Gauss-Seidel method for multi-leader-follower games

M2016SS004 Atsushi Hori
Supervisor: Masao Fukushima

1 Introduction

The noncooperative game has been studied for a long time. An equilibrium point of the noncooperative game proposed by J.F. Nash [7] is called a Nash equilibrium, in which no one can improve his/her utility by changing his/her strategy unilaterally. The game is called the Nash equilibrium problem (NEP) or the Nash game. On the other hand, when several of the players, called the leaders, has the initiative, or it can decide before the other players, called the followers, make decision, the game is called the multi-leader-follower (multi-L/F) game. Applications of multi-L/F games are for example, deregulated electricity markets or telecommunication markets. For the details, see [6].

A special case of the game is the Stackelberg game, or the single-leader-follower game, which has been studied for many years. The bilevel game may be reformulated as a mathematical program with equilibrium constraints (MPEC), which is a single-level optimization problem and has also been studied extensively in recent years. The multi-leader-follower game may also be reformulated as an equilibrium problem with equilibrium constraints (EPEC), in which each leader's problem is an MPEC. However, finding an equilibrium point of an EPEC is much more difficult than solving a single MPEC, because each leader's MPEC contains those variables which are common to other players' MPECs. Moreover, the constraints of each leader's MPEC depend on the other rival leaders' strategies.

For solving multi-L/F games or EPECs, Hu and Fukushima [5] proposed a variational inequality (VI) formulation approach to multi-L/F game. However, the model does not contain the inequality constraints in the followers' optimization problems. Then, Tsuyuguchi [9] extended the VI formulation to deal with followers' inequality constraints, and then showed convergence of the approach to a Clarke stationary equilibrium, which is one of the solution concepts of an EPEC.

In this paper, we propose a Gauss-Seidel type algorithm with a penalty technique for solving an EPEC associated with the multi-leader-follower game, and then suggest a refinement procedure to obtain more accurate solutions. Furthermore, we discuss convergence of the algorithm to a strong stationary equilibrium, which is a stronger solution concept than the Clarke stationary equilibrium, and report some numerical results to illustrate the behavior of the algorithm.

2 Multi-L/F game and its reformulation

In this section, we first recall some fundamental concepts about multi-L/F games. For details, refer to the survey paper by Hu and Fukushima [6].

Consider a multi-L/F game consisting of N leaders and one follower. The leaders are labeled $\nu(= 1, ..., N)$. Let $x^\nu \in \Re^{n_\nu}$ denote the strategy vector, and $\theta^\nu : \Re^{n+m} \to \Re$ denote the cost function of leader ν and be C^2. Let $y \in \Re^m$ and $\gamma : \Re^{n+m} \to \Re$ denote the strategy vector and the C^3 cost function of the follower, respectively. Here, $n := n_1 + \cdots + n_N$. For given $x^{-\nu} := (x^1, ..., x^{\nu-1}, x^{\nu+1}, ..., x^N) \in \Re^{n-n_\nu}$, leader ν solves the following optimization problem:

$$\begin{array}{ll} \min\limits_{x^\nu \in \Re^{n_\nu}} & \theta^\nu(x^\nu, x^{-\nu}, y) \\ \text{s.t.} & g^\nu(x^\nu) \le 0, h^\nu(x^\nu) = 0, \end{array} \tag{1}$$

where $g^\nu : \Re^{n_\nu} \to \Re^{r_\nu}$ and $h^\nu : \Re^{n_\nu} \to \Re^{s_\nu}$ are C^2 functions.

For a given tuple of the leaders' strategies $x := (x^1, ..., x^N) \in \Re^n$, the follower solves the following optimization problem:

$$\begin{array}{ll} \min\limits_{y \in \Re^m} & \gamma(x, y) \\ \text{s.t.} & u(x, y) \le 0, \end{array} \tag{2}$$

where $u : \Re^{n+m} \to \Re^p$ is a C^3 function. We assume that problem (2) is convex with respect to y. We define a basic solution concept of the multi-L/F game.

Definition 1 A tuple of strategies (x^*, y^*) is called a L/F Nash equilibrium if those strategies simultaneously achieve global optimality in the players' optimization problems.

However, to find a L/F Nash equilibrium is not easy in general. We extend the solution concepts which will be defined later.

The multi-L/F game can be regarded as a bilevel game which consists of the leaders' upper level problems and the follower's lower one. To follow the reformulation approach for the single-L/F game, we reformulate the multi-L/F game as an EPEC, or a single level game.

Because of the convexity of the follower's problem (2), it can be equivalently dealt with using the Karush-Kuhn-Tucker (KKT) conditions under an appropriate constraint qualification, which can be written as the mixed complementarity system:

$$\begin{array}{c} \psi(x, y, z, \lambda) = 0, \\ 0 \le z \perp \lambda \ge 0, \end{array} \tag{3}$$

where

$$\psi(x, y, z, \lambda) := \begin{bmatrix} \nabla_y \gamma(x, y) + \nabla_y u(x, y)\lambda \\ u(x, y) + z \end{bmatrix} \in \Re^{m+p}.$$

Here, $\lambda \in \Re^p$ is the Lagrange multiplier and $z \in \Re^p$ is a vector of slack variables for the inequality constraints $u(x, y) \le 0$.

By incorporating (3) into each leader's optimization problem (1), we have the following parametrized mathematical program with complementarity constraints (PMPCC) for leader ν:

$$\begin{array}{lll} \text{PMPCC}^\nu(x^{-\nu}) : & \min\limits_{x^\nu, y, z, \lambda} & \theta^\nu(x^\nu, x^{-\nu}, y) \\ & \text{s.t.} & g^\nu(x^\nu) \le 0, \ h^\nu(x^\nu) = 0, \\ & & \psi(x^\nu, x^{-\nu}, y, z, \lambda) = 0, \\ & & 0 \le z \perp \lambda \ge 0. \end{array}$$

Thus the multi-L/F game is reduced to an EPEC, which seeks an equilibrium point that simultaneously achieves optimality in $(\text{PMPCC}^\nu(x^{-\nu}))_{\nu=1}^N$. We call $(y, z, \lambda) \in \Re^{m+2p}$ shared variables, because all leaders have those as decision variables.

Now, we define an extended solution concept of the multi-L/F game called an S-stationary equilibrium point.

Definition 2 A tuple $(x^*, y^*, z^*, \lambda^*) \in \Re^{n+m+2p}$ is called a strong (S-) stationary equilibrium point of the EPEC (or multi-L/F game), if for each leader ν, $(x^{\nu,*}, y^*, z^*, \lambda^*)$ is an S-stationary point (see [8]) of $\mathrm{PMPCC}^\nu(x^{-\nu,*})$.

3 Method for multi-L/F games

In this section, we propose a numerical method for multi-L/F games by way of EPECs. First, we elaborate on a Gauss-Seidel type penalty method for EPECs and then a refinement procedure to obtain more accurate solutions.

3.1 Gauss-Seidel penalty method

In leader ν's problem $\mathrm{PMPCC}^\nu(x^{-\nu})$, the complementarity constraints can be replaced with the equality constraints by means of the FB-function $\phi\colon \Re^2 \to \Re$, which is defined by

$$\phi(a,b) := a + b - \sqrt{a^2+b^2}.$$

Specifically, $\mathrm{PMPCC}^\nu(x^{-\nu})$ can be rewritten as

$$\begin{array}{lll} \mathrm{P}^\nu(x^{-\nu}): & \displaystyle\min_{x^\nu,y,z,\lambda} & \theta^\nu(x^\nu, x^{-\nu}, y) \\ & \text{s.t.} & g^\nu(x^\nu) \le 0,\ h^\nu(x^\nu) = 0, \\ & & \Psi(x^\nu, x^{-\nu}, y, z, \lambda) = 0, \end{array}$$

where

$$\Psi(x^\nu, x^{-\nu}, y, z, \lambda) := \begin{bmatrix} \psi(x^\nu, x^{-\nu}, y, z, \lambda) \\ \phi(z_1, \lambda_1) \\ \vdots \\ \phi(z_p, \lambda_p) \end{bmatrix}.$$

However, $\mathrm{P}^\nu(x^{-\nu})$ is nonsmooth because of the nondifferentiability of the FB-function. To avoid this difficulty, we use the property that the squared FB-function is differentiable everywhere [2]. Define the penalty function associated with problem $\mathrm{P}^\nu(x^{-\nu})$ by

$$\bar\theta^\nu_\rho(x^\nu, x^{-\nu}, y, z, \lambda) := \theta^\nu(x^\nu, x^{-\nu}, y) + \frac{\rho}{2}\left[\sum_{i=1}^{r_\nu}[g_i^\nu(x^\nu)]_+^2 + \sum_{i=1}^{s_\nu}|h_i^\nu(x^\nu)|^2 + \sum_{j=1}^{m+2p}|\Psi_j(x^\nu, x^{-\nu}, y, z, \lambda)|^2\right],$$

where $\rho > 0$ is a penalty parameter and $[g_i^\nu(x^\nu)]_+ := \max\{0, g_i^\nu(x^\nu)\}$. The penalized problem for leader ν's problem $\mathrm{P}^\nu(x^{-\nu})$ is written as

$$\overline{\mathrm{P}}^\nu_\rho(x^{-\nu}): \quad \min_{x^\nu,y,z,\lambda} \quad \bar\theta^\nu_\rho(x^\nu, x^{-\nu}, y, z, \lambda),$$

which is a differentiable unconstrained optimization problem. The proposed algorithm is formally stated as follows.

Algorithm I: Gauss-Seidel Penalty Method

1. Set a tolerance $\varepsilon > 0$, and the maximum number of major iterations $K_{\max}$. Choose an initial point
$$x^{(0)} := (x^{1,(0)}, ..., x^{N,(0)}), y^{(0)}, z^{(0)}, \lambda^{(0)},$$
and an increasing positive sequence $\{\rho_k\}$. Set $k := 0$.
2. Set $\nu := 1$.
3. Solve $\overline{\mathrm{P}}^\nu_{\rho_k}(\bar x^{-\nu,(k)})$ to obtain the solution
$$\bar w^{\nu,(k+1)} := (\bar x^{\nu,(k+1)}, \bar y^{\nu,(k+1)}, \bar z^{\nu,(k+1)}, \bar\lambda^{\nu,(k+1)}),$$
where
$$\bar x^{-\nu,(k)} := (\bar x^{1,(k+1)}, ..., \bar x^{\nu-1,(k+1)}, \bar x^{\nu+1,(k)}, ..., \bar x^{N,(k)}).$$
4. If $\nu < N$, set $\nu := \nu+1$ and go to Step 3. Otherwise, go to Step 5.
5. If
$$\max\left\{\max_{1\le i\le r_\nu}[g_i^\nu(\bar x^{\nu,(k+1)})]_+, \max_{1\le i\le s_\nu}|h_i^\nu(\bar x^{\nu,(k+1)})|, \max_{1\le i\le m+2p+q}|\Psi_i(\bar w^{\nu,(k+1)}, \bar x^{-\nu,(k)})|\right\} < \varepsilon$$
holds for all ν, terminate.
6. If $k < K_{\max}$, set $k := k+1$ and go to Step 2. If $k = K_{\max}$, terminate.

In Step 3, we use the notation $\bar y^\nu, \bar z^\nu, \bar\lambda^\nu$ to distinguish among leaders, because all leaders do not necessarily output the same solutions y, z, λ.

To argue convergence of the algorithm with $\varepsilon = 0$ and $K_{\max} = \infty$, we give some results below.

Lemma 1 Let $\rho_k \to \infty$, and for each ν, $\bar w^{\nu,(k)} \to \bar w^{\nu,(\infty)}, \bar x^{-\nu,(k)} \to \bar x^{-\nu,(\infty)}$. Assume that the sequence $\{\bar\theta^\nu_{\rho_k}(\bar w^{\nu,(k+1)}, \bar x^{-\nu,(k)})\}$ of the objective values of problems $\overline{\mathrm{P}}^\nu_{\rho_k}(\bar x^{-\nu,(k)})$ is bounded above. Then, $\bar w^{\nu,(\infty)}$ is a feasible solution to $\mathrm{P}^\nu(\bar x^{-\nu,(\infty)})$, i.e., $\bar w^{\nu,(\infty)}$ is feasible to $\mathrm{PMPCC}^\nu(\bar x^{-\nu,(\infty)})$.

Lemma 2 Assume that the conditions of Lemma 1 hold. Suppose the sequence $\{(\bar x^{\nu,(k)}, \bar x^{-\nu,(k)}, \bar y^{\nu,(k)}, \bar z^{\nu,(k)}, \bar\lambda^{\nu,(k)})\}_{\nu=1}^N$ generated by the algorithm converges to $\{(\bar x^{(\infty)}, \bar y^{\nu,(\infty)}, \bar z^{\nu,(\infty)}, \bar\lambda^{\nu,(\infty)})\}_{\nu=1}^N$, and the function γ is strongly convex for any fixed x. Then the shared variables $\bar y^{\nu,(k)}$ converge to the same limit $\bar y^{(\infty)}$ independent of ν. Furthermore, if the linear independence constraint qualification (LICQ) holds at $\bar y^{(\infty)}$ in the followers' problems (2), then $\bar\lambda^{\nu,(k)}$ also converge to the same limit $\bar\lambda^{(\infty)}$.

We showed that if the algorithm converges, then the limit is an S-stationary point of the EPEC under appropriate assumptions.

Theorem 1 Let $\rho_k \to \infty$. Suppose that, for each $\nu = 1, ..., M$, $(\bar w^{\nu,(k)}, \bar x^{-\nu,(k)}) \to (\bar x^{(\infty)}, \bar y^{\nu,(\infty)}, \bar z^{\nu,(\infty)}, \bar\lambda^{\nu,(\infty)})$, where $\bar w^{\nu,(k+1)}$ is a local optimal solution of problem $\overline{\mathrm{P}}^\nu_{\rho_k}(\bar x^{-\nu,(k)})$ for each k. Assume that the conditions in Lemmas 1 and 2 hold. Moreover, suppose that, for each ν, the MPCC-LICQ for $\mathrm{PMPCC}^\nu(\bar x^{-\nu,(\infty)})$ and the upper level strict complementarity (ULSC) (see [8]) hold at the limit point $(\bar x^{\nu,(\infty)}, \bar y^{\nu,(\infty)}, \bar z^{\nu,(\infty)}, \bar\lambda^{\nu,(\infty)})$. Then, those limit points are identical and the limit point is an S-stationary point of $\mathrm{PMPCC}^\nu(\bar x^{-\nu,(\infty)})$ for each ν. Consequently, it constitutes an S-stationary equilibrium point of the multi-L/F game.

3.2 Refined Gauss-Seidel method

From the numerical viewpoint, the squared penalty method has some drawbacks. The main issue is that the penalized problem becomes ill-conditioned as the penalty parameter ρ_k increases, and so it is difficult to find an accurate solution even for a sufficient large k. Nevertheless, it may provide useful information about active sets in the complementarity constraints. In fact, if the active sets are correctly identified, we may further refine the solution produced by Algorithm I. To this end, we present another Gauss-Seidel based method for obtaining a more accurate solution.

Let $\bar w^* = (\bar w^{1,*}, ..., \bar w^{N,*})$ be a solution obtained by

Algorithm I, and define the index sets:

$$\bar{\mathcal{I}}^\nu := \{ i \mid |\bar{z}_i^{\nu,*}| < \delta,\ |\bar{\lambda}_i^{\nu,*}| \geq \delta \},$$
$$\bar{\mathcal{J}}^\nu := \{ i \mid |\bar{z}_i^{\nu,*}| < \delta,\ |\bar{\lambda}_i^{\nu,*}| < \delta \}, \qquad (4)$$
$$\bar{\mathcal{K}}^\nu := \{ i \mid |\bar{z}_i^{\nu,*}| \geq \delta,\ |\bar{\lambda}_i^{\nu,*}| < \delta \},$$

where $\delta > 0$ is a sufficiently small number. We assume that those index sets are independent of ν, i.e., $\bar{\mathcal{I}} := \bar{\mathcal{I}}^\nu, \bar{\mathcal{J}} := \bar{\mathcal{J}}^\nu, \bar{\mathcal{K}} := \bar{\mathcal{K}}^\nu$ for all ν. We define the following optimization problem for each leader ν:

$$\begin{array}{lll} \widetilde{\mathrm{P}}^\nu(x^{-\nu}): & \displaystyle\min_{x^\nu, y, z, \lambda} & \theta^\nu(x^\nu, x^{-\nu}, y) \\ & \text{s.t.} & g^\nu(x^\nu) \leq 0,\ h^\nu(x^\nu) = 0, \\ & & \psi(x^\nu, x^{-\nu}, y, z, \lambda) = 0, \\ & & z_i = 0,\ \lambda_i \geq 0\ \ (i \in \bar{\mathcal{I}}), \\ & & z_i = 0,\ \lambda_i = 0\ \ (i \in \bar{\mathcal{J}}), \\ & & z_i \geq 0,\ \lambda_i = 0\ \ (i \in \bar{\mathcal{K}}). \end{array}$$

Now the algorithm is stated as follows.

Algorithm II: Refined Gauss-Seidel Method

1. Set the maximum number of iterations $K_{\max}$, and the step size tolerance $\varepsilon' > 0$. Let the initial point $\tilde{w}^{(0)} := (\tilde{w}^{1,(0)}, ..., \tilde{w}^{N,(0)})$ be the last point obtained by Algorithm I. Set $k := 0$.
2. Set $\nu := 1$.
3. Solve $\widetilde{\mathrm{P}}^\nu(\tilde{x}^{-\nu,(k)})$ to obtain the solution $\tilde{w}^{\nu,(k+1)}$, where $\tilde{x}^{-\nu,(k)} := (\tilde{x}^{1,(k+1)}, ..., \tilde{x}^{\nu-1,(k+1)}, \tilde{x}^{\nu+1,(k)}, ..., \tilde{x}^{N,(k)})$.
4. If $\nu < N$, set $\nu := \nu+1$ and go to Step 3. Otherwise, go to Step 5.
5. If $||\tilde{w}^{\nu,(k+1)} - \tilde{w}^{\nu,(k)}|| < \varepsilon'$ holds for all ν, go to Step 6.
6. If $k < K_{\max}$, set $k := k+1$ and go to Step 2. If $k = K_{\max}$, terminate.

If a tuple of solutions $(\tilde{w}^{1,*}, ..., \tilde{w}^{N,*})$ is obtained and, for each leader ν, the KKT conditions of $\widetilde{\mathrm{P}}^\nu(\tilde{x}^{1,(k+1)}, ..., \tilde{x}^{\nu-1,(k+1)}, \tilde{x}^{\nu+1,(k+1)}, ..., \tilde{x}^{N,(k+1)})$ are sufficiently satisfied for all ν, then the algorithm successfully terminates and the point is a stationary equilibrium.

4 Application

In this section, we introduce an application of multi-L/F games. In the middle of 1990s, deregulation of electricity markets by governments stated mainly in Europe and the United States. Since then, the study of electricity markets has become popular [1, 3]. We introduce a wholesale market of electricity in terms of multi-L/F games or EPECs. The model we discuss is a simple model of competitive bidding under some macroeconomic regulation, which is an extension of [5].

In this model, we assume that there are two electricity firms labeled $\nu \in \{\mathrm{I}, \mathrm{II}\}$ and one market maker, called the independent system operator (ISO), who tries to collect the balance of demand and supply of electricity by paying the bid costs under the market clearing mechanism. The ISO also determines the price of electricity, and then sells it to consumers. The two firms are competing each other for market power in an electricity network with M nodes (consumers), and determine the bid price.

Let $x^\nu := (x_1^\nu, ..., x_M^\nu) \in \Re^M$ be the bid parameter of firm ν in which the firm indirectly determines how much it sells the electricity to each node. Let $y := (y_1^{\mathrm{I}}, ..., y_M^{\mathrm{I}}, y_1^{\mathrm{II}}, ..., y_M^{\mathrm{II}}) \in \Re^{2M}$ be the quantity of electricity, where y_i^ν means how much quantity of electricity the ISO buys from firm ν and supplies it to consumer i. The bid price function of firm ν is defined by $b^\nu(x^\nu, y) := \sum_{i=1}^M x_i^\nu y_i^\nu$. We assume that two firms produce the electricity up to quantities a^{I} and a^{II}, and then send it to all nodes at the price $p_i(y_i^{\mathrm{I}}, y_i^{\mathrm{II}}) := \alpha_i - \beta_i(y_i^{\mathrm{I}} + y_i^{\mathrm{II}})$, where α_i and β_i are positive constants. The revenue for the ISO by selling electricity to node i is given as the cumulative sum from zero to $y_i^{\mathrm{I}} + y_i^{\mathrm{II}}$. Thus, the ISO makes a profit given by $q_i(y_i^{\mathrm{I}}, y_i^{\mathrm{II}}) := \alpha_i(y_i^{\mathrm{I}} + y_i^{\mathrm{II}}) - \frac{\beta_i}{2}(y_i^{\mathrm{I}} + y_i^{\mathrm{II}})^2$.

Firm ν needs to pay the transaction cost according to the bid parameter x_i^ν, which is defined by $t^\nu(x^\nu) := \frac{1}{2}\sum_{i=1}^M \tau_i^\nu (x_i^\nu)^2$ with a constant $\tau_i^\nu > 0$, and tries to maximize its revenue by bidding from the ISO minus transaction costs. Then the optimization problem of firm ν can be written as follows:

$$\begin{array}{ll} \displaystyle\min_{x^\nu \in \Re^M} & t^\nu(x^\nu) - b^\nu(x^\nu, y) \\ \text{s.t.} & x^\nu \in X^\nu, \end{array} \qquad (5)$$

where X^ν is a nonempty strategy set.

On the other hand, the ISO also tries to maximize its revenue by selling electricity to consumers. Furthermore, we assume that some economic interventionism by governments works in the market to maintain the equilibrium between the quantities of electricity at each node i, or to reflect the ratio of quantities a^{I} and a^{II}, which is denoted by $\frac{\zeta_i}{2}\left(\frac{y_i^{\mathrm{I}}}{a^{\mathrm{I}}} - \frac{y_i^{\mathrm{II}}}{a^{\mathrm{II}}}\right)^2$, where $\zeta_i > 0$ is the interventionism parameter. Hence, the optimization problem of ISO can be written as follows:

$$\begin{array}{ll} \displaystyle\min_{y \in \Re^{2M}} & \sum_{i=1}^M \left[\frac{\zeta_i}{2}\left(\frac{y_i^{\mathrm{I}}}{a^{\mathrm{I}}} - \frac{y_i^{\mathrm{II}}}{a^{\mathrm{II}}}\right)^2 - q_i(y_i^{\mathrm{I}}, y_i^{\mathrm{II}})\right] \\ & + b^{\mathrm{I}}(x^{\mathrm{I}}, y) + b^{\mathrm{II}}(x^{\mathrm{II}}, y) \\ \text{s.t.} & \sum_{i=1}^M y_i^{\mathrm{I}} - a^{\mathrm{I}} \leq 0,\ \sum_{i=1}^M y_i^{\mathrm{II}} - a^{\mathrm{II}} \leq 0, \\ & y \geq 0. \end{array}$$

By the strong convexity of the ISO's problem, the solution is uniquely determined for any given x by Lemma 2. Furthermore, the response is piecewise linear for the variable x.

5 Numerical experiments

In this section, we present some numerical results to demonstrate the validity of the proposed method. We coded the algorithm in MATLAB 9.1.0 (2016b). We consider a multi-L/F game consisting of three leaders and one follower. Leader $\nu \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ solves the following optimization problem:

$$\begin{array}{ll} \displaystyle\min_{x^\nu \in \Re^3} & \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + \sum_{\substack{\nu'=1 \\ \nu' \neq \nu}}^N (x^\nu)^\top G_{\nu,\nu'} x^{\nu'} + (x^\nu)^\top D_\nu y \\ \text{s.t.} & A_\nu x^\nu \leq b^\nu. \end{array}$$

On the other hand, the follower solves the following optimization problem:

$$\begin{array}{ll} \displaystyle\min_{y \in \Re^3} & \frac{1}{2} y^\top M y + q^\top y - \sum_{\nu=1}^3 (x^\nu)^\top D_\nu y \\ \text{s.t.} & c^\top y + \sum_{\nu=1}^3 (d^\nu)^\top x^\nu + a \geq 0, \end{array}$$

where the matrix M is positive definite. The dimension of x^ν and y are three, respectively. Due to space limitations, we omit the numerical data (see [4]) and the reformulation. We set $\varepsilon = \delta = 10^{-2}$ in Algorithm I and (4), and $\varepsilon' = 10^{-6}$ in Algorithm II. Since the follower's problem is strongly convex, the solution y is uniquely determined for any x by Lemma 2.

To confirm the validity of the algorithm, we show the maximum distance between the N leaders' shared variables y^{I}, y^{II} and y^{III}. The maximum distance was reduced to 0.0051 by using Algorithm I and the number of iterations was 5. Then we proceeded to Algorithm II, and then the maximum distance was further reduced to 4.4136e-07 after 22 iterations. Furthermore, we confirmed that Algorithm II found more accurate solutions and the S-stationarity condition was satisfied at the final point obtained by Algorithm II. The behavior of Algorithms I and II are shown in Figure 1. The red curve represents the sequence $\{\max_{\nu\neq\nu'} ||y^{\nu,(k)} - y^{\nu',(k)}||\}$ generated by the algorithms, which shows linear convergence.

Next we applied a successive over-relaxation (SOR) method, which is a variant of the Gauss-Seidel method, to improve the convergence speed. Specifically, in Step 3 of Algorithms I and II, we changed $\bar{w}^{\nu,(k+1)}$ and $\tilde{w}^{\nu,(k+1)}$ to

$$\bar{w}^{\nu,(k+1)} := \bar{w}^{\nu,(k)} + \omega_k(\hat{w}^{\nu,(k+1)} - \bar{w}^{\nu,(k)}),$$
$$\tilde{w}^{\nu,(k+1)} := \tilde{w}^{\nu,(k)} + \omega_k(\hat{w}^{\nu,(k+1)} - \tilde{w}^{\nu,(k)}),$$

respectively, where $\hat{w}^{\nu,(k+1)}$ is the solution obtained by the Gauss-Seidel method. In general, the relaxation parameter ω_k is chosen so that $\omega_k \in (0, 2)$, and $\omega_k = 1.5$ $(k \in \{1, 2, ...\})$ is often used in practice.

The sequence generated by the algorithms converges to the same point as before, and the total number of iterations is reduced to 17 when ω_k are chosen as $\omega_k = 1 + (0.95\omega_{k-1} - 1)$ with $\omega_0 = 1.5$, as shown by the blue curve in Figure 1.

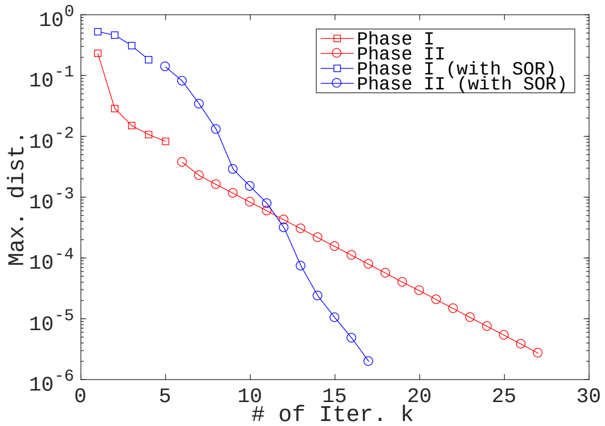

Figure 1 Maximum distance between shared variables $\max_{\nu\neq\nu'} ||y^{\nu,(k)} - y^{\nu',(k)}||$

Next we consider the electricity model introduced in Section 4. The strategy set X^ν in (5) is given by $X^\nu := \{x^\nu \in \Re^M \,|\, 0 \leq x^\nu \leq \xi^\nu\}$. The numerical data are found in the first example in [5]. The number of nodes M is 2. The variables of the players are, $x^\nu \in \Re^2$ for firm ν, and $y \in \Re^4$ for ISO.

When we set the interventionism parameters as $\zeta = (0.05, 0.05)$, we obtained an S-stationary equilibrium point with Algorithms I and II after 11 iterations, and the final distance between $(y^{\mathrm{I},*}, \lambda^{\mathrm{I},*})$ and $(y^{\mathrm{II},*}, \lambda^{\mathrm{II},*})$ was 1.0795e-05.

We observe the ratio of the electric supplies by the two firms $y_i^{\mathrm{I}} : y_i^{\mathrm{II}}$ at each node i. In this example, the ratio of the total electric supplies is $a^{\mathrm{I}} : a^{\mathrm{II}} = 1 : 1.5$. Let y^{I} and y^{II} denote $y^{\mathrm{I}} := (y_1^{\mathrm{I},*}, y_2^{\mathrm{I},*})$, $y^{\mathrm{II}} := (y_3^{\mathrm{I},*}, y_4^{\mathrm{I},*})$. As Table 1 shows, we found that the ratios $y_1^{\mathrm{I}} : y_1^{\mathrm{II}}$ and

Table 1 ratio of quantities

ζ	$(0.05, 0.05)$	$(0.05, 0.5)$	$(0.5, 0.05)$	$(0.5, 0.5)$
$y_1^{\mathrm{I}} : y_1^{\mathrm{II}}$	1 : 1.36601	1 : 1.38229	1 : 1.3823	1 : 1.39730
$y_2^{\mathrm{I}} : y_2^{\mathrm{II}}$	1 : 1.62862	1 : 1.61336	1 : 1.6134	1 : 1.59831

$y_2^{\mathrm{I}} : y_2^{\mathrm{II}}$ are getting closer to $1 : 1.5$ as ζ_i, $i = 1, 2$, increase. The objective function value of each firm was almost unchanged in these three cases.; However, that of ISO was is increased as the sum of ζ_i increases.

6 Conclusion

In this paper, we proposed a numerical method for solving multi-L/F games based on the penalty method and the nonlinear diagonalized Gauss-Seidel method. The method consists of two phases. The first phase of the method may be regarded as the identification of the active sets in the complementarity constraints, and the second phase is to find more accurate solutions with the active sets identified in the first phase. We discussed convergence of the Gauss-Seidel penalty method to an S-stationary equilibrium point under the feasibility, MPCC-LICQ, and ULSC assumptions. Furthermore, we confirmed the validity of the algorithm through numerical experiments. In particular, we applied the SOR method and succeeded to reduce computation time.

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