On exact models for the formulas with only one variable in **S4**

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In the classical propositional logic, we can know mutual relation between formulas by comparing their truth tables. Exact models are a kind of extension of truth tables into non-classical logics. So, exact models give an effective way to understand mutual relation between formulas like truth tables in the classical propositional logic. Here we construct exact models in **S4** for the sets of formulas with only one propositional variable p and the finite depth of $\Box$.

1 Introduction

We use lower case Latin letters $p, q, \cdots$ for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\bot$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation). By $\mathbf{S}(p)$, we mean the set of formulas constructed from p and $\bot$ by using $\wedge$, $\vee$, $\supset$ and $\Box$. We put $\mathbf{S}^n(p) = \{A \in \mathbf{S}(p) \mid d(A) \leq n\}$, where $d(A)$, the depth of $\Box$, of a formula $A \in \mathbf{S}(p)$ is defined as follows.

(1) $d(D) = 0$, for an atomic formula D,
(2) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
(3) $d(\Box B) = d(B) + 1$.

By **S4**, we mean the smallest set of formulas containing all the tautologies and the axioms
$K : \Box(A \supset B) \supset (\Box A \supset \Box B)$,
$T : \Box A \supset A$,
$4 : \Box A \supset \Box\Box A$,
and closed under modus ponens and necessitation. We say that a triple $\langle W, R, V\rangle$ is an **S4**-model if the following three hold:
(1) W is a non-empty set,
(2) R is a reflexive and transitive binary relation on W,
(3) V is a function from the set of propositional variables to $\mathcal{P}(W)(= \{W' \mid W' \subseteq W\})$.
Also we extend the domain of a function V to the set of all formulas in the usual way.

Definition 1.1. Let $\mathbf{S}$ be a set of formulas. We say that an **S4**-model $\langle W, R, V\rangle$ is exact if the following two hold:
(1) $\{V(B) \mid B \in \mathbf{S}\} = \mathcal{P}(W)$,
(2) $A \in \mathbf{S4}$ if and only if $V(A) = W$.

For example, a triple
$$\langle \{\{p, q\}, \{p\}, \{q\}, \emptyset\}, \emptyset, V\rangle$$
is an exact **S4**-model for $\mathbf{S}^0(p, q)$ if $V(A) = \{w \mid A \in w\}$ for $A \in \{p, q\}$, where $\mathbf{S}^0(p, q)$ is the set of non-modal formulas with only two variables p and q. Here we note that each world, a member of the first component of an **S4**-model, corresponds to a line in a truth table in classical logic. We can see the

correspondence below.

	p	q	a given formula
$\{p, q\} \rightarrow$	$true$	$true$	
$\{p\} \rightarrow$	$true$	$false$	
$\{q\} \rightarrow$	$false$	$true$	
$\emptyset \rightarrow$	$false$	$false$	

Also we can see that to check the truth value of a formula A on each line of a truth table is basically same as to check the validity of A at each world of an exact model.

Here we construct an exact **S4**-model for $\mathbf{S}^n(p)$. To construct it, we use the concrete representatives, given in [Sas05], of equivalent classes in $\mathbf{S}^n(p)/\equiv$, the quotient set modulo the provability of **S4**. It is known the structure $\langle \mathbf{S}^n(p)/\equiv, \leq\rangle$ is boolean and corresponds to exact models for $\mathbf{S}^n(p)$, where $[A] \leq [B]$ denotes that $B \supset A \in \mathbf{S4}$ (cf. Chagrov and Zakharyaschev [CZ97] and Hendriks [Hen96]). In the next section, we define representatives of the equivalent classes following [Sas05]. In section 3, we construct exact models. Section 4 is devoted to show a concrete exact models.

2 Construction of representatives

In this section, we define representatives of the equivalent classes following [Sas05]. To do so, we use a sequent.

We use Greek letters, Γ and Δ, possibly with suffixes, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. By a sequent, we mean the expression $(\Gamma \rightarrow \Delta)$. We often write $\Gamma \rightarrow \Delta$ instead of the expression with the parenthesis. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \rightarrow \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \rightarrow \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

We put

$$f(\Gamma \rightarrow \Delta) = \begin{cases} \bigwedge\Gamma \supset \bigvee\Delta & \text{if } \Gamma \neq \emptyset \\ \bigvee\Delta & \text{if } \Gamma = \emptyset, \end{cases} \qquad \mathbf{ant}(\Gamma \rightarrow \Delta) = \Gamma, \qquad \mathbf{suc}(\Gamma \rightarrow \Delta) = \Gamma$$

and for a set $\mathcal{S}$ of sequents,

$$f(\mathcal{S}) = \{f(X) \mid X \in \mathcal{S}\}.$$

Definition 2.1.
(1) $\mathbf{G}_0 = \{(p \rightarrow), (\rightarrow p)\}, \mathbf{G}_0^* = \emptyset$
(2) For $X \in \mathbf{G}_n$,
$\mathbf{G}^+(X) = \{\Box\Gamma, \mathbf{ant}(X) \rightarrow \mathbf{suc}(X), \Box\Delta \mid \Gamma \cup \Delta = f(\mathbf{G}_n), \Gamma \cap \Delta = \emptyset\}$,
$\mathbf{G}(X) = \{Y \in \mathbf{G}^+(X) \mid f(Y) \notin \mathbf{S4}\}$,
$\mathbf{G}^*(X) = \{\Gamma \rightarrow \Delta \in \mathbf{G}(X) \mid \Box\mathbf{G}_n \cap \Delta \text{ is minimal in } \langle\{\Box\mathbf{G}_n \cap \Lambda \mid \Pi \rightarrow \Lambda \in \mathbf{G}(X)\}, \subseteq\rangle\}$,
(3) $\mathbf{G}_{n+1} = \{Y \in \mathbf{G}(X) \mid X \in \mathbf{G}_n - \mathbf{G}_n^*\}$, $\mathbf{G}_{n+1}^* = \{Y \in \mathbf{G}^*(X) \mid X \in \mathbf{G}_n - \mathbf{G}_n^*\}$.

We put

$$\mathcal{G}_n = \mathbf{G}_n \cup \bigcup_{k=0}^{n-1} \mathbf{G}_k^*.$$

Lemma 2.2.
(1) $\mathbf{S}^n(p)/\equiv\; = \;\{[\bigwedge_{A \in \mathbf{S}} A] \mid \mathbf{S} \subseteq \mathcal{G}_n\}$.
(2) *For subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathcal{G}_n$,
(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge_{A \in \mathbf{S}_1} A] \leq [\bigwedge_{A \in \mathbf{S}_2} A]$,

(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge_{A\in\mathbf{S}_1} A] = [\bigwedge_{A\in\mathbf{S}_2} A]$.

By the above lemma, $\mathcal{G}_n$ is the set of representatives we want.

According to Definition 2.1, we have to use the provability in **S4** in order to construct concrete representatives. [Sas05], however, gave another conditions equivalent the provability in **S4** of the sequents. Using the conditions, we can construct concrete representatives, and as a result, we can give an example of exact models(see section 4).

3 Construction of exact models

Here we construct an exact model for $\mathbf{S}^n(p)$. Considering the correspondence between the structure $\langle \mathbf{S}^n(p)/\equiv, \leq\rangle$ and the exact model. We can define worlds of the exact model from representatives in section 2.

It is not hard to see that every sequent in $\mathcal{G}_n$ is either one of the forms

$$(\Box\Gamma \to p, \Box\Delta) \text{ or } (\Box\Gamma, p \to \Box\Delta).$$

Also a relation between these two forms will be important to express clusters consisting of two worlds in exact models. We put

$$\mathbf{rel}(\Box\Gamma \to p, \Box\Delta) = (\Box\Gamma, p \to \Box\Delta), \quad \mathbf{rel}(\Box\Gamma, p \to \Box\Delta) = (\Box\Gamma \to p, \Box\Delta).$$

We define a triple, which corresponds to a representative $X \in \mathbf{G}_n^*$ and will be a world in exact models. On the other hand, it seems very difficult or impossible to define a triple, which corresponds to a representative $X \in \mathbf{G}_n - \mathbf{G}_n^*$ in a similar way. Instead of such triple, however, we can use a triple, which corresponds to $Y \in \mathbf{G}^*(X)$. So, we have only to define a triple to $X \in \mathbf{G}_n^*$, and as a result, there are several exact models for $\mathbf{S}^n(p)$.

Definition 3.1.
(1) $w(p \to) = \langle\emptyset, \emptyset, \emptyset\rangle$, $w(\to p) = \langle\emptyset, \{p\}, \emptyset\rangle$
(2) Let X be in $\mathbf{G}_n$. For $Y \in \mathbf{G}^*(X)$,

$$w(Y) = \left\langle \bigcup_{\Box Z\in\mathbf{suc}(X)\cap\Box\mathbf{G}_n} (1st(Z) \cup \{w(Z)\}), \{p\}\cap\mathbf{suc}(Y), \{\Box\mathbf{rel}(X)\}\cap\mathbf{suc}(Y) \right\rangle.$$

We define a structure, which will be an exact model for $\mathbf{S}^n(p)$. Let **ENU** be an enumeration of sequents. For a set of $\mathcal{S}$ of sequents, the expression $X \in_1 \mathcal{S}$ means $X \in \mathcal{S}$ and, among $\mathcal{S}$, X is the first to occur in **ENU**.

Definition 3.2.

$$EM_n = \langle W_n, R_n, V_n\rangle,$$

where
(1) $W_n = \{w(X) \mid X \in \mathbf{G}_n^*\} \cup \{w(Y) \mid Y \in_1 \mathbf{G}^*(X), 3rd(Y) = \emptyset, X \in \mathbf{G}_n - \mathbf{G}_n^*\}$,
(2) $w_1 R_n w_2$ if and only if either one of the following holds:
 (2.1) $w_1 = w_2$,
 (2.2) $w_2 \in 1st(w_1)$,
 (2.3) $1st(w_1) = 1st(w_2)$, $2nd(w_1) \neq 2nd(w_2)$ and $3rd(w_1) \neq 3rd(w_2)$,
(3) $V_n(p) = \{w \mid p \in 2nd(w)\}$.

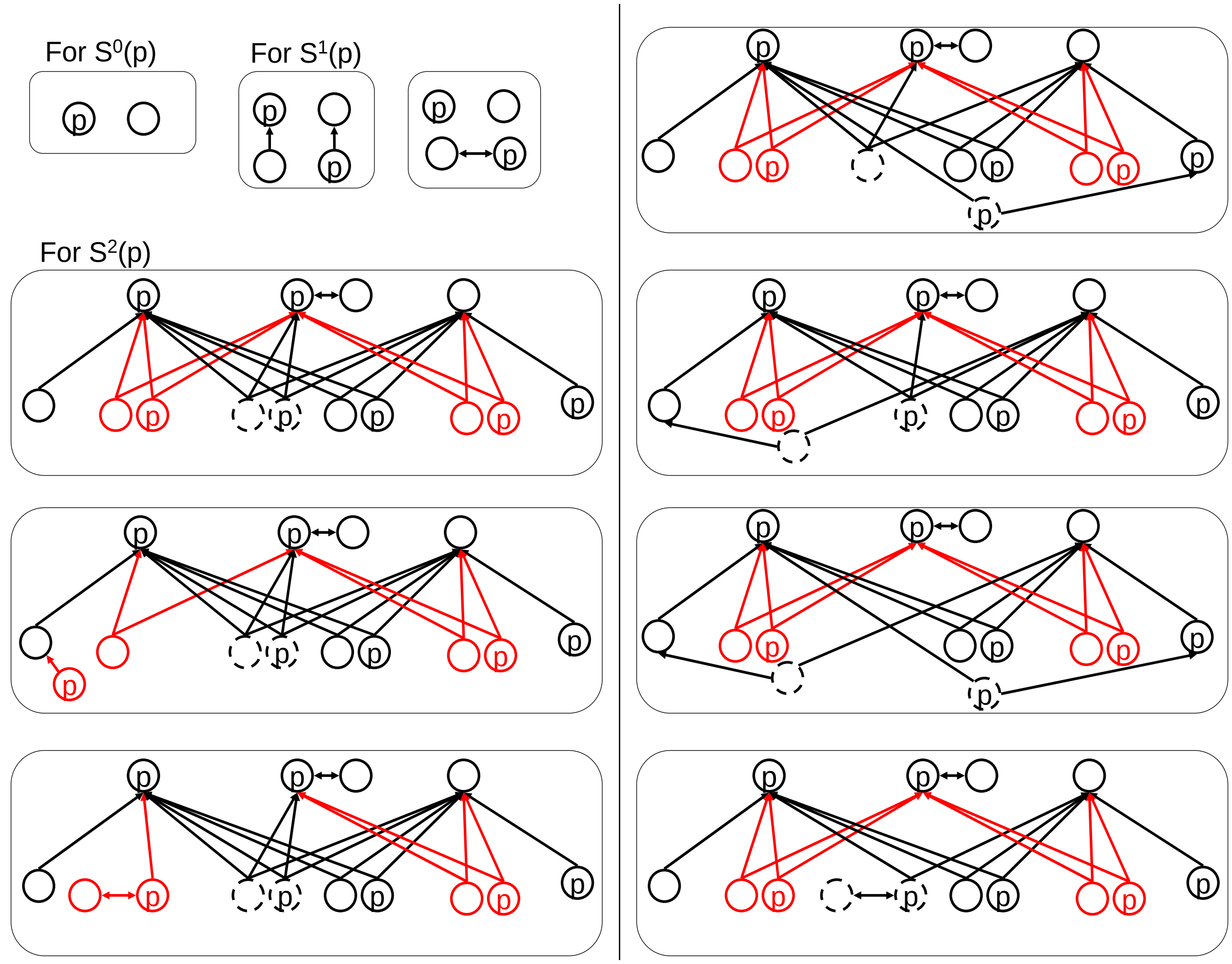
Theorem 3.3. *EM_n is an exact model for $\mathbf{S}^n(p)$.*

We can also construct other exact models for $\mathbf{S}^n(p)$ by replacing the condition (1) in Definition 3.2.

4 Examples

Here we list some concrete exact models for $\mathbf{S}^n(p)$ in Diagram. Let $\langle W, R, V\rangle$ be an exact model like EM_n. We represent worlds $w \in W$ by p if $p \in 2nd(w)$; by ○ if not, but we sometimes use circles in broken lines to clarify difference between two diffrent models. We draw an arrow from w_1 to w_2 if $w_1 \neq w_2$ and $w_1 R w_2$, but we do not draw from w_1 to w_3 if there are arrows from w_1 to w_2 and from w_2 to w_3.

There is basically only one exact model for $\mathbf{S}^0(p)$, two for $\mathbf{S}^1(p)$, and forty five for $\mathbf{S}^2(p)$. Below we only show diagrams of typical ones among them.



References

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