Design and Verification of Nonlinear Optimal Controller Considering Acceleration Constraints based on Invariant Manifold Calculation

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1 Introduction

The ISO 2631-1[1] states that there is a relationship between ride comfort and acceleration of vehicles. The smaller vehicle acceleration is the better ride comfort becomes. Therefore, in vehicle industry, it is necessary to impose acceleration constraints to acquire a reasonable ride comfort.

Conventionally, a control design that satisfies acceleration constraints by tuning weight matrices and gains has been done. However, this often leads to a degradation on control performance.

In this paper, we consider the optimal control problem of the nonlinear system satisfying acceleration constraints. We propose a nonlinear optimal control design method and a nonlinear servo control design method via stable and center-stable manifold methods[2], [3] introducing Lagrange multiplier to handle acceleration constraints. The stable and center-stable manifold methods are recently proposed, which are iterative calculation methods to calculate the approximated solution of a Hamilton-Jacobi equation subject to some constraints[4]. We verify the effectiveness of the proposed methods on a magnetic levitation system and a numerical example.

2 Optimal Regulation Problem with Acceleration Constraints

In this section, we consider the nonlinear optimal regulation problem for nonlinear system subject to acceleration constraints based on the stable manifold theory.

2.1 Problem Definition

Let us consider a nonlinear system of the physical system of the form

$$\dot{x} = f(x) + g(x)u, x \in \mathbb{R}^n, u \in \mathbb{R}^m, x(0) = x_0, \quad (1)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. We will design a controller that minimize the following cost function

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt, Q \geqslant 0, R > 0,$$

subject to some constraints on acceleration

$$a_{\min} \leqslant a \leqslant a_{\max},$$

where a is acceleration of the system (1), $a_{\max}$ and $a_{\min}$ are upper and lower limit values of acceleration, respectively.

2.2 Application of Dynamic Programming

In this section, the acceleration constraints are considered as follows

$$h_1 = a - a_{\max} = l(f + gu) - a_{\max} \leqslant 0,$$
$$h_2 = a_{\min} - a = a_{\min} - l(f + gu) \leqslant 0,$$

where $a_{\min} \leqslant 0 \leqslant a_{\max}$. l is a vector for extracting acceleration from the state space representation of the system (1). There are 3 possible cases that might happen at a time t, which are

- Case 1: all constraints are inactive.
- Case 2: h_1 is active, which is $a = a_{\max}$.
- Case 3: h_2 is active, which is $a = a_{\min}$.

The optimal control problem considering acceleration constraints can be handled as the conditional minimization problem by using Lagrange multiplier. Therefore, the pre Hamiltonian H_i, the optimal input $\bar{u}_i$ and the value of the Lagrange multiplier $\lambda_i (i = 1, \cdots, 3)$ for each case are calculated.

- Case 1: all constraints are inactive.

$$H_1 = p^T (f + gu) + x^T Qx + u^T Ru,$$
$$\bar{u}_1 = -\frac{1}{2} R^{-1} g^T p,$$
$$\lambda_1 = 0.$$

- Case 2: h_1 is active, which is $a = a_{\max}$.

$$H_2 = H_1 + \lambda_2 (l(f + gu) - a_{\max}),$$
$$\bar{u}_2 = \bar{u}_1 - \frac{1}{2} \lambda_2 R^{-1} g^T l^T,$$
$$\lambda_2 = \frac{-2(a_{\max} - l(f + g\bar{u}_1))}{lgR^{-1} g^T l^T}.$$

- Case 3: h_2 is active, which is $a = a_{\min}$.

$$H_3 = H_1 + \lambda_3 (a_{\min} - l(f + gu)),$$
$$\bar{u}_3 = \bar{u}_1 + \frac{1}{2} \lambda_3 R^{-1} g^T l^T,$$
$$\lambda_3 = \frac{2(a_{\min} - l(f + g\bar{u}_1))}{lgR^{-1} g^T l^T}.$$

Next, we substitute $\bar{u}_i$ into the pre Hamiltonian H_i to obtain Hamilton-Jacobi equation $\bar{H}_i = 0 (i = 1, \cdots, 3)$. Then, an associated Hamiltonian system is derived. And we apply the stable manifold method using Case-choosing algorithm[4] for Hamilton's canonical equation and obtain an optimal feedback controller $u(x)$.

2.3 Application for Magnetic Levitation System

We verify the effectiveness of the proposed method by application for a magnetic levitation system.

2.3.1 Modeling

The schematic diagram of the magnetic levitation system is shown Figure. 1. x_b is position of the ball and i_c is input of the system. The equation of motion of the system can be derived via Newton's second law. Considering a new variable $x_b^* := x_b - x_{b0}, u^* := u^2 - u_0^2$

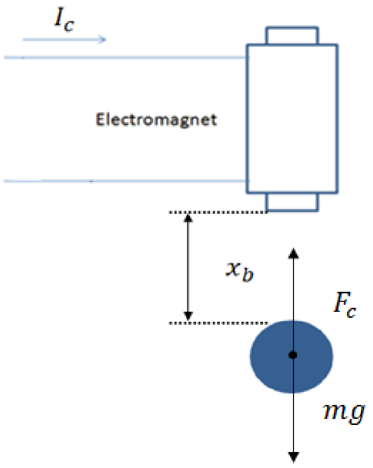

Figure 1 Schematic Diagram of Magnetic Levitation System

based on the equilibrium point and choosing state variable $x = [x_1 \; x_2]^T = [x_b^* \; \dot{x}_b^*]^T$ and input u^*, the equation of motion of the system is shown as follows

$$\dot{x} = f(x) + g(x)u^*, \tag{2}$$

where

$$f(x) = \begin{bmatrix} x_2 \\ g - \frac{C}{m_b}\left(\frac{u_0}{x_1 + x_{b0} + d}\right)^2 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ -\frac{C}{m_b}\left(\frac{1}{x_1 + x_{b0} + d}\right)^2 \end{bmatrix}.$$

Parameters of the magnetic levitation system are listed in Table. 1.

Table 1 Physical parameters of the magnetic levitation system

Gravitational acceleration : g	9.81[m/s^2]
Mass of ball : m_b	66[g]
Electromagnetic force constant : C	1.532×10^{-4}[Nm^2/A^2]
Actuator parameter : d	9.445×10^{-3}[m]

2.3.2 Control Design

In control design, the weighting matrices of the cost function are $Q = \text{diag}([1, \; 1]), R = 1$ and acceleration constraints are $a_{\max} = 0.1[\text{m/s}^2], a_{\min} = -0.1[\text{m/s}^2]$. Applying the stable manifold method algorithm 3 times, the optimal feedback controller $u(x)$ is approximated by linear interpolation (Figure 2).

2.3.3 Simulation Result

The proposed method is compared with a LQ controller using the same weighting matrices Q, R. To satisfy the acceleration constraints, the algorithm in Figure. 3 is applied to modify the LQ controller. The simulation results can be seen in the Fig. 4, 5 and 6. Here, the initial conditions are $x_1(0) = 0.013[\text{m}], x_2(0) = 0[\text{m/s}]$ and the equilibrium points $x_{1\text{ep}} = 0.007[\text{m}], x_{2\text{ep}}(0) = 0[\text{m/s}]$. It is shown in Fig. 4, 5 and 6 that the response of proposed method satisfies the acceleration constraints and the nonlinear controller has better convergence to the origin than the LQ controller with input modification.

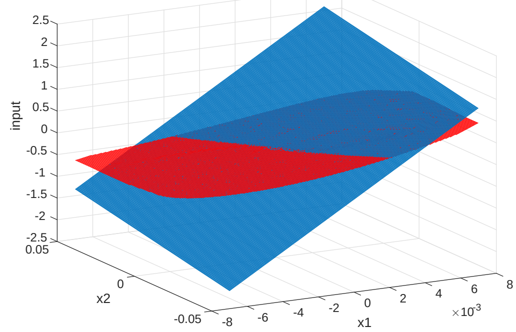

Figure 2 The nonlinear controller(red surface) and the LQ controller using the weighting matrices of the same cost function as the nonlinear controller(blue surface).

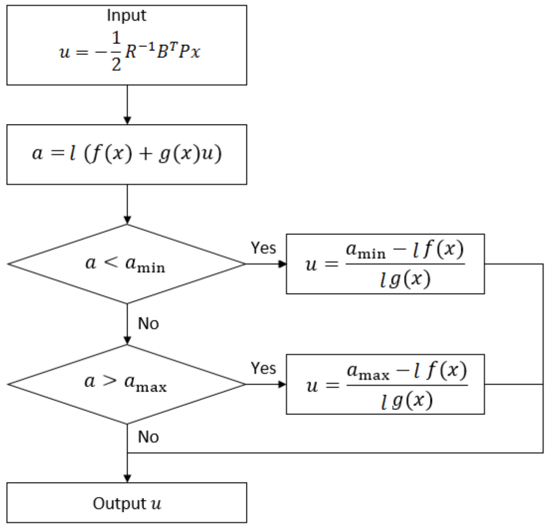

Figure 3 Input modification algorithm to satisfy acceleration constraint for LQ controller.

3 Optimal Servo Problem with Acceleration Constraints

We consider nonlinear optimal servo problem for nonlinear system with acceleration constraints based on the center-stable manifold theory and Lagrange multiplier.

3.1 Problem Definition

We consider the optimal servo problem for the nonlinear system (1) and an error equation is as follows

$$e = h(x, w),$$

where the system has relative degree 2. The reference signal is generated from the exosystem shown as follows

$$\dot{w} = s(w), w \in \mathbb{R}^p, s(0) = 0, \tag{3}$$

where $s : \mathbb{R}^p \to \mathbb{R}^p, h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^r$. Since the system has relative degree of 2, the following cost function is chosen

$$J = \frac{1}{2}\int_0^\infty (|e|^2 + |\dot{e}|^2 + |\ddot{e}|^2)dt. \tag{4}$$

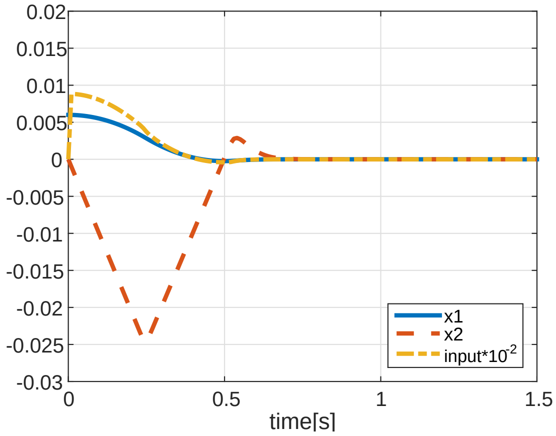

Figure 4 Responses of state and input of constrained nonlinear controller

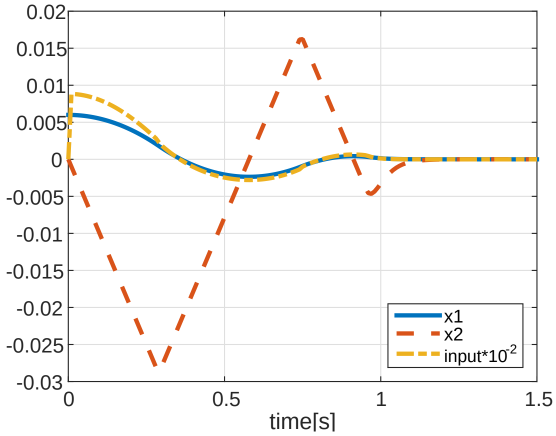

Figure 5 Responses of acceleration of constrained linear controller

We will design a controller that minimizes the cost function (4). J can be written as

$$J = \frac{1}{2}\int_0^\infty L(x,w,u)dt,$$
$$L(x,w,u) = |h(x,w)|^2 + |L_f h(x,w) + L_s h(x,w)|^2 + |L_f^2 h(x,w) + L_g L_f h(x,w)u + L_s^2 h(x,w)|^2,$$

where $L_f h, L_s h, L_f^2 h, L_g L_f h, L_s^2 h$ are the Lie differentiations and these are defined as follows

$$L_f h = \frac{\partial h}{\partial x} f, L_s h = \frac{\partial h}{\partial w} s,$$
$$L_f^2 h = \frac{\partial L_f h}{\partial x} f, L_g L_f h = \frac{\partial L_f h}{\partial x} g, L_s^2 h = \frac{\partial L_s h}{\partial w} s.$$

Then, we consider the control design problem of system (1), (3) subject to acceleration constraints (2).

3.2 Application of Dynamic Programming

The pre Hamiltonian H_i, the optimal input $\bar{u}_i$ and the value of the Lagrange multiplier $\lambda_i (i = 1, \cdots, 3)$ for each case are calculated.

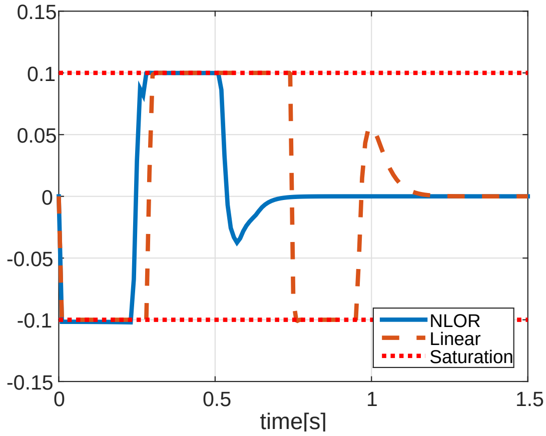

Figure 6 Response of acceleration of constrained nonlinear and linear controllers

- Case 1: all constraints are inactive.

$$H_1 = p_x^T(f + gu) + p_w^T s + \frac{1}{2}L(x,w,u),$$
$$\bar{u}_1 = -(L_g L_f h)^{-1}(L_f^2 h + L_s^2 h + (L_g L_f h)^{-T} g^T p_x),$$
$$\lambda_1 = 0.$$

- Case 2 : h_1 is active, which is $a = a_{\max}$.

$$H_2 = H_1 + \lambda_2(l(f + gu) - a_{\max}),$$
$$\bar{u}_2 = \bar{u}_1 - \lambda_2(L_g L_f h)^{-1}(L_g L_f h)^{-T} g^T l^T,$$
$$\lambda_2 = \frac{-a_{\max} + l(f + g\bar{u}_1)}{lg(L_g L_f h)^{-1}(L_g L_f h)^{-T} g^T l^T}.$$

- Case 3 : h_2 is active, which is $a = a_{\min}$.

$$H_3 = H_1 + \lambda_3(a_{\min} - l(f + gu)),$$
$$\bar{u}_3 = \bar{u}_1 + \lambda_3(L_g L_f h)^{-1}(L_g L_f h)^{-T} g^T l^T,$$
$$\lambda_3 = \frac{a_{\min} - l(f + g\bar{u}_1)}{lg(L_g L_f h)^{-1}(L_g L_f h)^{-T} g^T l^T}.$$

Next, we apply the center-stable manifold method using case-choosing algorithm for Hamilton's canonical equation and obtain a optimal feedback controller $u(x,w)$.

3.3 Numerical Example

We verify the effectiveness of the proposed method on a nonlinear spring system. The equation of motion of the system is given by

$$m\ddot{x}_m + kx_m + \epsilon x_m^3 = u \tag{5}$$

where k is the spring constant, ϵ is the nonlinear spring constant and m is the mass. For simplicity, all of these parameters have the value of 1. Then, Eq. (5) is rewritten as follows

$$\begin{cases} \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{\epsilon}{m}x_1^3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m}\end{bmatrix} u \\ e = x_1 - w, \end{cases} \tag{6}$$

where $x = [x_1 \ x_2]^T = [x_m \ \dot{x}_m]^T$. Here, the reference is step signal, is therefore the exosystem is

$$\dot{w} = 0.$$

In control design, acceleration constraints are $a_{\max} = 0.1[\mathrm{m/s^2}], a_{\min} = -0.1[\mathrm{m/s^2}]$. Applying the center-stable manifold method algorithm 10 times, the optimal feedback controller $u(x, w)$ is approximated by linear interpolation. The proposed method is compared with an PID controller using the gains $K_P = 10, K_I = 1, K_D = 15$. To satisfy acceleration constraints, the algorithm in Figure. 7 is applied to modify the PID controller.

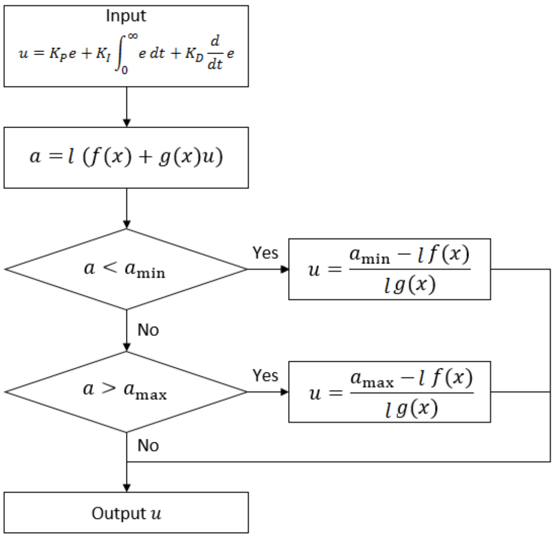

Figure 7 Input modification algorithm to satisfy acceleration constraints for PID controller.

The simulation results are shown in Figure. 8, 9 and 10. Here, the initial conditions are $x_1(0) = 0, x_2(0) = 0$ and $w(0) = 1$ which is the reference value of the state x_1. It is shown in Fig. 8, 9 and 10 that the response of proposed method satisfies the acceleration constraints and correctly follows the reference and the proposed method has no overshoot and better response to the reference than the PID controller.

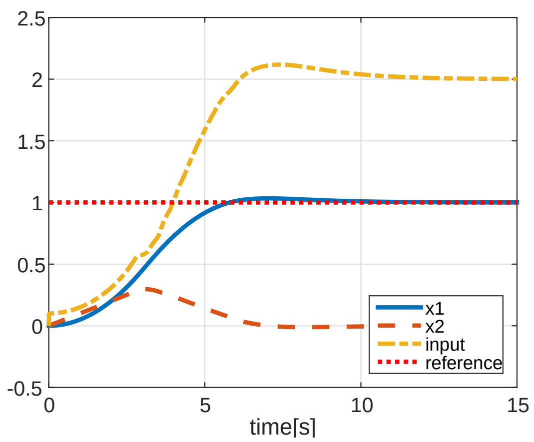

Figure 8 Responses of state and input of constrained nonlinear controller

4 Conclusion

In this paper, we proposed a nonlinear optimal controller and a nonlinear optimal servo controller designs for systems with acceleration constraints. The nonlinear controllers were designed via stable manifold and center-stable manifold methods including Lagrange multiplier to satisfy acceleration constraints. We verified the effectiveness of the proposed methods on a magnetic levitation system and a numerical example.

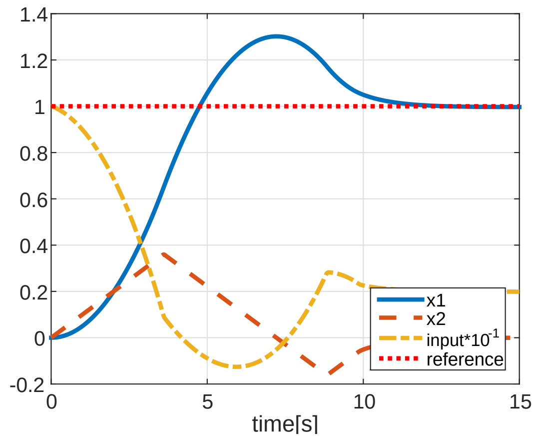

Figure 9 Responses of state and input of constrained PID controller

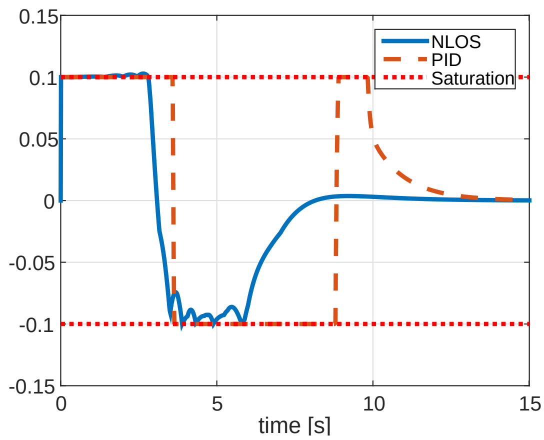

Figure 10 Response of acceleration of constrained nonlinear and PID controllers

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