H.J. Nussbaumer 'Fast Fourier Transform and Convolution Algorithms' ξ1)

The operations are executed in the order t_l, m_l, s_l, $\bar{X}_l$, with indices in natural order. For DFTs of lengths 5, 7, 9, 16, the operations can also be executed using the form shown in Sects. 5.6.4, 5, 7, 8 which embeds the various polynomial products.

The figures between parentheses indicate trivial multiplications by ± 1, $\pm j$

At the end of each of algorithm description for $N = 3, 5, 7, 9$, we give the number of operations for the corresponding algorithm in which the number of nontrivial multiplications is minimized and the output is scaled by a constant factor.

5.5.1 2-Point DFT

$\bar{X}_k = \sum_{n=0}^{N-1} x_n W^{nk}, \; k = 0, \cdots, N-1$

$W = e^{-2\pi i/N}, \; i = \sqrt{-1}$

2 multiplications (2), 2 additions

$m_0 = 1 \cdot (x_0 + x_1)$ $\qquad m_1 = 1 \cdot (x_0 - x_1)$

$\bar{X}_0 = m_0$

$\bar{X}_1 = m_1$

5.5.2 3-Point DFT

$u = 2\pi/3$, 3 multiplications (1), 6 additions

$t_1 = x_1 + x_2$

$m_0 = 1 \cdot (x_0 + t_1)$ $\qquad m_1 = (\cos u - 1) \cdot t_1$

$m_2 = \mathrm{j} \sin u \cdot (x_2 - x_1)$

$s_1 = m_0 + m_1$

$\bar{X}_0 = m_0$

$\bar{X}_1 = s_1 + m_2$

$\bar{X}_2 = s_1 - m_2$

Corresponding algorithm with scaled output:
3 multiplications (2), 8 additions, scaling factor: 2

5.5.3 4-Point DFT

4 multiplications (4), 8 additions

$t_1 = x_0 + x_2$ $\qquad t_2 = x_1 + x_3$

$m_0 = 1 \cdot (t_1 + t_2)$ $\qquad m_1 = 1 \cdot (t_1 - t_2)$

$m_2 = 1 \cdot (x_0 - x_2)$ $\qquad m_3 = \mathrm{j}\,(x_3 - x_1)$

$\bar{X}_0 = m_0$

$\bar{X}_1 = m_2 + m_3$

$\bar{X}_2 = m_1$

$\bar{X}_3 = m_2 - m_3$

5.5.4 5-Point DFT

$u = 2\pi/5$, 6 multiplications (1), 17 additions

$t_1 = x_1 + x_4$ $t_2 = x_2 + x_3$ $t_3 = x_1 - x_4$ $t_4 = x_3 - x_2$

$t_5 = t_1 + t_2$

$m_0 = 1 \cdot (x_0 + t_5)$

$m_1 = [(\cos u + \cos 2u)/2 - 1]t_5$

$m_2 = [(\cos u - \cos 2u)/2](t_1 - t_2)$

Polynomial product modulo $(z^2 + 1)$

$m_3 = -\mathrm{j}(\sin u)(t_3 + t_4)$

$m_4 = -\mathrm{j}(\sin u + \sin 2u) \cdot t_4$

$m_5 = \mathrm{j}(\sin u - \sin 2u)t_3$

$s_3 = m_3 - m_4$

$s_5 = m_3 + m_5$

$s_1 = m_0 + m_1$ $s_2 = s_1 + m_2$

$s_4 = s_1 - m_2$

$\bar{X}_0 = m_0$ $\bar{X}_3 = s_4 - s_5$

$\bar{X}_1 = s_2 + s_3$ $\bar{X}_4 = s_2 - s_3$

$\bar{X}_2 = s_4 + s_5$

Corresponding algorithm with scaled output:

6 multiplications (2), 21 additions, scaling factor: 4

5.5.5 7-Point DFT

$u = 2\pi/7$, 9 multiplications (1), 36 additions

$t_1 = x_1 + x_6$ $t_2 = x_2 + x_5$ $t_3 = x_3 + x_4$

$t_4 = t_1 + t_2 + t_3$ $t_5 = x_1 - x_6$ $t_6 = x_2 - x_5$

$t_7 = x_4 - x_3$ $t_8 = t_1 - t_3$ $t_9 = t_3 - t_2$

$t_{10} = t_5 + t_6 + t_7$ $t_{11} = t_7 - t_5$ $t_{12} = t_6 - t_7$

$m_0 = 1 \cdot (x_0 + t_4)$

$m_1 = [(\cos u + \cos 2u + \cos 3u)/3 - 1]\, t_4$

$t_{13} = -t_8 - t_9$

$m_2 = [(2\cos u - \cos 2u - \cos 3u)/3]\, t_8$

$m_3 = [(\cos u - 2\cos 2u + \cos 3u)/3]\, t_9$

$m_4 = [(\cos u + \cos 2u - 2\cos 3u)/3]\, t_{13}$

$s_0 = -m_2 - m_3$ Polynomial product

$s_1 = -m_2 - m_4$ modulo $(z^2 + z + 1)$

$m_5 = -j[(\sin u + \sin 2u - \sin 3u)/3]t_{10}$

$t_{14} = -t_{11} - t_{12}$

$m_6 = j[(2\sin u - \sin 2u + \sin 3u)/3]t_{11}$

$m_7 = j[\sin u - 2\sin 2u - \sin 3u)/3]t_{12}$

$m_8 = j[(\sin u + \sin 2u + 2\sin 3u)/3]t_{14}$

$s_2 = -m_6 - m_7$ Polynomial product

$s_3 = m_6 + m_8$ modulo $(z^2 - z + 1)$

$s_4 = m_0 + m_1$ $s_5 = s_4 - s_0$ $s_6 = s_4 + s_1$

$s_7 = s_4 + s_0 - s_1$ $s_8 = m_5 - s_2$ $s_9 = m_5 - s_3$

$s_{10} = m_5 + s_2 + s_3$

$\bar{X}_0 = m_0$ $\bar{X}_1 = s_5 + s_8$ $\bar{X}_2 = s_6 + s_9$ $\bar{X}_3 = s_7 - s_{10}$

$\bar{X}_4 = s_7 + s_{10}$ $\bar{X}_5 = s_6 - s_9$ $\bar{X}_6 = s_5 - s_8$

Corresponding algorithm with scaled output:

9 multiplications (2), 43 additions, scaling factor: 6

5.5.6 8-Point DFT

$u = 2\pi/8$, 8 multiplications (6), 26 additions

$t_1 = x_0 + x_4$ $t_2 = x_2 + x_6$ $t_3 = x_1 + x_5$

$t_4 = x_1 - x_5$ $t_5 = x_3 + x_7$ $t_6 = x_3 - x_7$

$t_7 = t_1 + t_2$ $t_8 = t_3 + t_5$

$m_0 = 1\cdot(t_7 + t_8)$ $m_1 = 1\cdot(t_7 - t_8)$

$m_2 = 1\cdot(t_1 - t_2)$ $m_3 = 1\cdot(x_0 - x_4)$

$m_4 = \cos u\cdot(t_4 - t_6)$ $m_5 = j(t_5 - t_3)$

$m_6 = j(x_6 - x_2)$ $\quad m_7 = -j \sin u \cdot (t_4 + t_6)$

$s_1 = m_3 + m_4$ $\quad s_2 = m_3 - m_4$ $\quad s_3 = m_6 + m_7$

$s_4 = m_6 - m_7$

$\bar{X}_0 = m_0$ $\quad \bar{X}_1 = s_1 + s_3$ $\quad \bar{X}_2 = m_2 + m_5$

$\bar{X}_3 = s_2 - s_4$ $\quad \bar{X}_4 = m_1$ $\quad \bar{X}_5 = s_2 + s_4$

$\bar{X}_6 = m_2 - m_5$ $\quad \bar{X}_7 = s_1 - s_3$

5.5.7 9-Point DFT

$u = 2\pi/9$, 11 multiplications (1), 44 additions

$t_1 = x_1 + x_8$ $\quad t_2 = x_2 + x_7$ $\quad t_3 = x_3 + x_6$

$t_4 = x_4 + x_5$ $\quad t_5 = t_1 + t_2 + t_4$ $\quad t_6 = x_1 - x_8$

$t_7 = x_7 - x_2$ $\quad t_8 = x_3 - x_6$ $\quad t_9 = x_4 - x_5$

$t_{10} = t_6 + t_7 + t_9$ $\quad t_{11} = t_1 - t_2$ $\quad t_{12} = t_2 - t_4$

$t_{13} = t_7 - t_6$ $\quad t_{14} = t_7 - t_9$

$m_0 = 1 \cdot (x_0 + t_3 + t_5)$

$m_1 = (3/2)t_3$

$m_2 = -t_5/2$

$t_{15} = -t_{12} - t_{11}$

$m_3 = [(2 \cos u - \cos 2u - \cos 4u)/3]t_{11}$

$m_4 = [(\cos u + \cos 2u - 2 \cos 4u)/3]t_{12}$

$m_5 = [(\cos u - 2 \cos 2u + \cos 4u)/3]t_{15}$

$s_0 = -m_3 - m_4$ $\quad$ Polynomial product

$s_1 = m_5 - m_4$ $\quad$ modulo $(z^2 + z + 1)$

$m_6 = -j \sin 3u \cdot t_{10}$

$m_7 = -j \sin 3u \cdot t_8$

$t_{16} = -t_{13} + t_{14}$

$m_8 = j \sin u \cdot t_{13}$

$m_9 = j \sin 4u \cdot t_{14}$

$m_{10} = j \sin 2u \cdot t_{16}$

$s_2 = -m_8 - m_9$ $\quad$ Polynomial product

$s_3 = m_9 - m_{10}$ $\quad$ modulo $(z^2 - z + 1)$

$s_4 = m_0 + m_2 + m_2$ $\quad s_5 = s_4 - m_1$

$s_6 = s_4 + m_2$ $\quad s_7 = s_5 - s_0$

$s_8 = s_1 + s_5$ $\quad s_9 = s_0 - s_1 + s_5$

$s_{10} = m_7 - s_2$ $\quad s_{11} = m_7 - s_3$

$s_{12} = m_7 + s_2 + s_3$

$\bar{X}_0 = m_0$ $\quad \bar{X}_1 = s_7 + s_{10}$ $\quad \bar{X}_2 = s_8 - s_{11}$

$\bar{X}_3 = s_6 + m_6$ $\quad \bar{X}_4 = s_9 + s_{12}$ $\quad \bar{X}_5 = s_9 - s_{12}$

$\bar{X}_6 = s_6 - m_6$ $\quad \bar{X}_7 = s_8 + s_{11}$ $\quad \bar{X}_8 = s_7 - s_{10}$

Corresponding algorith with scaled output:
11 multiplications (3), 45 additions, scaling factor: 2

5.5.8 16-Point DFT

$u = 2\pi/16$, 18 multiplications (8), 74 additions

$t_1 = x_0 + x_8$ $\quad t_2 = x_4 + x_{12}$ $\quad t_3 = x_2 + x_{10}$

$t_4 = x_2 - x_{10}$ $\quad t_5 = x_6 + x_{14}$ $\quad t_6 = x_6 - x_{14}$

$t_7 = x_1 + x_9$ $\quad t_8 = x_1 - x_9$ $\quad t_9 = x_3 + x_{11}$

$t_{10} = x_3 - x_{11}$ $\quad t_{11} = x_5 + x_{13}$ $\quad t_{12} = x_5 - x_{13}$

$t_{13} = x_7 + x_{15}$ $\quad t_{14} = x_7 - x_{15}$ $\quad t_{15} = t_1 + t_2$

$t_{16} = t_3 + t_5$ $\quad t_{17} = t_{15} + t_{16}$ $\quad t_{18} = t_7 + t_{11}$

$t_{19} = t_7 - t_{11}$ $\quad t_{20} = t_9 + t_{13}$ $\quad t_{21} = t_9 - t_{13}$

$t_{22} = t_{18} + t_{20}$ $\quad t_{23} = t_8 + t_{14}$ $\quad t_{24} = t_8 - t_{14}$

$t_{25} = t_{10} + t_{12}$ $\quad t_{26} = t_{12} - t_{10}$

$m_0 = 1 \cdot (t_{17} + t_{22})$ $\quad m_1 = 1 \cdot (t_{17} - t_{22})$

$m_2 = 1 \cdot (t_{15} - t_{16})$ $\quad m_3 = 1 \cdot (t_1 - t_2)$

$m_4 = 1 \cdot (x_0 - x_8)$ $\quad m_5 = \cos 2u \cdot (t_{19} - t_{21})$

$m_6 = \cos 2u \cdot (t_4 - t_6)$

$m_7 = \cos 3u \cdot (t_{24} + t_{26})$

$m_8 = (\cos u + \cos 3u) \cdot t_{24}$

$m_9 = (\cos 3u - \cos u) \cdot t_{26}$ $\quad$ Polynomial product

$s_7 = m_8 - m_7$ $\quad s_8 = m_9 - m_7$ $\quad$ modulo $(z^2 + 1)$

$m_{10} = \mathrm{j} \cdot (t_{20} - t_{18})$ $\quad m_{11} = \mathrm{j} \cdot (t_5 - t_3)$

$m_{12} = \mathrm{j} \cdot (x_{12} - x_4)$ $\quad m_{13} = -\mathrm{j} \sin 2u \cdot (t_{19} + t_{21})$

$m_{14} = -\text{j} \sin 2u \cdot (t_4 + t_6)$

$m_{15} = -\text{j} \sin 3u \cdot (t_{23} + t_{25})$

$m_{16} = \text{j} (\sin 3u - \sin u) \cdot t_{23}$

$m_{17} = -\text{j} (\sin u + \sin 3u) \cdot t_{25}$

$s_{15} = m_{15} + m_{16}$ $\quad$ $s_{16} = m_{15} - m_{17}$ $\quad$ Polynomial product modulo$(z^2 + 1)$

$s_1 = m_3 + m_5$	$s_2 = m_3 - m_5$	$s_3 = m_{11} + m_{13}$
$s_4 = m_{13} - m_{11}$	$s_5 = m_4 + m_6$	$s_6 = m_4 - m_6$
$s_9 = s_5 + s_7$	$s_{10} = s_5 - s_7$	$s_{11} = s_6 + s_8$
$s_{12} = s_6 - s_8$		
$s_{13} = m_{12} + m_{14}$	$s_{14} = m_{12} - m_{14}$	$s_{17} = s_{13} + s_{15}$
$s_{18} = s_{13} - s_{15}$	$s_{19} = s_{14} + s_{16}$	$s_{20} = s_{14} - s_{16}$
$\bar{X}_0 = m_0$	$\bar{X}_1 = s_9 + s_{17}$	$\bar{X}_2 = s_1 + s_3$
$\bar{X}_3 = s_{12} - s_{20}$	$\bar{X}_4 = m_2 + m_{10}$	$\bar{X}_5 = s_{11} + s_{19}$
$\bar{X}_6 = s_2 + s_4$	$\bar{X}_7 = s_{10} - s_{18}$	$\bar{X}_8 = m_1$
$\bar{X}_9 = s_{10} + s_{18}$	$\bar{X}_{10} = s_2 - s_4$	$\bar{X}_{11} = s_{11} - s_{19}$
$\bar{X}_{12} = m_2 - m_{10}$	$\bar{X}_{13} = s_{12} + s_{20}$	$\bar{X}_{14} = s_1 - s_3$
$\bar{X}_{15} = s_9 - s_{17}$		