第２章　Cooley-Tukeyのアルゴリズム

2.1　基本的アイデア

◎n項複素離散型フーリエ変換（DFT_n）

$$\boldsymbol{c}=\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}=\left(\omega_n{}^{kl}\right)\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}=W_n\boldsymbol{f},\ \omega_n=\exp\frac{-2\pi\mathrm{i}}{n},\ \mathrm{i}=\sqrt{-1} \tag{1.1}$$

要素ごとに書き下すと，

$$c_k=\sum_{l=0}^{n-1}\omega_n{}^{kl}f_l\quad(0\le k<n) \tag{1.2}$$

◎項数の因数分解$n=p_1p_2$

$k=k_2p_1+k_1,\ l=l_1p_2+l_2,\ 0\le k_1,l_1<p_1,\ 0\le k_2,l_2<p_2$とすると，

$$\omega_n{}^{kl}=\omega_n{}^{(k_2p_1+k_1)(l_1p_2+l_2)}=\omega_n{}^{k_2l_1n}\omega_n{}^{k_1l_1p_2}\omega_n{}^{k_2l_2p_1}\omega_n{}^{l_2k_1}=\omega_{p_1}{}^{k_1l_1}\omega_{p_2}{}^{k_2l_2}\omega_n{}^{l_2k_1}$$

ゆえに，

$$\begin{aligned} c_{k_2p_1+k_1}&=\sum_{l_1=0}^{p_1-1}\sum_{l_2=0}^{p_2-1}\omega_n{}^{(k_2p_1+k_1)(l_1p_2+l_2)}f_{l_1p_2+l_2} \\ &=\sum_{l_2=0}^{p_2-1}\omega_{n_2}{}^{k_2l_2}\left\{\omega_n{}^{l_2k_1}\sum_{l_1=0}^{p_1-1}\omega_{p_1}{}^{k_1l_1}f_{l_1p_2+l_2}\right\} \end{aligned} \tag{1.3}$$

これより次のアルゴリズムが導かれる．

◎基本アルゴリズム

(1) 内DFT（$p_2\times\mathrm{DFT}_{p_1}$）：$0\le l_2<p_2$で，

$$\tilde{f}_{k_1p_2+l_2}=\sum_{l_1=0}^{n_1-1}\omega_{p_1}{}^{k_1l_1}f_{l_1p_2+l_2}\quad(0\le k_1<p_1) \tag{1.4}$$

(2) 回転:$0\le l_2<p_2,\ 0\le k_1<p_1$で，

$$\tilde{c}_{k_1p_2+l_2}=\omega_n{}^{l_2k_1}\tilde{f}_{k_1p_2+l_2} \tag{1.5}$$

(3) 外DFT（$p_1\times\mathrm{DFT}_{p_2}$）：$0\le k_1<p_1$で，

$$c_{k_2p_1+k_1}=\sum_{l_2=0}^{p_2-1}\omega_{p_2}{}^{k_2l_2}\tilde{c}_{k_1p_2+l_2}\quad(0\le k_2<p_2) \tag{1.6}$$

◎計算量

乗算数のみ比べる．式(1.2)の（複素）乗算数n^2．基本アルゴリズムおける乗算数は，およそ，内DFT $p_2\times p_1{}^2$,回転p_1p_2，外DFT $p_1\times p_2{}^2$,計

$$p_1 {p_2}^2 + p_1 p_2 + {p_1}^2 p_2 = n(p_1 + p_2 + 1).$$

基本アルゴリズムは p_1, p_2 が大きいとき有利． p_1, p_2 がさらに因数分解できるなら，基本アルゴリズムを再帰的に用いて， $\mathrm{DFT}_{p_1}, \mathrm{DFT}_{p_2}$ を高速化できる．

2.2　計算量

複素乗算1回は実乗算4回と実加減算2回である．複素加算1回は実加算2回である．式(1.5)で， $l_2 = 0$ または $k_1 = 0$ のとき， ${\omega_n}^{l_2 k_1} = 1$ ゆえ演算不要．ゆえに，

規約：回転に要する複素乗算数を $(p_1 - 1)(p_2 - 1)$ とみなす．　(2.1)

［定理2.1］ $n = p_1 p_2 \cdots p_m$ とし， DFT_{p_l} $(1 \le l \le m)$ に要する実乗算数を μ_l，実加減算数を α_l とする．このとき，規約(2.1)のもとで， DFT_n の計算量は実乗算回数

$$\mu(n) = n \sum_{l=1}^{m} \frac{\mu_l + 4(p_l - 1)}{p_l} - 4(n-1)$$

実加減算回数

$$\alpha(n) = n \sum_{l=1}^{m} \frac{\alpha_l + 2(p_l - 1)}{p_l} - 2(n-1)$$

である．

（証明） m に関する帰納法． $m = 1$ のとき， $\mu(p_1) = \mu_1, \alpha(p_1) = \alpha_1$ ゆえ成立． $m > 1$ で， $n' = p_1 \cdots p_{m-1}$ に関し実乗算回数が

$$\mu(n') = n' \sum_{l=1}^{m-1} \frac{\mu_l + 4(p_l - 1)}{p_l} - 4(n'-1)$$

と仮定する． $n = n' p_m$ として基本アルゴリズムを用いると，内DFTで乗算数

$$n' \mu_m = n \frac{\mu_m}{p_m},$$

回転では規約(2.1)により乗算数

$$4(n'-1)(p_m - 1) = n \frac{4(p_m - 1)}{p_m} - 4(p_m - 1),$$

外DFTで乗算数

$$p_m \mu(n') = p_m \left\{ n' \sum_{l=1}^{m-1} \frac{\mu_l + 4(p_l - 1)}{p_l} - 4n' + 4 \right\}$$
$$= n \sum_{l=1}^{m-1} \frac{\mu_l + 4p_l - 4}{p_l} - 4n + 4p_m$$

である．ゆえに，乗算総数

$$\mu(n)=\left(n\frac{\mu_m}{p_m}\right)+n\sum_{l=1}^{m-1}\frac{\mu_l+4(p_l-1)}{p_l}-4n+4p_m+\left(n\frac{4(p_m-1)}{p_m}-4p_m+4\right)$$
$$=n\sum_{l=1}^{m}\frac{\mu_l+4(p_l-1)}{p_l}-4(n-1)$$

加算数についても同様である．／／

2.3 $n=p^m$ 型

項数 p の小さい DFT_p $(p=2,3,4,5,7,8,9,16)$ については，計算量の小さいアルゴリズムが個別に工夫されている．それらが要する実乗算数 $\mu(p)$，実加減算数 $\alpha(p)$ は以下の通り．

p	2	3	4	5	7	8	9	16
$\mu(p)$	0	4	0	10	16	4	20	20
$\alpha(p)$	4	12	16	34	72	52	88	148

これらのみを因数とする n を項数とする DFT_n は効率的に計算できる．

［定理3.1］　$n=p^m$ のとき，

$$\mu(n)=M(p)n\log_2 n+O(n),\ M(p)=\frac{\mu(p)+4(p-1)}{p\log_2 p}, \quad (3.1)$$

$$\alpha(n)=A(p)n\log_2 n+O(n), A(p)=\frac{\alpha(p)+2(p-1)}{p\log_2 p}. \quad (3.2)$$

（証明）$m=\log_2 n/\log_2 p$ ゆえ，定理2.1より

$$\mu(n)=n\sum_{l=1}^{m}\frac{\mu(p)+4(p-1)}{p}-4(n-1)=nm\frac{\mu(p)+4(p-1)}{p}+O(n)$$
$$=\left(\frac{\mu(p)+4(p-1)}{p\log_2 p}\right)n\log_2 n+O(n)=M(p)n\log_2 n+O(n).$$

$\alpha(n)$ についても同様．／／

$M(p),A(p)$ を効率係数と呼ぶ．各 p に付いての効率係数を示す．

p	2	3	4	5	7	8	9	16
$M(p)$	2.00	2.52	1.50	2.24	2.04	1.33	1.82	1.25
$A(p)$	3.00	3.36	2.75	3.62	4.27	2.79	3.65	2.78

H.J. Nussbaumer 'Fast Fourier Transform and Convolution Algorithms' (?1)

The operations are executed in the order t_l, m_l, s_l, $\bar{X}_l$, with indices in natural order. For DFTs of lengths 5, 7, 9, 16, the operations can also be executed using the form shown in Sects. 5.6.4, 5, 7, 8 which embeds the various polynomial products.

The figures between parentheses indicate trivial multiplications by ± 1, $\pm j$

At the end of each of algorithm description for $N = 3, 5, 7, 9$, we give the number of operations for the corresponding algorithm in which the number of nontrivial multiplications is minimized and the output is scaled by a constant factor.

5.5.1 2-Point DFT

$\bar{X}_k = \sum_{n=0}^{N-1} x_n W^{nk}, \quad k = 0, \cdots, N-1$

$W = e^{-2\pi i/N}, \quad i = \sqrt{-1}$

2 multiplications (2), 2 additions

$m_0 = 1 \cdot (x_0 + x_1)$ $\qquad m_1 = 1 \cdot (x_0 - x_1)$

$\bar{X}_0 = m_0$

$\bar{X}_1 = m_1$

5.5.2 3-Point DFT

$u = 2\pi/3$, 3 multiplications (1), 6 additions

$t_1 = x_1 + x_2$

$m_0 = 1 \cdot (x_0 + t_1)$ $\qquad m_1 = (\cos u - 1) \cdot t_1$

$m_2 = \mathrm{j} \sin u \cdot (x_2 - x_1)$

$s_1 = m_0 + m_1$

$\bar{X}_0 = m_0$

$\bar{X}_1 = s_1 + m_2$

$\bar{X}_2 = s_1 - m_2$

Corresponding algorithm with scaled output:
3 multiplications (2), 8 additions, scaling factor: 2

5.5.3 4-Point DFT

4 multiplications (4), 8 additions

$t_1 = x_0 + x_2$ $\qquad t_2 = x_1 + x_3$

$m_0 = 1 \cdot (t_1 + t_2)$ $\qquad m_1 = 1 \cdot (t_1 - t_2)$

$m_2 = 1 \cdot (x_0 - x_2)$ $\qquad m_3 = \mathrm{j}\,(x_3 - x_1)$

$\bar{X}_0 = m_0$

$\bar{X}_1 = m_2 + m_3$

$\bar{X}_2 = m_1$

$\bar{X}_3 = m_2 - m_3$

5.5.4 5-Point DFT

$u = 2\pi/5$, 6 multiplications (1), 17 additions

$t_1 = x_1 + x_4$ $t_2 = x_2 + x_3$ $t_3 = x_1 - x_4$ $t_4 = x_3 - x_2$

$t_5 = t_1 + t_2$

$m_0 = 1 \cdot (x_0 + t_5)$

$m_1 = [(\cos u + \cos 2u)/2 - 1]t_5$

$m_2 = [(\cos u - \cos 2u)/2](t_1 - t_2)$

Polynomial product modulo $(z^2 + 1)$

$m_3 = -j(\sin u)(t_3 + t_4)$

$m_4 = -j(\sin u + \sin 2u) \cdot t_4$

$m_5 = j(\sin u - \sin 2u)t_3$

$s_3 = m_3 - m_4$

$s_5 = m_3 + m_5$

$s_1 = m_0 + m_1$ $s_2 = s_1 + m_2$

$s_4 = s_1 - m_2$

$\bar{X}_0 = m_0$ $\bar{X}_3 = s_4 - s_5$

$\bar{X}_1 = s_2 + s_3$ $\bar{X}_4 = s_2 - s_3$

$\bar{X}_2 = s_4 + s_5$

Corresponding algorithm with scaled output:

6 multiplications (2), 21 additions, scaling factor: 4

5.5.5 7-Point DFT

$u = 2\pi/7$, 9 multiplications (1), 36 additions

$t_1 = x_1 + x_6$ $t_2 = x_2 + x_5$ $t_3 = x_3 + x_4$

$t_4 = t_1 + t_2 + t_3$ $t_5 = x_1 - x_6$ $t_6 = x_2 - x_5$

$t_7 = x_4 - x_3$ $t_8 = t_1 - t_3$ $t_9 = t_3 - t_2$

$t_{10} = t_5 + t_6 + t_7$ $t_{11} = t_7 - t_5$ $t_{12} = t_6 - t_7$

$m_0 = 1 \cdot (x_0 + t_4)$

$m_1 = [(\cos u + \cos 2u + \cos 3u)/3 - 1]\, t_4$

$t_{13} = -t_8 - t_9$

$m_2 = [(2\cos u - \cos 2u - \cos 3u)/3]\, t_8$

$m_3 = [(\cos u - 2\cos 2u + \cos 3u)/3]\, t_9$

$m_4 = [(\cos u + \cos 2u - 2\cos 3u)/3]\, t_{13}$

$s_0 = -m_2 - m_3$ Polynomial product

$s_1 = -m_2 - m_4$ modulo $(z^2 + z + 1)$

$m_5 = -\mathrm{j}[(\sin u + \sin 2u - \sin 3u)/3]t_{10}$

$t_{14} = -t_{11} - t_{12}$

$m_6 = \mathrm{j}[(2 \sin u - \sin 2u + \sin 3u)/3]t_{11}$

$m_7 = \mathrm{j}[\sin u - 2 \sin 2u - \sin 3u)/3]t_{12}$

$m_8 = \mathrm{j}[(\sin u + \sin 2u + 2 \sin 3u)/3]t_{14}$

$s_2 = -m_6 - m_7$ Polynomial product

$s_3 = m_6 + m_8$ modulo $(z^2 - z + 1)$

$s_4 = m_0 + m_1$ $s_5 = s_4 - s_0$ $s_6 = s_4 + s_1$

$s_7 = s_4 + s_0 - s_1$ $s_8 = m_5 - s_2$ $s_9 = m_5 - s_3$

$s_{10} = m_5 + s_2 + s_3$

$\bar{X}_0 = m_0$ $\bar{X}_1 = s_5 + s_8$ $\bar{X}_2 = s_6 + s_9$ $\bar{X}_3 = s_7 - s_{10}$

$\bar{X}_4 = s_7 + s_{10}$ $\bar{X}_5 = s_6 - s_9$ $\bar{X}_6 = s_5 - s_8$

Corresponding algorithm with scaled output:

9 multiplications (2), 43 additions, scaling factor: 6

5.5.6 8-Point DFT

$u = 2\pi/8$, 8 multiplications (6), 26 additions

$t_1 = x_0 + x_4$ $t_2 = x_2 + x_6$ $t_3 = x_1 + x_5$

$t_4 = x_1 - x_5$ $t_5 = x_3 + x_7$ $t_6 = x_3 - x_7$

$t_7 = t_1 + t_2$ $t_8 = t_3 + t_5$

$m_0 = 1 \cdot (t_7 + t_8)$ $m_1 = 1 \cdot (t_7 - t_8)$

$m_2 = 1 \cdot (t_1 - t_2)$ $m_3 = 1 \cdot (x_0 - x_4)$

$m_4 = \cos u \cdot (t_4 - t_6)$ $m_5 = \mathrm{j}(t_5 - t_3)$

$m_6 = j(x_6 - x_2)$ $\quad m_7 = -j \sin u \cdot (t_4 + t_6)$

$s_1 = m_3 + m_4$ $\quad s_2 = m_3 - m_4$ $\quad s_3 = m_6 + m_7$

$s_4 = m_6 - m_7$

$\bar{X}_0 = m_0$ $\quad \bar{X}_1 = s_1 + s_3$ $\quad \bar{X}_2 = m_2 + m_5$

$\bar{X}_3 = s_2 - s_4$ $\quad \bar{X}_4 = m_1$ $\quad \bar{X}_5 = s_2 + s_4$

$\bar{X}_6 = m_2 - m_5$ $\quad \bar{X}_7 = s_1 - s_3$

5.5.7 9-Point DFT

$u = 2\pi/9$, 11 multiplications (1), 44 additions

$t_1 = x_1 + x_8$ $\quad t_2 = x_2 + x_7$ $\quad t_3 = x_3 + x_6$

$t_4 = x_4 + x_5$ $\quad t_5 = t_1 + t_2 + t_4$ $\quad t_6 = x_1 - x_8$

$t_7 = x_7 - x_2$ $\quad t_8 = x_3 - x_6$ $\quad t_9 = x_4 - x_5$

$t_{10} = t_6 + t_7 + t_9$ $\quad t_{11} = t_1 - t_2$ $\quad t_{12} = t_2 - t_4$

$t_{13} = t_7 - t_6$ $\quad t_{14} = t_7 - t_9$

$m_0 = 1 \cdot (x_0 + t_3 + t_5)$

$m_1 = (3/2)t_3$

$m_2 = -t_5/2$

$t_{15} = -t_{12} - t_{11}$

$m_3 = [(2 \cos u - \cos 2u - \cos 4u)/3]t_{11}$

$m_4 = [(\cos u + \cos 2u - 2 \cos 4u)/3]t_{12}$

$m_5 = [(\cos u - 2 \cos 2u + \cos 4u)/3]t_{15}$

$s_0 = -m_3 - m_4$ Polynomial product

$s_1 = m_5 - m_4$ modulo $(z^2 + z + 1)$

$m_6 = -j \sin 3u \cdot t_{10}$

$m_7 = -j \sin 3u \cdot t_8$

$t_{16} = -t_{13} + t_{14}$

$m_8 = j \sin u \cdot t_{13}$

$m_9 = j \sin 4u \cdot t_{14}$

$m_{10} = j \sin 2u \cdot t_{16}$

$s_2 = -m_8 - m_9$ Polynomial product

$s_3 = m_9 - m_{10}$ modulo $(z^2 - z + 1)$

$s_4 = m_0 + m_2 + m_2$ $\qquad s_5 = s_4 - m_1$

$s_6 = s_4 + m_2$ $\qquad s_7 = s_5 - s_0$

$s_8 = s_1 + s_5$ $\qquad s_9 = s_0 - s_1 + s_5$

$s_{10} = m_7 - s_2$ $\qquad s_{11} = m_7 - s_3$

$s_{12} = m_7 + s_2 + s_3$

$\bar{X}_0 = m_0$ $\qquad \bar{X}_1 = s_7 + s_{10}$ $\qquad \bar{X}_2 = s_8 - s_{11}$

$\bar{X}_3 = s_6 + m_6$ $\qquad \bar{X}_4 = s_9 + s_{12}$ $\qquad \bar{X}_5 = s_9 - s_{12}$

$\bar{X}_6 = s_6 - m_6$ $\qquad \bar{X}_7 = s_8 + s_{11}$ $\qquad \bar{X}_8 = s_7 - s_{10}$

Corresponding algorith with scaled output:
11 multiplications (3), 45 additions, scaling factor: 2

5.5.8 16-Point DFT

$u = 2\pi/16$, 18 multiplications (8), 74 additions

$t_1 = x_0 + x_8$ $\qquad t_2 = x_4 + x_{12}$ $\qquad t_3 = x_2 + x_{10}$

$t_4 = x_2 - x_{10}$ $\qquad t_5 = x_6 + x_{14}$ $\qquad t_6 = x_6 - x_{14}$

$t_7 = x_1 + x_9$ $\qquad t_8 = x_1 - x_9$ $\qquad t_9 = x_3 + x_{11}$

$t_{10} = x_3 - x_{11}$ $\qquad t_{11} = x_5 + x_{13}$ $\qquad t_{12} = x_5 - x_{13}$

$t_{13} = x_7 + x_{15}$ $\qquad t_{14} = x_7 - x_{15}$ $\qquad t_{15} = t_1 + t_2$

$t_{16} = t_3 + t_5$ $\qquad t_{17} = t_{15} + t_{16}$ $\qquad t_{18} = t_7 + t_{11}$

$t_{19} = t_7 - t_{11}$ $\qquad t_{20} = t_9 + t_{13}$ $\qquad t_{21} = t_9 - t_{13}$

$t_{22} = t_{18} + t_{20}$ $\qquad t_{23} = t_8 + t_{14}$ $\qquad t_{24} = t_8 - t_{14}$

$t_{25} = t_{10} + t_{12}$ $\qquad t_{26} = t_{12} - t_{10}$

$m_0 = 1\cdot(t_{17} + t_{22})$ $\qquad m_1 = 1\cdot(t_{17} - t_{22})$

$m_2 = 1\cdot(t_{15} - t_{16})$ $\qquad m_3 = 1\cdot(t_1 - t_2)$

$m_4 = 1\cdot(x_0 - x_8)$ $\qquad m_5 = \cos 2u\cdot(t_{19} - t_{21})$

$m_6 = \cos 2u\cdot(t_4 - t_6)$

$m_7 = \cos 3u\cdot(t_{24} + t_{26})$

$m_8 = (\cos u + \cos 3u)\cdot t_{24}$

$m_9 = (\cos 3u - \cos u)\cdot t_{26}$ $\qquad$ Polynomial product

$s_7 = m_8 - m_7$ $\qquad s_8 = m_9 - m_7$ $\qquad$ modulo $(z^2 + 1)$

$m_{10} = \mathrm{j}\cdot(t_{20} - t_{18})$ $\qquad m_{11} = \mathrm{j}\cdot(t_5 - t_3)$

$m_{12} = \mathrm{j}\cdot(x_{12} - x_4)$ $\qquad m_{13} = -\mathrm{j}\sin 2u\cdot(t_{19} + t_{21})$

$m_{14} = -\mathrm{j} \sin 2u \cdot (t_4 + t_6)$

$m_{15} = -\mathrm{j} \sin 3u \cdot (t_{23} + t_{25})$

$m_{16} = \mathrm{j}\,(\sin 3u - \sin u) \cdot t_{23}$

$m_{17} = -\mathrm{j}\,(\sin u + \sin 3u) \cdot t_{25}$

$s_{15} = m_{15} + m_{16}$	$s_{16} = m_{15} - m_{17}$	Polynomial product modulo$(z^2 + 1)$

$s_1 = m_3 + m_5$	$s_2 = m_3 - m_5$	$s_3 = m_{11} + m_{13}$
$s_4 = m_{13} - m_{11}$	$s_5 = m_4 + m_6$	$s_6 = m_4 - m_6$
$s_9 = s_5 + s_7$	$s_{10} = s_5 - s_7$	$s_{11} = s_6 + s_8$
$s_{12} = s_6 - s_8$		
$s_{13} = m_{12} + m_{14}$	$s_{14} = m_{12} - m_{14}$	$s_{17} = s_{13} + s_{15}$
$s_{18} = s_{13} - s_{15}$	$s_{19} = s_{14} + s_{16}$	$s_{20} = s_{14} - s_{16}$
$\bar{X}_0 = m_0$	$\bar{X}_1 = s_9 + s_{17}$	$\bar{X}_2 = s_1 + s_3$
$\bar{X}_3 = s_{12} - s_{20}$	$\bar{X}_4 = m_2 + m_{10}$	$\bar{X}_5 = s_{11} + s_{19}$
$\bar{X}_6 = s_2 + s_4$	$\bar{X}_7 = s_{10} - s_{18}$	$\bar{X}_8 = m_1$
$\bar{X}_9 = s_{10} + s_{18}$	$\bar{X}_{10} = s_2 - s_4$	$\bar{X}_{11} = s_{11} - s_{19}$
$\bar{X}_{12} = m_2 - m_{10}$	$\bar{X}_{13} = s_{12} + s_{20}$	$\bar{X}_{14} = s_1 - s_3$
$\bar{X}_{15} = s_9 - s_{17}$		