3^- \\ (\Gamma_k \to \Delta_k) & \text{otherwise} \end{cases}$$

where

$S_1 = (\Gamma_k \to \Delta_k, \Box B, \Box C)$,
$S_2 = (\Box B \prec \Box C, \Gamma_k \to \Delta_k)$,
$S_3 = (\Box C \prec \Box B, \Gamma_k \to \Delta_k)$ and
$S_4 = (\Box B \preceq \Box C, \Box C \preceq \Box B, \Gamma_k \to \Delta_k)$.

If $A_{k \bmod n} = \Box B \prec \Box B$, then

$$(\Gamma_{k+1} \to \Delta_{k+1}) = (\Gamma_k \to \Delta_k)$$

If $A_{k \bmod n} = \Box B \wedge \Box C \wedge \Box D \wedge p$, then

$$(\Gamma_{k+1} \to \Delta_{k+1}) = \begin{cases} S_1 & \text{if } S_1 \notin \mathbf{GR}_1^- \\ S_2 & \text{if } S_1 \in \mathbf{GR}_1^- \text{ and } S_2 \notin \mathbf{GR}_1^- \\ S_3 & \text{if } S_1, S_2 \in \mathbf{GR}_1^- \text{ and } S_3 \notin \mathbf{GR}_1^- \\ (\Gamma_k \to \Delta_k) & \text{otherwise} \end{cases}$$

where
$S_1 = (\Gamma_k \to \Delta_k, \Box B \preceq \Box C)$,
$S_2 = (\Gamma_k \to \Delta_k, \Box C \preceq \Box D)$ and
$S_3 = (\Box B \preceq \Box D, \Gamma_k \to \Delta_k)$.
If $A_{k \bmod n}$ is a $\Box$-formula, then $(\Gamma_{k+1} \to \Delta_{k+1}) = (\Gamma_k \to \Delta_k)$. In the other cases, $(\Gamma_{k+1} \to \Delta_{k+1})$ is defined in the usual way.

Also in the usual way, we can prove that $\bigcup_{i=1}^{\infty} \Gamma_i \to \bigcup_{i=1}^{\infty} \Delta_i$ is a sequent and satisfies the conditions (1),(2) and (3). ⊣

Definition 3.5. For a sequent $S \notin \mathbf{GR}_3^-$, we fix a sequent satisfying the three conditions in the above lemma and call it *a saturation* of S, write $sat(S)$. For $S \in \mathbf{GR}_3^-$, we put $sat(S) = S$.

Remark 3.6. *For a sequent $S \notin \mathbf{GR}_3^-$,*
(1) *$sat(S) \notin \mathbf{GR}_3^-$,*
(2) *$sat(S)$ is saturated,*
(3) *$\mathsf{Sub}(sat(S)) = \mathsf{Sub}(S)$.*

A sequence of formulas is defined as follows:
(1) [ ] is a sequence of formulas,
(2) if $[A_1, \cdots, A_n]$ is a sequence of formulas, then so is $[A_1, \cdots, A_n, B]$.
A binary operator $\circ$ is defined by

$$[A_1, \cdots, A_m] \circ [B_1, \cdots, B_n] = [A_1, \cdots, A_m, B_1, \cdots, B_n].$$

We use τ and σ, possibly with suffixes, for sequences of formulas.

Definition 3.7. Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. We define the set $\mathbf{W}(S_0)$ of pairs of a sequent and a sequence of formulas as follows:
(1) $(sat(S_0); [\,]) \in \mathbf{W}(S_0)$,
(2) if a pair $(\Gamma \to \Delta, \Box A; \tau)$ belongs to $\mathbf{W}(S_0)$, then so does the pair

$$(sat(\Box A, \{D \mid \Box D \in \Gamma\}, \Gamma \cap \Sigma \to A); \tau \circ [\Box A]).$$

Lemma 3.8. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$ and let $(S; \tau)$ be a pair in $\mathbf{W}(S_0)$. Then*
(1) *S is saturated,*
(2) *$S \notin \mathbf{GR}_3^-$,*
(3) *S consists of only formulas in $\mathsf{Sub}^+(S_0)$,*
(4) *τ consists of only $\Box$-formulas in $\mathsf{Sub}(S_0)$.*

Proof. We use an induction on $(S; \tau)$ as an element in $\mathbf{W}(S_0)$. If $(S; \tau) = (sat(S_0); [\,])$, then the lemma is clear from Definition 3.5. Suppose that $(S; \tau) \neq (sat(S_0); [\,])$. Then by Definition 3.7, there exists a pair $(\Gamma \to \Delta, \Box A; \sigma) \in \mathbf{W}(S_0)$ such that

$$(S; \tau) = (sat(\Box A, \{D \mid \Box D \in \Gamma\}, \Gamma \cap \Sigma \to A); \sigma \circ [\Box A]).$$

So, we obtain (1). By the induction hypothesis, we have the following three:
(5) $\Gamma \to \Delta, \Box A \notin \mathbf{GR}_3^-$,
(6) $\Gamma \to \Delta, \Box A$ consists only formulas in $\mathsf{Sub}^+(S_0)$

(7) σ consists of only $\Box$-formulas in $\mathsf{Sub}(S_0)$.
From (6) and (7), we obtain (4). By Remark 3.6(3) and (6), we have (3). Also by (5), we have $\Gamma\cap\Sigma \to \Box A \notin \mathbf{GR}_3^-$. Using $(\to\Box)$, we have $\Box A, \{D \mid \Box D \in \Gamma\}, \Gamma\cap\Sigma \to A \notin \mathbf{GR}_3^-$, and by Remark 3.6(1), neither is its saturation S. We have (2). $\dashv$

Lemma 3.9. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. Then*
(1) *$S_1 = S_2$ for any $(S_1;\tau), (S_2;\tau) \in \mathbf{W}(S_0)$,*
(2) *$(\Gamma_1 \to \Delta_1;\tau), (\Gamma_2 \to \Delta_2;\tau\circ\sigma) \in \mathbf{W}(S_0)$ implies $\Gamma_1\cap\Sigma \subseteq \Gamma_2$,*
(3) *if there exists a $\Box$-formula in the antecedent of $sat(S_0)$, then $\mathsf{Sub}^+(S_0)\cap\Sigma = \mathsf{Sub}^+(S)\cap\Sigma$, for any $(S;\tau) \in \mathbf{W}(S_0)$,*
(4) *$(\Gamma \to \Delta;\tau\circ\sigma) \in \mathbf{W}(S_0)$ implies $(\Gamma_1 \to \Delta_1;\tau) \in \mathbf{W}(S_0)$ for some $\Gamma_1 \to \Delta_1$,*
(5) *$(\Gamma \to \Delta;\tau\circ[\Box A]\circ\sigma) \in \mathbf{W}(S_0)$ implies $\Box A \in \Gamma$,*
(6) *$(S;\tau_1\circ[\Box A]\circ\tau_2\circ[\Box A]\circ\tau_3) \notin \mathbf{W}(S_0)$, for any A and S,*
(7) *$\mathbf{W}(S_0)$ is finite.*

Proof. For (1). We use an induction on τ. If $\tau = [\ ]$, then we have $S_1 = S_2 = sat(S_0)$. Suppose that $\tau = \sigma\circ[\Box A]$. Then by Definition 3.7, there exist

$$(\Gamma_1 \to \Delta_1, \Box A;\sigma), (\Gamma_2 \to \Delta_2, \Box A;\sigma) \in \mathbf{W}(S_0)$$

such that

$$S_1 = sat(\Box A, \{D \mid \Box D \in \Gamma_1\}, \Gamma_1\cap\Sigma \to A)$$

and

$$S_2 = sat(\Box A, \{D \mid \Box D \in \Gamma_2\}, \Gamma_2\cap\Sigma \to A)$$

By the induction hypothesis, we have $(\Gamma_1 \to \Delta_1) = (\Gamma_2 \to \Delta_2)$, and so $\Gamma_1 = \Gamma_2$. Hence we obtain $S_1 = S_2$.

For (2). We use an induction on σ. If $\sigma = [\ ]$, then by (1), we have $\Gamma_1 = \Gamma_2$. Suppose that $\sigma = \sigma'\circ[\Box A]$. Then by Definition 3.7, there exists $(\Gamma_3 \to \Delta_3, \Box A;\tau\circ\sigma') \in \mathbf{W}(S_0)$ such that $(\Gamma_2 \to \Delta_2) = sat(\Box A, \{D \mid \Box D \in \Gamma_3\}, \Gamma_3\cap\Sigma \to A)$. By the induction hypothesis, $\Gamma_1\cap\Sigma \subseteq \Gamma_3$. Using Remark 3.6(3), $\Gamma_1\cap\Sigma \subseteq \Gamma_3\cap\Sigma \subseteq \Gamma_2$.

For (3). By Lemma 3.8(2), we have $\mathsf{Sub}^+(S_0)\cap\Sigma \supseteq \mathsf{Sub}^+(S)\cap\Sigma$. We show $\mathsf{Sub}^+(S_0)\cap\Sigma \subseteq \mathsf{Sub}^+(S)\cap\Sigma$. We put $S_0 = \Gamma_0 \to \Delta_0$. Suppose that $\Box C \in \Gamma_0$ and $E \in \mathsf{Sub}^+(S_0)\cap\Sigma$.

If $E = \Box A$, then either one of the following four holds since $sat(S_0)$ is saturated:
(3a) $\Box C, \Box A \in \Delta_0$,
(3b) $\Box C \prec \Box A, \Box C \preceq \Box A \in \Gamma_0$,
(3c) $\Box A \prec \Box C, \Box A \preceq \Box C \in \Gamma_0$,
(3d) $\Box C \preceq \Box A, \Box A \preceq \Box C \in \Gamma_0$.
By Lemma 3.8(1) and $\Box C \in \Gamma_0$, we note that (3a) does not hold. So, one of the formulas $\Box C \prec \Box A$ and $\Box A \preceq \Box C$ belongs to Γ_0, and by (2), it also belongs to the antecedent of S. Hence $\Box A \in \mathsf{Sub}^+(S)\cap\Sigma$.

If E is either a $\preceq$-formula $\Box A \preceq \Box B$ or a $\prec$-formula $\Box A \prec \Box B$, then $\Box A, \Box B \in \mathsf{Sub}^+(S_0)\cap\Sigma$. So similarly to the proof for the case that $E = \Box A$, we have $\Box A, \Box B \in \mathsf{Sub}^+(S)\cap\Sigma$, and so, $E \in \mathsf{Sub}^+(S)\cap\Sigma$.

For (4). We use an induction on σ. If $\sigma = [\ ]$, then the lemma is clear. Suppose that $\sigma = \sigma'\circ[A]$. Then by Definition 3.7, there exists $(\Gamma_1 \to \Delta_1, \Box A;\tau\circ\sigma') \in \mathbf{W}(S_0)$ such that

$$(\Gamma \to \Delta) = sat(\Box A, \{D \mid \Box D \in \Gamma_1\}, \Gamma_1\cap\Sigma \to A).$$

By the induction hypothesis, $(\Gamma_2 \to \Delta_2;\tau) \in \mathbf{W}(S_0)$ for some $\Gamma_2 \to \Delta_2$.

For (5). By (4), $(\Gamma_1 \to \Delta_1;\tau\circ[\Box A]) \in \mathbf{W}(S_0)$ for some $\Gamma_1 \to \Delta_1$. Using Definition 3.7, we have $\Box A \in \Gamma_1$. Using (2), we have $\Box A \in \Gamma$.

For (6). Suppose that $(S;\tau_1\circ[\Box A]\circ\tau_2\circ[\Box A]\circ\tau_3) \in \mathbf{W}(S_0)$. Then by (4), $(\Gamma \to \Delta;\tau_1\circ[\Box A]\circ\tau_2\circ[\Box A]) \in \mathbf{W}(S_0)$ for some $\Gamma \to \Delta$. By Definition 3.7, there exists $(\Gamma_1 \to \Delta_1, \Box A;\tau_1\circ[\Box A]\circ\tau_2) \in \mathbf{W}(S_0)$. Using (5), $\Box A \in \Gamma_1$. So, $\Gamma_1 \to \Delta_1, \Box A \in \mathbf{GR}_3^-$. This is contradictory to Lemma 3.8.

For (7). By (1), (6) and Lemma 3.8(4). $\dashv$

Definition 3.10. Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. We define a structure $\mathcal{K}(S_0) = \langle \mathbf{W}(S_0), <, \models \rangle$ as follows:

(1) $(\Gamma_1 \to \Delta_1; \tau_1) < (\Gamma_2 \to \Delta_2; \tau_2)$ if and only if $\tau_2 = \tau_1 \circ \sigma$ for some $\sigma \neq [\,]$,

(2) $\models$ is a valuation satisfying, in addition to the conditions in Definition 1.1(3),

(2.1) $p \in \Gamma$ if and only if $(\Gamma \to \Delta; \tau) \models p$, for any propositional variable p,

(2.2) $A \in \Gamma$ if and only if $(\Gamma \to \Delta; \tau) \models A$, for any $\prec$-formula $A \in \mathsf{Sub}^+(S_0)$,

(2.3) $(\Gamma \to \Delta; \tau) \models \Box A$ if and only if $(\Gamma \to \Delta; \tau) \models \Box A \preceq \Box A$, for any $\Box A \in \mathsf{Sub}(S_0)$,

(2.4) $\Box A \preceq \Box B \in \Gamma$ if and only if $(\Gamma \to \Delta; \tau) \models \Box A \preceq \Box B$, for any $\Box A, \Box B \in \mathsf{Sub}(S_0)$ such that $A \neq B$.

Lemma 3.11. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. Then for any A and for any $(\Gamma \to \Delta; \tau) \in \mathbf{W}(S_0)$,*

(1) $A \in \Gamma$ *implies* $(\Gamma \to \Delta; \tau) \models A$,

(2) $A \in \Delta$ *implies* $(\Gamma \to \Delta; \tau) \not\models A$.

Proof. We use an induction on A.

If $A = \bot$, then by Lemma 3.8(1), $A \notin \Gamma$. So we have (1). On the other hand, from $(\Gamma \to \Delta; \tau) \not\models A$, we have (2).

If A is a propositional variable, then (1) is clear. Suppose that $p \in \Delta$. By Lemma 3.8(1), $p \notin \Gamma$, and so, we have (2).

Suppose that A is not a propositional variable. If A is a $\preceq$-formula $\Box B \preceq \Box C$ with $B \neq C$ or a $\prec$-formula, then the lemma can be shown similarly to the case that A is a propositional variable. Other cases can be shown in the usual way (cf. [Avr84]). Here we show the case that $A = \Box B \preceq \Box B$ and the case that $A = \Box B$.

For the case that $A = \Box B \preceq \Box B$. Suppose that $\Box B \preceq \Box B \in \Gamma$. Since $\Gamma \to \Delta$ is saturated, we have $\Box B \in \Gamma$. By the induction hypothesis, $(\Gamma \to \Delta; \tau) \models \Box B$. From Definition 2.8(2.4), we obtain $(\Gamma \to \Delta; \tau) \models \Box B \preceq \Box B$.

Suppose that $\Box B \preceq \Box B \in \Delta$. Since $\Gamma \to \Delta$ is saturated, we have $\Box B \in \Delta$. By the induction hypothesis, $(\Gamma \to \Delta; \tau) \not\models \Box B$. From Definition 2.8(2.4), we obtain $(\Gamma \to \Delta; \tau) \not\models \Box B \preceq \Box B$.

For the case that $A = \Box B$. Suppose that $\Box B \in \Gamma$ and $(\Gamma \to \Delta; \tau) < (\Gamma_1 \to \Delta_1; \tau_1)$. Then $\tau_1 = \tau \circ \sigma \circ [\Box C]$ for some σ and C. Hence there exists $(\Gamma_2 \to \Delta_2, \Box C; \tau \circ \sigma) \in \mathbf{W}(S_0)$ such that

$$(\Gamma_1 \to \Delta_1) = sat(\Box C, \{D \mid \Box D \in \Gamma_2\}, \Gamma_2 \cap \Sigma \to C).$$

By Lemma 3.9(2), we have $\Box B \in \Gamma_2$. Using Definition 3.5, $B \in \Gamma_1$. By the induction hypothesis, we have $(\Gamma_1 \to \Delta_1; \tau_1) \models B$. Hence $(\Gamma \to \Delta; \tau) \models \Box B$.

Suppose that $\Box B \in \Delta$. Then $(\Gamma \to \Delta; \tau) < (sat(\Box B, \{D \mid \Box D \in \Gamma\}, \Gamma \cap \Sigma \to B); \tau \circ [\Box B]) \in \mathbf{W}(S_0)$. By Definition 3.5, B belongs to the succeedent of the above saturation. By the induction hypothesis, B is false at the new pair above. Hence $(\Gamma \to \Delta; \tau) \not\models \Box B$. $\dashv$

Corollary 3.12. *Let $A_1, \cdots, A_m \to B_1, \cdots, B_n$ be a sequent, which is not provable in $\mathbf{GR}_3^-$. Then in $\mathcal{K}(A_1, \cdots, A_m \to B_1, \cdots, B_n)$,*

$$(sat(A_1, \cdots, A_m \to B_1, \cdots, B_n); [\,]) \not\models A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n.$$

Lemma 3.13. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. If there exists a $\Box$-formula in the antecedent of $sat(S_0)$, then $\mathcal{K}(S_0)$ is a Kripke pseudo-model for $\mathbf{R}^-$ satisfying the two conditions in Definition 1.2 for any D such that $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathsf{Sub}^+(S_0)$.*

Proof. By Lemma 3.9(7), $\mathbf{W}(S_0)$ is finite. The irreflexivity and the transitivity of $<$ can be shown easily. We show

$$\alpha < \gamma \text{ and } \beta < \gamma \text{ imply either one of } \alpha = \beta,\ \alpha < \beta \text{ or } \beta < \alpha.$$

Suppose that $(S_1;\tau_1) < (S_3;\tau_3)$ and $(S_2;\tau_2) < (S_3;\tau_3)$. Then $\tau_3 = \tau_1 \circ \sigma_1 = \tau_2 \circ \sigma_2$ for some non-empty sequences σ_1 and σ_2. Hence either $\tau_1 = \tau_2 \circ \sigma'_2$ or $\tau_1 \circ \sigma'_1 = \tau_2$ holds. Using Lemma 3.9(1), we have either one of $(S_1;\tau_1) = (S_2;\tau_2)$, $(S_1;\tau_1) < (S_2;\tau_2)$ or $(S_2;\tau_2) < (S_1;\tau_1)$. Hence $\mathcal{K}(S_0)$ is a Kripke pseudo-model for $\mathbf{R}^-$.

We show the two conditions in Definition 1.2 for any D such that $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathsf{Sub}^+(S_0)$.

For (1). If D is $\Box$-formula, then (1) is clear by the definition of $\models$. Suppose that D is either a $\preceq$-formula or a $\prec$-formula, $(\Gamma_1 \to \Delta_1;\tau_1) \models D$ and $(\Gamma_1 \to \Delta_1;\tau_1) < (\Gamma_2 \to \Delta_2;\tau_2)$. Then $D \in \Gamma_1$. By Lemma 3.9(2), $D \in \Gamma_2$. Hence $(\Gamma_2 \to \Delta_2;\tau_2) \models D$.

For (2). We divide the cases.

For the case that D is $A5$ (i.e., $D = (\Box A \preceq \Box B) \supset \Box A$). Suppose that $(\Gamma \to \Delta;\tau) \models \Box A \preceq \Box B$. If $A = B$, then immediately $(\Gamma \to \Delta;\tau) \models \Box A$. So, we assume that $A \neq B$. Since $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathsf{Sub}^+(S_0)$, we have $\Box A \preceq \Box B \in \mathsf{Sub}^+(S_0)$, and so $\Box A \preceq \Box B \in \Gamma$. Since $\Gamma \to \Delta$ is saturated, we have $\Box A \in \Gamma$. Using Lemma 3.11, $(\Gamma \to \Delta;\tau) \models \Box A$.

For the case that D is $A6$ (i.e., $D = ((\Box A \preceq \Box B) \wedge (\Box B \preceq \Box C)) \supset (\Box A \preceq \Box C)$. Suppose that $(\Gamma \to \Delta;\tau) \models \Box A \preceq \Box B$ and $(\Gamma \to \Delta;\tau) \models \Box B \preceq \Box C$. If either $A = B$ or $B = C$, then $(\Gamma \to \Delta;\tau) \models \Box A \preceq \Box C$ is clear. So, we assume that $A \neq B$ and $B \neq C$. Since $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathsf{Sub}^+(S_0)$, we have $\Box A \preceq \Box B, \Box B \preceq \Box C \in \mathsf{Sub}^+(S_0)$, and so, $\Box A \preceq \Box B, \Box B \preceq \Box C \in \Gamma$.

If $A \neq C$, then $\Box A \preceq \Box C \in \Gamma$ since $\Gamma \to \Delta$ is saturated. Using Lemma 3.11, $(\Gamma \to \Delta;\tau) \models \Box A \preceq \Box C$.

If $A = C$, then $\Box A \in \Gamma$ since $\Gamma \to \Delta$ is saturated. Using Lemma 3.11, $(\Gamma \to \Delta;\tau) \models \Box A$, and so $(\Gamma \to \Delta;\tau) \models \Box A \preceq \Box C$.

For the case that D is $A7$ (i.e., $D = (\Box A \vee \Box B) \supset ((\Box A \preceq \Box B) \vee (\Box B \prec \Box A)))$. Suppose that $(\Gamma \to \Delta;\tau) \models \Box A \vee \Box B$, i. e. ,either $\Box A$ or $\Box B$ is true at the pair.

If $A = B$, then $\Box A$ is true at $(\Gamma \to \Delta;\tau)$, and so are $\Box A \preceq \Box B$ and $(\Box A \preceq \Box B) \vee (\Box B \prec \Box A)$.

So, we assume that $A \neq B$. Since $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathsf{Sub}^+(S_0)$, we have $\Box A, \Box B \in \mathsf{Sub}^+(S_0)$, and using Lemma 3.9(3), we also have $\Box A, \Box B \in \mathsf{Sub}^+(\Gamma \to \Delta)$, Since $\Gamma \to \Delta$ is saturated, either one of the following four conditions holds:

$\Box A, \Box B \in \Delta$,

$\Box A \prec \Box B, \Box A \preceq \Box B \in \Gamma$,

$\Box B \prec \Box A, \Box B \preceq \Box A \in \Gamma$,

$\Box A \preceq \Box B, \Box B \preceq \Box A \in \Gamma$.

If $\Box A, \Box B \in \Delta$, then by Lemma 3.11, $\Box A$ and $\Box B$ are false at $(\Gamma \to \Delta;\tau)$, which is in contradiction with $(\Gamma \to \Delta;\tau) \models \Box A \vee \Box B$. So, either one of $\Box A \preceq \Box B$ or $\Box B \prec \Box A$ is true at $(\Gamma \to \Delta;\tau)$, and so is $(\Box A \preceq \Box B) \vee (\Box B \prec \Box A)$.

For the case that D is $A8$ (i.e., $D = (\Box A \prec \Box B) \supset (\Box A \preceq \Box B))$. Suppose that $(\Gamma \to \Delta;\tau) \models \Box A \prec \Box B$. Since $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathsf{Sub}^+(S_0)$, we have $\Box A \prec \Box B \in \mathsf{Sub}^+(S_0)$, and so, $\Box A \prec \Box B \in \Gamma$. Since $\Gamma \to \Delta$ is saturated, $\Box A \preceq \Box B \in \Gamma$, and by Lemma 3.11, $(\Gamma \to \Delta;\tau) \models \Box A \preceq \Box B$.

For the case that D is $A9$ (i.e., $D = ((\Box A \preceq \Box B) \wedge (\Box B \prec \Box A)) \supset \bot)$. Suppose that $(\Gamma \to \Delta;\tau) \models \Box A \preceq \Box B$ and $(\Gamma \to \Delta;\tau) \models \Box B \prec \Box A$. Since $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathsf{Sub}^+(S_0)$, we have $\Box B \prec \Box A \in \Gamma$. Since $\Gamma \to \Delta$ is saturated, $\Box A \preceq \Box B \in \Delta$. and by Lemma 3.11, $(\Gamma \to \Delta;\tau) \not\models \Box A \preceq \Box B$, which is contradictory to $(\Gamma \to \Delta;\tau) \models \Box A \preceq \Box B$. $\dashv$

Lemma 3.14([GS79]). *Let* $\mathbf{S}$ *be a set of formulas satisfying*

$$A \in \mathbf{S} \textit{ implies } \mathsf{Sub}^+(A) \subseteq \mathbf{S}$$

and Let $\mathcal{K}^*$ *be a Kripke pseudo-model for* $\mathbf{R}^-$ *satisfying the two conditions in Definition 1.2 for any* D *such that* $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathbf{S}$. *Then there exists a Kripke model* $\mathcal{K}$ *for* $\mathbf{R}^-$ *such that for any* $A \in \mathbf{S}$,

$$A \textit{ is valid in } \mathcal{K}^* \textit{ if and only if } A \textit{ is valid in } \mathcal{K}$$

The lemma above sometimes called "extension lemma".

Theorem 3.15. *Let* $A_1, \cdots, A_m \to B_1, \cdots, B_n$ *be a sequent, which is not provable in* $\mathbf{GR}_3^-$. *Then there exists a Kripke model* $\mathbf{K}$ *for* $\mathbf{R}^-$, *in which the formula* $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ *is not valid.*

Proof. Let Γ and Δ be the antecedent and succeedent of $sat(A_1, \cdots, A_m \to B_1, \cdots, B_n)$, respectively. For any $\Box A \in \Delta$, $S(A)$ denotes the sequent $\Box A, \{D \mid \Box D \in \Gamma\}, \Gamma \cap \Sigma \to A$. By $(\to \Box)$, we note $S(A) \notin \mathbf{GR}_3^-$. Also we note that there exists a $\Box$-formula in the antecedent of the saturation of $S(A)$. Using Corollary 3.12, Lemma 3.13 and Lemma 3.14, there exists a Kripke model $\mathcal{K}_A = \langle \mathbf{W}_A, <_A, \models_A \rangle$ for $\mathbf{R}^-$ such that for any $B \in \mathsf{Sub}^+(S(A))$,

$$B \text{ is valid in } \mathcal{K}(S(A))) \text{ if and only if } B \text{ is valid in } \mathcal{K}_A.$$

We construct a structure $\mathcal{K} = \langle \mathbf{W}, <, \models \rangle$ as follows:

(1) $\mathbf{W} = \{r\} \cup \bigcup_{\Box A \in \Delta} \{(w, A) \mid w \in \mathbf{W}_A\}$,

(2) $<$ is a binary relation on $\mathbf{W}$ satisfying

(2.1) $r < \alpha$ if and only if $\alpha \neq r$,

(2.2) $(w_1, A_1) < (w_2, A_2)$ if and only if $A_1 = A_2$ and $w_1 <_{A_1} w_2$,

(3) $\models$ is a valuation satisfying, in addition to the conditions in Definition 1.1(3),

(3.1) $r \models \Box B \preceq \Box B$ if and only if $r \models \Box B$,

(3.2) $(w, A) \models \Box B \preceq \Box B$ if and only if $w \models_A \Box B$,

(3.3) $r \models C$ if and only if $C \in \Gamma$,

(3.4) $(w, A) \models C$ if and only if $w \models_A C$,

where C is either one of a propositional variable, a $\prec$-formula $\Box B \preceq \Box C$ with $B \neq C$ or a $\preceq$-formula. We note that for any formula D and for any $w \in \mathbf{W}_A$,

$$w \models_A D \text{ if and only if } (w, A) \models D. \cdots\cdots (*)$$

We show

(4) $B \in \Gamma$ implies $r \models B$

and

(5) $B \in \Delta$ implies $r \not\models B$

by an induction on B.

If B is either one of a propositional variable, a $\prec$-formula $\Box C \preceq \Box D$ with $C \neq D$ or a $\preceq$-formula, then (4) is clear. Suppose that $B \in \Delta$. Then we have $B \notin \Gamma$ since $\Gamma \to \Delta \notin \mathbf{GR}_3^-$. By (3.3), we have $r \not\models B$.

Among the other cases, we only show the case that $B = \Box C \preceq \Box C$ and the case that $B = \Box C$.

For the case that $B = \Box C \preceq \Box C$.

	$\Box C \preceq \Box C \in \Gamma$	
$\Rightarrow$	$\Box C \in \Gamma$	since $\Gamma \to \Delta$ is saturated
$\Rightarrow$	$r \models \Box C$	by the induction hypothesis
$\Rightarrow$	$r \models \Box C \preceq \Box C$	by (3.3)

and

	$\Box C \preceq \Box C \in \Delta$	
$\Rightarrow$	$\Box C \in \Delta$	since $\Gamma \to \Delta$ is saturated
$\Rightarrow$	$r \not\models \Box C$	by the induction hypothesis
$\Rightarrow$	$r \not\models \Box C \preceq \Box C$	by (3.3).

For the case that $B = \Box C$.

Suppose that $\Box C \in \Gamma$ and let $\Box A$ be an formula in Δ. Then $C, \Box C$ belong to the antecedent of $S(A)$. By Lemma 3.11, in $\mathcal{K}(S(A))$,

$$(sat(S(A)); [\,]) \models C \text{ and } (sat(S(A)); [\,]) \models \Box C.$$

Since $(sat(S(A)); [\,])$ is the root of $\mathcal{K}(S(A))$, C is valid in $\mathcal{K}(S(A))$. Hence C is valid in $\mathcal{K}_A$. Using $(*)$, in $\mathcal{K}$, we have $(w, A) \models B$ for any $w \in \mathbf{W}_A$. Hence $r \models \Box B$.

Suppose that $\Box C \in \Delta$. Then C belongs the succeedent of $S(C)$. By Lemma 3.11, in $\mathcal{K}(S(C))$,

$$(sat(S(A)); [\,]) \not\models C.$$

So, C is not valid in $\mathcal{K}_B$, and there exists $w \in \mathbf{W}_B$ such that $w \not\models C$. Using $(*)$, in $\mathcal{K}$, $(w, C) \not\models C$. Since $r < (w, B)$, we have $r \not\models \Box C$.

Hence we obtain (4) and (5), and so,

$$r \not\models (A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n).$$

We show $\mathcal{K}$ is a pseudo-Kripke model for $\mathbf{R}^-$ satisfying two conditions in Definition 1.2 for any D such that $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathsf{Sub}^+(A_1, \cdots, A_m \to B_1, \cdots, B_n)$. Since $\mathcal{K}_A$ is a Kripke model, we can see that $\mathcal{K}$ is a Kripke pseudo-model for $\mathbf{R}^-$ from the definition of $\mathcal{K}$. Also since $\mathcal{K}_A$ is a Kripke model, two conditions in Definition 1.2 are clear if $\alpha \neq r$. So, it is sufficient to show the following two:

(6) if $D \in \Sigma$ and $r \models D$, then for any $\Box A \in \Delta$ and for any $w \in \mathbf{W}_A$, $(w, A) \models D$,

(7) if D is either one of the axioms $A5, A6, A7, A8$ and $A9$, then $r \models D$.

For (6): If $D = \Box E$, then (6) is clear by the definition of $\models$. If $D = \Box E \prec \Box E$, then by (3.1), we obtain (6). So, we assume that $D = \Box E \preceq \Box F$ with $E \neq F$ or D is a $\prec$-formula. Suppose that $r \models D$ and let it be that $\Box A \in \Delta$. Then by (3.3), $D \in \Gamma \cap \Sigma$. Hence D belongs the antecedent of $sat(S(A))$. Using Lemma 3.9(2) and $(sat(S(A)); [\,]) \in \mathbf{W}(S(A))$, $D \in \Phi$ for any $(\Phi \to \Psi; \sigma) \in \mathbf{W}(S(A))$. Using Lemma 3.11, D is valid in $\mathcal{K}(S(A))$, and hence, D is valid in $\mathcal{K}_A$. Using $(*)$, we obtain for any $w \in \mathbf{W}_A$, $(w, A) \models D$.

For (7): We divide the cases.

For the case that D is $A5$ (i.e., $D = (\Box E \preceq \Box F) \supset \Box E$). If $E = F$, then by (3.1),

$$r \models \Box E \preceq \Box F \Rightarrow r \models \Box E.$$

If $E \neq F$, then

$$\begin{array}{lll} & r \models \Box E \preceq \Box F & \\ \Rightarrow & \Box E \preceq \Box F \in \Gamma & \text{by (3.3)} \\ \Rightarrow & \Box E \in \Gamma & \text{since } \Gamma \to \Delta \text{ is saturated} \\ \Rightarrow & r \models \Box E & \text{by (4)}. \end{array}$$

For the case that D is $A6$ (i.e., $D = ((\Box E \preceq \Box F) \wedge (\Box F \preceq \Box G)) \supset (\Box E \preceq \Box G)$. If $E = F$ or $F = G$, then

$$r \models \Box E \preceq \Box F, r \models \Box F \preceq \Box G \Rightarrow r \models \Box E \preceq \Box G.$$

If $E \neq F$ and $E = G$, then

$$\begin{array}{lll} & r \models \Box E \preceq \Box F, r \models \Box F \preceq \Box E & \\ \Rightarrow & \Box E \preceq \Box F \in \Gamma & \text{by (3.3)} \\ \Rightarrow & \Box E \in \Gamma & \text{since } \Gamma \to \Delta \text{ is saturated} \\ \Rightarrow & r \models \Box E & \text{by (4)} \\ \Rightarrow & r \models \Box E \preceq \Box E & \text{by (3.1)}. \end{array}$$

If $E \neq F$,$F \neq G$ and $E \neq G$, then

$$\begin{array}{lll} & r \models \Box E \preceq \Box F, r \models \Box F \preceq \Box G & \\ \Rightarrow & \Box E \preceq \Box F, \Box F \preceq \Box G \in \Gamma & \text{by (3.3)} \\ \Rightarrow & \Box E \prec \Box G \in \Gamma & \text{since } \Gamma \to \Delta \text{ is saturated} \\ \Rightarrow & r \models \Box E \prec \Box G & \text{by (4)}. \end{array}$$

For the case that D is $A7$ (i.e., $D = (\Box E \vee \Box F) \supset ((\Box E \preceq \Box F) \vee (\Box F \prec \Box E)))$. If $E = F$, then

$$\begin{array}{lll} & r \models \Box E \text{ or } r \models \Box E & \\ \Rightarrow & r \models \Box E \preceq \Box E & \text{by (3.1)} \\ \Rightarrow & r \models (\Box E \prec \Box F) \vee (\Box F \prec \Box E). & \end{array}$$

If $E \neq F$, then

$$\begin{array}{lll} & r \models \Box E \text{ or } r \models \Box F & \\ \Rightarrow & \Box E \notin \Delta \text{ or } \Box F \notin \Delta & \text{by (5)} \\ \Rightarrow & \Box E \preceq \Box F \in \Gamma \text{ or } \Box F \preceq \Box E \in \Gamma & \text{since } \Gamma \to \Delta \text{ is saturated} \\ \Rightarrow & r \models \Box E \preceq \Box F \in \Gamma \text{ or } r \models \Box F \preceq \Box E \in \Gamma & \text{by (4)}. \end{array}$$

For the case that D is $A8$ (i.e., $D = (\Box E \prec \Box F) \supset (\Box F \preceq \Box E)$).

$$\begin{array}{lll} & r \models \Box E \prec \Box F & \\ \Rightarrow & \Box E \prec \Box F \in \Gamma & \text{by (3.3)} \\ \Rightarrow & \Box E \preceq \Box F \in \Gamma & \text{since } \Gamma \to \Delta \text{ is saturated} \\ \Rightarrow & r \models \Box E \preceq \Box F & \text{by (4).} \end{array}$$

For the case that D is $A9$ (i.e., $D = ((\Box E \preceq \Box F) \wedge (\Box F \prec \Box E)) \supset \bot$).

$$\begin{array}{lll} & r \models \Box F \prec \Box E & \\ \Rightarrow & \Box F \prec \Box E \in \Gamma & \text{by (3.3)} \\ \Rightarrow & \Box E \preceq \Box F \in \Delta & \text{since } \Gamma \to \Delta \text{ is saturated} \\ \Rightarrow & r \not\models \Box E \preceq \Box F & \text{by (5).} \end{array}$$

Hence using Lemma 3.14, we obtain the theorem. ⊣

Corollary 3.16. *The following conditions are equivalent:*
(1) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_3^-$,
(2) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_2^-$,
(3) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n \in \mathbf{R}^-$,
(4) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ *is valid in any Kripke model for* $\mathbf{R}^-$.

Corollary 3.17. *If $S \in \mathbf{GR}^-$, then there exists a proof figure in $\mathbf{GR}^-$ whose cuts are of the form of cuts occurring in page 5.*

Proof. Suppose that $S \in \mathbf{GR}^-$. By Lemma 3.1 and Corollary 3.16, $S \in \mathbf{GR}_3^-$. So, there exists a proof figure $\mathcal{P}$ for S in $\mathbf{GR}_3^-$. Let $\mathcal{Q}$ be the figure obtained from $\mathcal{P}$ by replacing ($\prec\to$), ($tran$) and (lin) with the corresponding figure in page 5. We note that $\mathcal{Q}$ is a proof figure for S in $\mathbf{GR}^-$ and each cut is of the form of cuts occurring in page 5. ⊣

4 A proof without extension lemma

In the previous section, we proved Theorem 3.15 using the extension lemma (Lemma 3.14) several times. So, for a sequent $S_0 \notin \mathbf{GR}_3^-$, a concrete Kripke model, in which the corresponding formula to S_0 is not valid, is not clearly given. Here we show Theorem 3.15 without the extension lemma by modifying $\mathcal{K}(S_0)$ in Definition 3.10. The proof also gives a concrete Kripke model, in which the corresponding formula to a sequent $S_0 \notin \mathbf{GR}_3^-$ is not valid.

Definition 4.1. Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. We define a structure $\mathcal{K}^*(S_0) = \langle \mathbf{W}(S_0), <, \models^* \rangle$ as follows:
(1) $<$ is as in Definition 3.10, we also write $\alpha \le \beta$ if $\alpha < \beta$ or $\alpha = \beta$,
(2) $\models^*$ is a valuation satisfying, in addition to the conditions in Definition 1.1(3),
(2.1) $(\Gamma \to \Delta; \tau) \models^* p$ if and only if $p \in \Gamma$, for any propositional variable p,
(2.2) $(\Gamma \to \Delta; \tau) \models^* \Box A \prec \Box B$ if and only if either one of the following three holds:
(2.2.1) $\Box A \prec \Box B \in \Gamma$,
(2.2.2) there exists $\alpha \le (\Gamma \to \Delta; \tau)$ such that $\alpha \models^* \Box A$ and $\alpha \not\models^* \Box B$,
(2.2.3) $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$ and there exist C and $(\Gamma_1 \to \Delta_1; \tau_1) \le (\Gamma \to \Delta; \tau)$ such that
$\Box A \prec \Box C \in \Gamma_1$,
$\Box A, \Box B, \Box C$ are true at $(\Gamma_1 \to \Delta_1; \tau_1)$,
for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$, $\alpha \not\models^* \Box A$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box C$,
(2.3) $(\Gamma \to \Delta; \tau) \models^* \Box A \preceq \Box B$ if and only if either one of the following two holds:
(2.3.1) $(\Gamma \to \Delta; \tau) \models^* \Box A \prec \Box B$,
(2.3.2) $(\Gamma \to \Delta; \tau) \models^* \Box A$ and $(\Gamma \to \Delta; \tau) \not\models^* \Box B \prec \Box A$.

Lemma 4.2. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. Then for any A and for any $(\Gamma \to \Delta; \tau) \in \mathbf{W}(S_0)$,*

(1) $A \in \Gamma$ *implies* $(\Gamma \to \Delta; \tau) \models^* A$,

(2) $A \in \Delta$ *implies* $(\Gamma \to \Delta; \tau) \not\models^* A$.

Proof. We use an induction on A.

We only show the case that A is a $\prec$-formula or a $\preceq$-formula.

For the case that $A = \Box B \prec \Box C$. From Definition 4.1(2.2), (1) is clear. Suppose that

(3) $\Box B \prec \Box C \in \Delta$.

By Lemma 3.8(1),

(4) $\Box B \prec \Box C \notin \Gamma$.

By (3), we have

(5) $\Box C \in \mathsf{Sub}(\Gamma \to \Delta)$.

So, we have only to show

(6) there does not exist $\alpha \leq (\Gamma \to \Delta; \tau)$ such that $\alpha \models^* \Box B$ and $\alpha \not\models^* \Box C$.

Suppose

(7) there exists $(\Phi \to \Psi; \sigma) \leq (\Gamma \to \Delta; \tau)$ such that $(\Phi \to \Psi; \sigma) \models^* \Box B$ and $(\Phi \to \Psi; \sigma) \not\models^* \Box C$.

Immediately, we have

(8) $B \neq C$.

Also by the induction hypothesis,

(9) $\Box C \notin \Phi$ and $\Box B \notin \Psi$.

Since $(\Phi \to \Psi; \sigma) \in \mathbf{W}(S_0)$, $\Phi \to \Psi$ is saturated, and hence

(10) $\Box C \preceq \Box B \notin \Phi$ and $\Box C \prec \Box B \notin \Phi$.

By Definition 3.3(11),

(11) $\Box B \prec \Box C \in \Phi$.

Using Lemma 3.9(2),

(12) $\Box B \prec \Box C \in \Gamma$.

This is contradictory to (3) and Lemma 3.8(1).

For the case that $A = \Box B \preceq \Box C$. Suppose

(13) $\Box B \preceq \Box C \in \Gamma$.

Since $\Gamma \to \Delta$ is saturated,

(14) $\Box B \in \Gamma$

and either

(15a) $\Box B \prec \Box C \in \Gamma$ or (15b) $\Box C \preceq \Box B \in \Gamma$.

holds. If (15a) holds, then by the definition,

(16) $(\Gamma \to \Delta; \tau) \models^* \Box B \prec \Box C$,

and hence

(17) $(\Gamma \to \Delta; \tau) \models^* \Box B \preceq \Box C$.

So, we assume that (15b) holds. By (15b), we have $\Box C \preceq \Box B \notin \Delta$, and hence

(18) $\Box B \prec \Box C \notin \Gamma$.

Similarly by (13), we have

(19) $\Box C \prec \Box B \notin \Gamma$.

Let it be that $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$. By (18), (19) and Lemma 3.9(2),

(20) $\Box C \prec \Box B \notin \Gamma_1$ and $\Box B \prec \Box C \notin \Gamma_1$.

By (13),

(21) $\Box B, \Box C \in \mathsf{Sub}(\Gamma \to \Delta) \subseteq \mathsf{Sub}(\Gamma_1 \to \Delta_1)$.

Since $\Gamma_1 \to \Delta_1$ is saturated,

(22) either $\Box C \preceq \Box B, \Box B \preceq \Box C, \Box B, \Box C \in \Gamma_1$ or $\Box B, \Box C \in \Delta_1$ holds.

By the induction hypothesis,

(23) at $(\Gamma_1 \to \Delta_1; \tau_1)$, both of $\Box B$ and $\Box C$ are true or none of $\Box B$ and $\Box C$ is true.

Hence

(24) for any $\alpha \leq (\Gamma \to \Delta; \tau)$, $\alpha \not\models^* \Box C$ or $\alpha \models^* \Box B$.

Also by (13), we have

(25) $\Box B \in \mathsf{Sub}(\Gamma \to \Delta)$.
By (18), (24) and (25), we have
(26) $(\Gamma \to \Delta;\tau) \not\models^* \Box C \prec \Box B$.
Using (14), we obtain
(26) $(\Gamma \to \Delta;\tau) \models^* \Box B \preceq \Box C$.

Suppose
(27) $\Box B \preceq \Box C \in \Delta$.
Then we have
(28) $\Box B, \Box C \in \mathsf{Sub}(\Gamma \to \Delta)$.
and
(29) $\Box B \preceq \Box C \notin \Gamma$.
Since $\Gamma \to \Delta$ is saturated,
(30) $\Box B \prec \Box C \notin \Gamma$,
and
(31) either $\Box B, \Box C \in \Delta$ or $\Box C \prec \Box B \in \Gamma$ holds.
Let it be that $(\Gamma_1 \to \Delta_1;\tau_1) \leq (\Gamma \to \Delta;\tau)$. Then by (29), (30) and Lemma 3.9(2), we have
(32) $\Box B \preceq \Box C \notin \Gamma_1$ and $\Box B \prec \Box C \notin \Gamma_1$.
By (28), we have
(33) $\Box B, \Box C \in \mathsf{Sub}(\Gamma \to \Delta) \subseteq \mathsf{Sub}(\Gamma_1 \to \Delta_1)$.
Since $\Gamma_1 \to \Delta_1$ is saturated,
(33) either $\Box B, \Box C \in \Delta_1$ or $\Box C \prec \Box B, \Box C \in \Gamma_1$ holds.
By the induction hypothesis,
(33) either $(\Gamma_1 \to \Delta_1;\tau_1) \not\models^* \Box B$ or $(\Gamma_1 \to \Delta_1;\tau_1) \models^* \Box C$
Hence
(34) for any $\alpha \leq (\Gamma \to \Delta;\tau)$, $\alpha \not\models^* \Box B$ or $\alpha \models^* \Box C$.
Using (28) and (30), we have
(35) $(\Gamma \to \Delta;\tau) \not\models^* \Box B \prec \Box C$.
On the other hand, by (31), the induction hypothesis and a result in the above case,
(36) either $(\Gamma \to \Delta;\tau) \not\models^* \Box B$ or $(\Gamma \to \Delta;\tau) \models^* \Box C \preceq \Box B$ holds.
Hence we obtain
(37) $(\Gamma \to \Delta;\tau) \not\models^* \Box B \preceq \Box C$. $\dashv$

Corollary 4.3. *Let $A_1, \cdots, A_m \to B_1, \cdots, B_n$ be a sequent, which is not provable in $\mathbf{GR}_3^-$. Then in $\mathcal{K}^*(A_1, \cdots, A_m \to B_1, \cdots, B_n)$,*

$$(sat(A_1, \cdots, A_m \to B_1, \cdots, B_n);[\,]) \not\models^* A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n.$$

Lemma 4.4. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. Then $(\Gamma \to \Delta;\tau) \models^* \Box A \preceq \Box B$ implies $(\Gamma \to \Delta;\tau) \models^* \Box A$.*

Proof. Suppose that $(\Gamma \to \Delta;\tau) \models^* \Box A \preceq \Box B$. If $(\Gamma \to \Delta;\tau) \not\models^* \Box A \prec \Box B$, then by the definition we have $(\Gamma \to \Delta;\tau) \models^* \Box A$. So, we assume that $(\Gamma \to \Delta;\tau) \models^* \Box A \prec \Box B$. Then either one of the following three holds:
(1) $\Box A \prec \Box B \in \Gamma$,
(2) there exists $(\Gamma_1 \to \Delta_1;\tau_1) \leq (\Gamma \to \Delta;\tau)$ such that $(\Gamma_1 \to \Delta_1;\tau_1) \models^* \Box A$ and $(\Gamma_1 \to \Delta_1;\tau_1) \not\models^* \Box B$,
(3) there exist a formula C and $(\Gamma_1 \to \Delta_1;\tau_1) \leq (\Gamma \to \Delta;\tau)$ such that
$\Box A \prec \Box C \in \Gamma_1$,
$\Box B \notin \mathsf{Sub}(\Gamma_1 \to \Delta_1)$,
$\Box A, \Box B, \Box C$ are true at $(\Gamma_1 \to \Delta_1;\tau_1)$,
for any $\alpha < (\Gamma_1 \to \Delta_1;\tau_1)$, $\alpha \not\models^* \Box A$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box C$.

If (1) holds, then $\Box A, \Box A \preceq \Box B \in \Gamma$. Using Lemma 4.2, we have $(\Gamma \to \Delta;\tau) \models^* \Box A$.

If either (2) or (3) holds, then there exists $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box A$. So, $(\Gamma_2 \to \Delta_2; \tau_2) \models^* A$ for any $(\Gamma_2 \to \Delta_2; \tau_2) \in (\Gamma_1 \to \Delta_1; \tau_1) \uparrow$. So, $(\Gamma_2 \to \Delta_2; \tau_2) \models^* A$ for any $(\Gamma_2 \to \Delta_2; \tau_2) \in (\Gamma \to \Delta; \tau) \uparrow$. Hence $(\Gamma \to \Delta; \tau) \models^* \Box A$. $\dashv$

Lemma 4.5. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$ Then $\mathcal{K}^*(S_0)$ is a Kripke pseudo-model for $\mathbf{R}^-$*

Proof. By Lemma 3.13. $\dashv$

Lemma 4.6. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. Then for any $D \in \Sigma$, $(\Gamma_1 \to \Delta_1; \tau_1) \models^* D$ and $(\Gamma_1 \to \Delta_1; \tau_1) < (\Gamma_2 \to \Delta_2; \tau_2)$ imply $(\Gamma_2 \to \Delta_2; \tau_2) \models^* D$.*

Proof. Suppose that D is either a $\preceq$-formula or a $\prec$-formula, $(\Gamma_1 \to \Delta_1; \tau_1) \models^* D$ and $(\Gamma_1 \to \Delta_1; \tau_1) < (\Gamma_2 \to \Delta_2; \tau_2)$.

If D is $\Box$-formula, then the lemma is clear by the definition of $\models^*$.

If $D = \Box A \prec \Box B$, then either one of the following three holds:

(1) $\Box A \prec \Box B \in \Gamma_1$,

(2) there exists $\alpha \leq (\Gamma_1 \to \Delta_1; \tau_1)$ such that $\alpha \models^* \Box A$ and $\alpha \not\models^* \Box B$,

(3) $\Box B \notin \mathsf{Sub}(\Gamma_1 \to \Delta_1)$ and there exist C and $(\Gamma_3 \to \Delta_3; \tau_3) \leq (\Gamma_1 \to \Delta_1; \tau_1)$ such that

$\Box A \prec \Box C \in \Gamma_3$,

$\Box A, \Box B, \Box C$ are true at $(\Gamma_3 \to \Delta_3; \tau_3)$,

for any $\alpha < (\Gamma_3 \to \Delta_3; \tau_3)$, $\alpha \not\models^* \Box A$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box C$.

If (1) holds, then by Lemma 3.9(2), $\Box A \prec \Box B \in \Gamma_2$, and by Lemma 4.2, $(\Gamma_2 \to \Delta_2; \tau_2) \models^* D$. If (2) holds, then we have $\alpha < (\Gamma_2 \to \Delta_2; \tau_2)$, and hence $(\Gamma_2 \to \Delta_2; \tau_2) \models^* D$. If (3) holds, then similarly, $(\Gamma_3 \to \Delta_3; \tau_3) < (\Gamma_2 \to \Delta_2; \tau_2)$. We also note that $\Box B \notin \mathsf{Sub}(\Gamma_1 \to \Delta_1) \supseteq \mathsf{Sub}(\Gamma_2 \to \Delta_2)$. Hence we obtain $(\Gamma_2 \to \Delta_2; \tau_2) \models^* D$.

If $D = \Box A \preceq \Box B$, then either one of the following two holds:

(4) $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box A \prec \Box B$,

(5) $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box A$ and $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box B \prec \Box A$.

If (4) holds, then by the case above, $(\Gamma_2 \to \Delta_2; \tau_2) \models^* \Box A \prec \Box B$, and hence $(\Gamma_2 \to \Delta_2; \tau_2) \models^* \Box A \preceq \Box B$. Assume that (5) holds. From the definition, we have $(\Gamma_2 \to \Delta_2; \tau_2) \models^* \Box A$. So, it is sufficient to show $(\Gamma_2 \to \Delta_2; \tau_2) \not\models^* \Box B \prec \Box A$. Suppose that $(\Gamma_2 \to \Delta_2; \tau_2) \models^* \Box B \prec \Box A$. Then either one of the following three holds:

(6) $\Box B \prec \Box A \in \Gamma_2$ and $A \neq B$,

(7) there exists $(\Gamma_3 \to \Delta_3; \tau_3) \leq (\Gamma_2 \to \Delta_2; \tau_2)$ such that $(\Gamma_3 \to \Delta_3; \tau_3) \models^* \Box B$ and $(\Gamma_3 \to \Delta_3; \tau_3) \not\models^* \Box A$,

(8) $\Box B \notin \mathsf{Sub}(\Gamma_2 \to \Delta_2)$ and there exist a formula C and $(\Gamma_3 \to \Delta_3; \tau_3) \leq (\Gamma_2 \to \Delta_2; \tau_2)$ such that

$\Box A \prec \Box C \in \Gamma_3$,

$\Box A, \Box B, \Box C$ are true at $(\Gamma_3 \to \Delta_3; \tau_3)$,

for any $\alpha < (\Gamma_3 \to \Delta_3; \tau_3)$, $\alpha \not\models^* \Box A$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box C$.

If (6) holds, then $\Box A \preceq \Box B \in \Delta_2$ and $\Box A \preceq \Box B \in \mathsf{Sub}^+(\Gamma_2 \to \Delta_2) \subseteq \mathsf{Sub}^+(\Gamma_1 \to \Delta_1)$. On the other hand, by (5) and Lemma 4.2, we have $\Box A \notin \Delta_1$ and $\Box B \prec \Box A \notin \Gamma_1$. Hence $\Box A \preceq \Box B \in \Gamma_1$. Using Lemma 3.9(2), $\Box A \preceq \Box B \in \Gamma_2$. This is in contradiction with $\Box A \preceq \Box B \in \Delta_2$ and Lemma 3.8.

If (7) holds, then by (5), $(\Gamma_3 \to \Delta_3; \tau_3) < (\Gamma_1 \to \Delta_1; \tau_1)$. So, we have $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box B \prec \Box A$. This is in contradiction with (5).

If (8) holds, then by (5), $(\Gamma_3 \to \Delta_3; \tau_3) < (\Gamma_1 \to \Delta_1; \tau_1)$. So, if $\Box B \notin \mathsf{Sub}(\Gamma_1 \to \Delta_1)$, then we have $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box B \prec \Box A$, and this is in contradiction with (5). Assume that $\Box B \in \mathsf{Sub}(\Gamma_1 \to \Delta_1)$. By (8), $\Box A \in \mathsf{Sub}(\Gamma_3 \to \Delta_3) \subseteq \mathsf{Sub}(\Gamma_1 \to \Delta_1)$. By (5), Lemma 4.2 and Lemma 3.9(3), we have $\Box A \preceq \Box B \in \Gamma_1 \cap \Sigma \subseteq \Gamma_2$, similarly to the case that (6) holds. Hence $\Box B \in \mathsf{Sub}(\Gamma_2 \to \Delta_2)$. This is contradictory to (8). $\dashv$

Lemma 4.7. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$ and let be that $\alpha \models^* \Box A \preceq \Box B$. Then for any $\beta \leq \alpha$, $\beta \models^* \Box B$ implies $\beta \models^* \Box A$.*

Proof. We put $\alpha = (\Gamma \to \Delta; \tau)$. Then either one of the following four holds:
(1) $\alpha \models^* \Box A$ and $\alpha \not\models^* \Box B \prec \Box A$.
(2) $\Box A \prec \Box B \in \Gamma$,
(3) there exists $\gamma \leq \alpha$ such that $\gamma \models^* \Box A$ and $\gamma \not\models^* \Box B$,
(4) $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$ and there exist C and $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that
$\Box A \prec \Box C \in \Gamma_1$,
$\Box A, \Box B, \Box C$ are true at $(\Gamma_1 \to \Delta_1; \tau_1)$,
for any $\delta < (\Gamma_1 \to \Delta_1; \tau_1)$, $\delta \not\models^* \Box A$, $\delta \not\models^* \Box B$, $\delta \not\models^* \Box C$.

If (1) holds, then we obtain the lemma by the definition.

If (2) holds, then $\Box B \preceq \Box A \in \Delta$. Since $\Gamma \to \Delta \notin \mathbf{GR}_3^-$, $\Box B \preceq \Box A \notin \Gamma$. Hence $\Box B \prec \Box A \notin \Gamma$. Using Lemma 3.9(2) for any $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ we have $\Box B \prec \Box A \notin \Gamma_1$. Using $\Box A, \Box B \in \mathsf{Sub}(\Gamma \to \Delta) \subseteq \mathsf{Sub}(\Gamma_1 \to \Delta_1)$, we have either $\Box B \in \Delta_1$, $\Box A \prec \Box B \in \Gamma_1$ or $\Box A \preceq \Box B \in \Gamma_1$. Hence $\Box B \in \Delta_1$ or $\Box A \in \Gamma_1$. Using Lemma 4.2, $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box B$ or $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box A$.

If (3) holds, then by Lemma 4.6, $\gamma_1 \models^* \Box A$ for any $\gamma_1 \in \gamma\uparrow$ and $\gamma_2 \not\models^* \Box B$ for any $\gamma_2 \leq \gamma$. We note that for any $\beta \leq \alpha$, either $\beta \leq \gamma$ or $\gamma < \beta$. So, we obtain the lemma.

If (4) holds, then $(\Gamma_1 \to \Delta_1; \tau_1) \leq \alpha$. Suppose that $\beta \leq \alpha$. By Lemma 4.5, $(\Gamma_1 \to \Delta_1; \tau_1) \leq \beta$ or $\beta < (\Gamma_1 \to \Delta_1; \tau_1)$. By (4), $\beta \models^* \Box A$ if $(\Gamma_1 \to \Delta_1; \tau_1) \leq \beta$; and $\beta \not\models^* \Box B$ if $\beta < (\Gamma_1 \to \Delta_1; \tau_1)$. So, $\beta \models^* \Box B$ implies $\beta \models^* \Box A$. ⊣

Lemma 4.8. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_3^-$. then $\mathcal{K}^*(S_0)$ is a Kripke model for $\mathbf{R}^-$.*

Proof. By Lemma 4.5 and Lemma 4.6, it is sufficient to show the condition (2) in Definition 1.2.

We divide the cases.

The case that D is $A5$ (i.e., $D = (\Box A \preceq \Box B) \supset \Box A$) is shown by Lemma 4.4.

For the case that D is $A6$ (i.e., $D = ((\Box A \preceq \Box B) \wedge (\Box B \preceq \Box C)) \supset (\Box A \preceq \Box C)$. If either $A = B$ or $B = C$, then $(\Gamma \to \Delta; \tau) \models D$ is clear. So, we assume that $A \neq B$ and $B \neq C$. Suppose that
(1) $(\Gamma \to \Delta; \tau) \models^* \Box A \preceq \Box B$,
(2) $(\Gamma \to \Delta; \tau) \models^* \Box B \preceq \Box C$ and
(3) $(\Gamma \to \Delta; \tau) \not\models^* \Box A \preceq \Box C$.
By (3), we have
(4) $(\Gamma \to \Delta; \tau) \not\models^* \Box A \prec \Box C$ and
(5) $(\Gamma \to \Delta; \tau) \not\models^* \Box A$ or $(\Gamma \to \Delta; \tau) \models^* \Box C \prec \Box A$.
By (1) and Lemma 4.4, we have
(6) $(\Gamma \to \Delta; \tau) \models^* \Box A$.
Using (5), we have $(\Gamma \to \Delta; \tau) \models^* \Box C \prec \Box A$. So, either one of the following three holds:
(3a) $\Box C \prec \Box A \in \Gamma$,
(3b) there exists $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box C$ and $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box A$,
(3c) $\Box A \notin \mathsf{Sub}(\Gamma \to \Delta)$, and there exist a formula D and $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that
$\Box C \prec \Box D \in \Gamma_1$,
$\Box A, \Box C, \Box D$ are true at $(\Gamma_1 \to \Delta_1; \tau_1)$,
for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$, $\alpha \not\models^* \Box A$, $\alpha \not\models^* \Box C$, $\alpha \not\models^* \Box D$.
By (1), either one of the following four holds:
(1a) $(\Gamma \to \Delta; \tau) \not\models^* \Box B \prec \Box A$.
(1b) $\Box A \prec \Box B \in \Gamma$,
(1c) there exists $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box A$ and $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box B$,
(1d) $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$, and there exist a formula D and $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that
$\Box A \prec \Box D \in \Gamma_1$,
$\Box A, \Box B, \Box D$ are true at $(\Gamma_1 \to \Delta_1; \tau_1)$,
for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$, $\alpha \not\models^* \Box A$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box D$.
By (2), either one of the following four holds:

(2a) $(\Gamma \to \Delta; \tau) \not\models^* \Box C \prec \Box B$.

(2b) $\Box B \prec \Box C \in \Gamma$,

(2c) there exists $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box B$ and $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box C$,

(2d) $\Box C \notin \mathsf{Sub}(\Gamma \to \Delta)$ and there exist a formula D and $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that

$\Box B \prec \Box D \in \Gamma_1$,

$\Box C, \Box C, \Box D$ are true at $(\Gamma_1 \to \Delta_1; \tau_1)$,

for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box C$, $\alpha \not\models^* \Box D$.

We divide the subcases.

The subcase that (3a) and $\Box B \in \mathsf{Sub}(\Gamma \to \Delta)$ hold. We note that $\Box A, \Box B, \Box C \in \mathsf{Sub}(\Gamma \to \Delta)$. By (1) and (2) and Lemma 4.2, we have $\Box A \preceq \Box B \notin \Delta$ and $\Box B \preceq \Box C \notin \Delta$. Since $\Gamma \to \Delta$ is saturated, we have $\Box A \preceq \Box C \in \Gamma$. Using Lemma 4.2, $(\Gamma \to \Delta; \tau) \models^* \Box A \preceq \Box C$. This is contradictory to (3).

The subcase that (3a), (2a) and $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$ hold. By (3a), we have $\Box A \preceq \Box C \in \Delta$, and hence $\Box A \preceq \Box C \notin \Gamma$ and $\Box A \prec \Box C \notin \Gamma$. Let $(\Gamma_1 \to \Delta_1; \tau_1)$ be a world in $(\Gamma \to \Delta; \tau) \downarrow \cup \{(\Gamma \to \Delta; \tau)\}$. By Lemma 3.9(2), $\Box A \preceq \Box C \notin \Gamma_1$ and $\Box A \prec \Box C \notin \Gamma_1$. So, either $\Box C \prec \Box A, \Box C \preceq \Box A, \Box C \in \Gamma_1$ or $\Box A, \Box C \in \Delta_1$. Using Lemma 4.2, either $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box C$ or $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box A$ Using (1),(2) and Lemma 4.7, either each of $\Box A, \Box B, \Box C$ is true or none of $\Box A, \Box B, \Box C$ is true at $(\Gamma_1 \to \Delta_1; \tau_1)$ By Lemma 4.5, $((\Gamma \to \Delta; \tau) \downarrow \cup \{(\Gamma \to \Delta; \tau)\}, <)$ is a linear ordered set and is not empty. So, there exists the minimal element $(\Gamma_2 \to \Delta_2; \tau_2)$ that makes each of $\Box A, \Box B, \Box C$ true. We note that

$(\Gamma_2 \to \Delta_2; \tau_2) \leq (\Gamma \to \Delta; \tau)$

$\Box C \prec \Box A \in \Gamma_2$,

$\Box C, \Box B, \Box A$ are true at $(\Gamma_2 \to \Delta_2; \tau_2)$,

for any $\alpha < (\Gamma_2 \to \Delta_2; \tau_2)$, $\alpha \not\models^* \Box C$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box A$.

Also by the assumption, $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$. So, we have $(\Gamma \to \Delta; \tau) \models^* \Box C \prec \Box B$. This is contradictory to (2a).

The subcase that (3a) and (2b) hold. We note that $\Box A, \Box B, \Box C \in \mathsf{Sub}(\Gamma \to \Delta)$. So, this subcase resolves into the first subcase.

The subcase that (3a) and (2c) hold. By (2c), we have $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box B$ and $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box C$. Using (1) and Lemma 4.7, $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box A$. Using Lemma 4.2, $\Box A \notin \Delta_1$ and $\Box C \notin \Gamma_1$. So, $\Box C \prec \Box A \notin \Gamma_1$ and $\Box C \preceq \Box A \notin \Gamma_1$. By (3a), we have $\Box A, \Box C \in \mathsf{Sub}(\Gamma \to \Delta) \subseteq \mathsf{Sub}(\Gamma_1 \to \Delta_1)$. Hence $\Box A \prec \Box C \in \Gamma_1$. Using Lemma 3.9(2), $\Box A \prec \Box C \in \Gamma$. This is contradictory to (3a).

The subcase that (3a) and (2d) hold. By (3a), we have $\Box C \prec \Box D \in \Gamma_1$. Using Lemma 3.9(2), $\Box C \prec \Box D \in \Gamma$. So, $\Box C \in \mathsf{Sub}(\Gamma \to \Delta)$. This is contradictory to (2d).

The subcase that (3b) holds. Using (1) and Lemma 4.7, $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box B$. Using (2) and Lemma 4.7, $(\Gamma_1 \to \Delta_1; \tau_1) \not\models^* \Box A$. This is contradictory to (3b).

The subcase that (3c), (1a) and $\Box B \in \mathsf{Sub}(\Gamma \to \Delta)$ hold. We note that $\Box B \in \mathsf{Sub}(\Gamma \to \Delta) \subseteq \mathsf{Sub}(\Gamma_1 \to \Delta_1)$ hold. Using (3c), we have $\Box B, \Box C, \Box D \in \mathsf{Sub}(\Gamma_1 \to \Delta_1)$. On the other hand, by (2) and Lemma 4.2, $\Box B \preceq \Box C \notin \Delta$, and hence $\Box C \prec \Box B \notin \Gamma$. Using Lemma 4.6, $\Box C \prec \Box B \notin \Gamma_1$. Also by (3c) and Lemma 4.2, $\Box C \notin \Delta_1$. Hence we have either $\Box B \prec \Box C, \Box B \preceq \Box C \in \Gamma_1$ or $\Box B \preceq \Box C, \Box C \preceq \Box B \in \Gamma_1$. So, we have $\Box B \preceq \Box C \in \Gamma_1$. By (3c), $\Box C \prec \Box D, \Box C \preceq \Box D \in \Gamma_1$. So, $\Box B \preceq \Box D \in \Gamma_1$. Also, $\Box B \in \Gamma_1$ and that $\Box D \prec \Box B \in \Gamma_1$ implies $\Box B \preceq \Box D \in \Delta_1$. Hence $\Box B \notin \Delta_1$ and $\Box D \prec \Box B \notin \Gamma_1$. Since $\Gamma_1 \to \Delta_1$ is saturated, we have either $\Box B \prec \Box D \in \Gamma_1$ or $\Box D \preceq \Box B \in \Gamma_1$. If $\Box D \preceq \Box B \in \Gamma_1$, then by $\Box B \preceq \Box C \in \Gamma_1$, we have $\Box D \preceq \Box C \in \Gamma_1$, and hence $\Box C \prec \Box D \in \Delta_1$. This is in contradiction with $\Box C \prec \Box D \in \Gamma_1$. Assume that $\Box B \prec \Box D \in \Gamma_1$. By $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box C$, (2) and Lemma 4.7, we have $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box B$. By $\alpha \not\models^* \Box A$ for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$,(1) and Lemma 4.7, we have $\alpha \not\models^* \Box B$ for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$. So, using (3c), we have $(\Gamma \to \Delta; \tau) \models^* \Box B \prec \Box A$. This is in contradiction with (1a).

The subcase that (3c)[,(1a)], $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$ and (2a) hold. By (3c), (1), (2) and Lemma 4.7, we have $(\Gamma_1 \to \Delta_1; \tau_1) \models^* \Box B$. $\alpha \not\models \Box B$ for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$.Using (3c) and $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$, we have $(\Gamma \to \Delta; \tau) \models^* \Box C \prec \Box B$. This is in contradiction with (2a).

The subcase that [(3c),(1a)], $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$ and (2b) hold. By (2b), $\Box B \in \mathsf{Sub}(\Gamma \to \Delta)$, which is in contradiction with $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$.

The subcase that (3c)[,(1a), $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$] and (2c) hold. By (2c), there exists $\alpha \leq (\Gamma \to \Delta; \tau)$ such that $\alpha \models^* \Box B$ and $\alpha \not\models^* \Box C$. By (3c), there exists $\beta \leq (\Gamma \to \Delta; \tau)$ such that $\beta \models^* \Box C$ and $\gamma \not\models^* \Box A$ for any $\gamma \in \beta \downarrow$. Using Lemma 4.5, $\alpha < \beta$ or $\beta \leq \alpha$. By $\alpha \not\models^* \Box C$, $\beta \models^* \Box C$ and Lemma 4.6, we have $\alpha < \beta$. Since $\gamma \not\models^* \Box A$ for any $\gamma \in \beta \downarrow$, $\alpha \not\models^* \Box A$. Using (1) and Lemma 4.7, we have $\alpha \not\models^* \Box B$. This is in contradiction in $\alpha \models^* \Box B$.

The subcase that (3c)[,(1a), $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$] and (2d) hold. By (3c) and Lemma 3.9(2), we have $\Box A \prec \Box C \in \Gamma_1 \subseteq \Gamma$, and hence $\Box A \in \mathsf{Sub}(\Gamma \to \Delta)$. This is in contradiction with $\Box C \notin \mathsf{Sub}(\Gamma \to \Delta)$ from (2d).

The subcase that (3c) and (1b) hold. By (3c), we have $\Box A \notin \mathsf{Sub}(\Gamma \to \Delta)$. By (1b), we have $\Box A \in \mathsf{Sub}(\Gamma \to \Delta)$. This is a contradiction.

The subcase that (3c) and (1c) hold. By (1c), there exists $\alpha \leq (\Gamma \to \Delta; \tau)$ such that $\alpha \models^* \Box A$ and $\alpha \not\models^* \Box B$. By (3c), there exists $\beta \leq (\Gamma \to \Delta; \tau)$ such that $\beta \models^* \Box C$ and $\gamma \not\models^* \Box A$ for any $\gamma \in \beta \downarrow$. Using Lemma 4.5, $\alpha < \beta$ or $\beta \leq \alpha$. Since $\alpha \models^* \Box A$ and $\gamma \not\models^* \Box A$ for any $\gamma \in \beta \downarrow$, we have $\beta \leq \alpha$. Using Lemma 4.6, we have $\alpha \models^* \Box C$. Using (2) and Lemma 4.7, we have $\alpha \models^* \Box B$. This is in contradiction in $\alpha \not\models^* \Box B$.

The subcase that (3c) and (1d) hold. By (1d) and Lemma 3.9(2), we have $\Box A \prec \Box D \in \Gamma_1 \subseteq \Gamma$, and hence $\Box A \in \mathsf{Sub}(\Gamma \to \Delta)$. This is in contradiction with $\Box A \notin \mathsf{Sub}(\Gamma \to \Delta)$ from (3c).

For the case that D is $A7$ (i.e., $D = (\Box A \vee \Box B) \supset ((\Box A \preceq \Box B) \vee (\Box B \prec \Box A))$). Suppose that $(\Gamma \to \Delta; \tau) \models \Box A \vee \Box B$, i. e. ,either $\Box A$ or $\Box A$ is true at the pair.

If $(\Gamma \to \Delta; \tau) \models^* \Box A$, then from the definition, $(\Gamma \to \Delta; \tau) \not\models^* \Box B \prec \Box A$ implies $(\Gamma \to \Delta; \tau) \models^* \Box A \preceq \Box B$. So, $(\Gamma \to \Delta; \tau) \models^* (\Box A \preceq \Box B) \vee (\Box B \prec \Box A)$.

If $(\Gamma \to \Delta; \tau) \not\models^* \Box A$, then $(\Gamma \to \Delta; \tau) \models^* \Box B$. So, there exists $\alpha \leq (\Gamma \to \Delta; \tau)$ such that $\alpha \models^* \Box B$. $\alpha \not\models^* \Box A$, and hence $(\Gamma \to \Delta; \tau) \models^* \Box B \prec \Box A$.

For the case that D is $A8$ (i.e., $D = (\Box A \prec \Box B) \supset (\Box A \preceq \Box B)$). From the definition, $(\Gamma \to \Delta; \tau) \models^* \Box A \prec \Box B$ implies $(\Gamma \to \Delta; \tau) \models^* \Box A \preceq \Box B$.

For the case that D is $A9$ (i.e., $D = ((\Box A \preceq \Box B) \wedge (\Box B \prec \Box A)) \supset \bot$). Suppose that

(7) $(\Gamma \to \Delta; \tau) \models \Box A \preceq \Box B$ and

(8) $(\Gamma \to \Delta; \tau) \models \Box B \prec \Box A$.

By (8), either one of the following three holds:

(8a) $\Box B \prec \Box A \in \Gamma$,

(8b) there exists $\alpha \leq (\Gamma \to \Delta; \tau)$ such that $\alpha \models^* \Box B$ and $\alpha \not\models^* \Box A$,

(8c) $\Box A \notin \mathsf{Sub}(\Gamma \to \Delta)$ and there exist C and $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that

$\Box B \prec \Box C \in \Gamma_1$,

$\Box A, \Box B, \Box C$ are true at $(\Gamma_1 \to \Delta_1; \tau_1)$,

for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$, $\alpha \not\models^* \Box A$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box C$.

If (8a) holds, then $\Box A \preceq \Box B \in \Delta$ since $\Gamma \to \Delta$ is saturated. Using Lemma 4.2, $(\Gamma \to \Delta; \tau) \not\models^* \Box A \preceq \Box B$. This is in contradiction with (7).

By (7) and Lemma 4.7, for any $\alpha \leq (\Gamma \to \Delta; \tau)$, $\alpha \models^* \Box B$ implies $\alpha \not\models^* \Box A$. This is in contradiction with (8b). So, (8b) does not hold.

Assume that (8c) holds. By (7), either one of the following four holds:

(7a) $(\Gamma \to \Delta; \tau) \models^* \Box A$ and $(\Gamma \to \Delta; \tau) \not\models^* \Box B \prec \Box A$,

(7b) $\Box A \prec \Box B \in \Gamma$,

(7c) there exists $\alpha \leq (\Gamma \to \Delta; \tau)$ such that $\alpha \models^* \Box A$ and $\alpha \not\models^* \Box B$,

(7d) $\Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$ and there exist C and $(\Gamma_1 \to \Delta_1; \tau_1) \leq (\Gamma \to \Delta; \tau)$ such that

$\Box A \prec \Box C \in \Gamma_1$,

$\Box A, \Box B, \Box C$ are true at $(\Gamma_1 \to \Delta_1; \tau_1)$,

for any $\alpha < (\Gamma_1 \to \Delta_1; \tau_1)$, $\alpha \not\models^* \Box A$, $\alpha \not\models^* \Box B$, $\alpha \not\models^* \Box C$.

(7a) is in contradiction with (8). (7b) is in contradiction with $\Box A \notin \mathsf{Sub}(\Gamma \to \Delta)$ from (8c). By (8) and

Lemma 4.7, for any $\alpha \leq (\Gamma \to \Delta; \tau)$, $\alpha \models^* \Box A$ implies $\alpha \not\models^* \Box B$, which is in contradiction with (7c). If (7d) holds, then $\Box A \prec \Box C \in \Gamma_1$, using Lemma 3.9(2), $\Box A \prec \Box C \in \Gamma \subseteq \mathsf{Sub}(\Gamma \to \Delta)$, which is in contradiction with $\Box A \notin \mathsf{Sub}(\Gamma \to \Delta)$ from (8c).

From Corollary 4.3 and Lemma 4.8, we obtain Theorem 3.15.

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