Construction of Reduced Counter-models for S4 *

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Abstract

We discuss an approach to reduce counter-models for **S4** based on sequent system. Our counter-models are constructed from failed proofs when a proof-search fails for a given sequent. Failed proof is like proof but not all of whose top sequents are initial sequents.

1 A sequent system for S4

Let small letters p, q etc. be propositional variables. Formulas are defined in the usual way with logical connectives $\wedge$, $\vee$, $\supset$, $\neg$ and $\Box$. Capital letters A, B etc. denote arbitrary formulas. Greek capital letters Γ, Δ etc. denote (finite, possibly empty) sets of formulas. Subscripts are used if necessary. The notation $\Box\Gamma$ denotes the set of formulas $\{\Box A_1, \cdots, \Box A_n\}$ when Γ is $\{A_1, \cdots, A_n\}$.

Now, we introduce a sequent system for **S4** proposed in [2]. Hereafter, we call it $\mathcal{S}$**S4**. A sequent of $\mathcal{S}$**S4** is an expression of the form $\Gamma \to \Delta\ \langle \Box\Sigma \mid \Box\Pi \rangle$. The pair $\langle \Box\Sigma \mid \Box\Pi \rangle$ of sets of $\Box$-formulas is called *history* which is for detecting loops in proof-search. Let $\mathcal{H}$ be a history. Initial sequents are of the form $\Gamma, p \to p, \Delta\ \mathcal{H}$. Rules of $\mathcal{S}$**S4** are given in Figure 1, though they are slightly modified so that we can easily incorporate a rule introduced later into $\mathcal{S}$**S4**.

$$\frac{\Gamma, A, B \to \Delta\ \mathcal{H}}{\Gamma, A \wedge B \to \Delta\ \mathcal{H}}\ (\wedge \to) \qquad \frac{\Gamma \to \Delta, A\ \mathcal{H} \quad \Gamma \to \Delta, B\ \mathcal{H}}{\Gamma \to \Delta, A \wedge B\ \mathcal{H}}\ (\to \wedge) \qquad \frac{\Gamma, A \to \Delta\ \mathcal{H} \quad \Gamma, B \to \Delta\ \mathcal{H}}{\Gamma, A \vee B \to \Delta\ \mathcal{H}}\ (\vee \to)$$

$$\frac{\Gamma \to \Delta, A, B\ \mathcal{H}}{\Gamma \to \Delta, A \vee B\ \mathcal{H}}\ (\to \vee) \qquad \frac{\Gamma \to \Delta, A\ \mathcal{H} \quad \Gamma, B \to \Delta\ \mathcal{H}}{\Gamma, A \supset B \to \Delta\ \mathcal{H}}\ (\supset\to) \qquad \frac{\Gamma, A \to \Delta, B\ \mathcal{H}}{\Gamma \to \Delta, A \supset B\ \mathcal{H}}\ (\to\supset)$$

$$\frac{\Gamma \to \Delta, A\ \mathcal{H}}{\Gamma, \neg A \to \Delta\ \mathcal{H}}\ (\neg \to) \qquad \frac{\Gamma, A \to \Delta\ \mathcal{H}}{\Gamma \to \Delta, \neg A\ \mathcal{H}}\ (\to \neg) \qquad \frac{A, \blacksquare A, \Gamma \to \Delta\ \mathcal{H}}{\Box A, \Gamma \to \Delta\ \mathcal{H}}\ (T)$$

$$\frac{\Box\Gamma \to A_1\ \langle \Box\Gamma \mid \Box\Theta, \Box\Sigma \rangle \quad \mid \quad \cdots \quad \mid \quad \Box\Gamma \to A_k\ \langle \Box\Gamma \mid \Box\Theta, \Box\Sigma \rangle}{\blacksquare\Gamma, p_1, \cdots, p_m \to \Box A_1, \cdots, \Box A_k, \Box\Delta, q_1, \cdots, q_n\ \langle \Box\Gamma \mid \Box\Sigma \rangle}\ (\Box)_s$$

$$\text{where } \Box\Theta = \{\Box A_i \mid \Box A_i \notin \Box\Sigma,\ 1 \le i \le k\},\quad \Box\Delta \subseteq \Box\Sigma$$

$$\frac{\Box\Gamma \to A_1\ \langle \Box\Gamma \mid \Box\Theta \rangle \quad \mid \quad \cdots \quad \mid \quad \Box\Gamma \to A_k\ \langle \Box\Gamma \mid \Box\Theta \rangle}{\blacksquare\Gamma, p_1, \cdots, p_m \to \Box A_1, \cdots, \Box A_k, q_1, \cdots, q_n\ \langle \Box\Pi \mid \Box\Sigma \rangle}\ (\Box)_t$$

$$\text{where } \Box\Theta = \{\Box A_1, \cdots, A_k\},\quad \Box\Pi \subset \Box\Gamma$$

Figure 1: Rules of $\mathcal{S}$**S4**

The $(\Box)_s$ and $(\Box)_t$ rules are called *transitional rules*, while the rules except them are called *static rules*. Applications of the rules are meant to be backward application, that is the upper sequent(s) are generated

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from the lower sequent. For a given sequent $\Gamma \to \Delta$, a proof-search with $\mathcal{S}\mathbf{S4}$ starts from $\Gamma \to \Delta \ \langle \emptyset \mid \emptyset \rangle$. In the (T) rule, the principal formula $\Box A$ is marked with $\blacksquare$ after its application so that (T) would never be re-applied to the same $\Box A$. Also, in the $(\Box)_s$ and $(\Box)_t$ rules, all possible choices of the upper sequents are shown with a vertical line $|$. This means existential branch, that is if one of the upper sequents is provable then so is lower sequent. Let $Sub(\Gamma)$ denote the set of the all subformulas of all formulas in Γ. Any formula occurring in all proofs of $\Gamma \to \Delta$ is in $(\Gamma \cup \Delta)^{\mathcal{S}\mathbf{S4}*}$, where $(\Gamma \cup \Delta)^{\mathcal{S}\mathbf{S4}*} = Sub(\Gamma \cup \Delta) \ \cup \ \{\blacksquare A \mid \Box A \in Sub(\Gamma \cup \Delta)\}$.

2 Construction of counter-models

Here, we will briefly introduce how to construct counter-models for **S4** from failed proofs for a given unprovable sequent in $\mathcal{S}\mathbf{S4}$. History of such a method can be seen in [1], though it is based on tableau system. The reader is supposed to be familiar with Kripke semantics of modal logics. For a sequent $\Gamma \to \Delta$, let $a(\Gamma \to \Delta) = \Gamma$ and $s(\Gamma \to \Delta) = \Delta$, respectively. Also, for a set $\Gamma = \{A_1, \cdots, A_n\}$ of formulas, let Γ_* and Γ^* denote $A_1 \wedge \cdots \wedge A_n$ and $A_1 \vee \cdots \vee A_n$, respectively. For simplicity's sake, we sometimes omit histories of sequents.

In constructing counter-models, we associate sequents with possible worlds. We denote possible worlds corresponding to sequents as w, u, v etc. In static rule, the upper sequent(s) and the lower sequent represent the same world in the same model, while in transitional rule, the upper sequent and the lower sequent represent different worlds in the same model. Suppose that failed proofs for a sequent $\Gamma_0 \to \Delta_0$ unprovable in $\mathcal{S}\mathbf{S4}$ is given. First of all, in order to generate possible worlds, we focus on its sub-proofs constructed only by applications of the static rules. Suppose that each of the end-sequents is $\Gamma_1 \to \Delta_1$. At the first application of a static rule to $\Gamma_1 \to \Delta_1$, since $\Gamma_0 \to \Delta_0$ is unprovable, at least one of the upper sequents, say $\Gamma_2 \to \Delta_2$, must be unprovable. Then put $w = \Gamma_1 \cup \Gamma_2 \to \Delta_1 \cup \Delta_2$. Note that w is still unprovable in $\mathcal{S}\mathbf{S4}$. Continue this step with w for the above applications of static rules. By the iteration, we can obtain sequents as possible worlds. We have to take all sub-proofs from all possible upper sequents of the $(\Box)_s$ and $(\Box)_t$ rules. We call this procedure *saturation*. Let W consist of all w generated by saturation. Then we can construct the following:

Definition 2.1 (S4–Model Graphs) *Let W be a nonempty set and R be a binary relation on W, that is $R \subseteq W \times W$. Then a* **S4-model graph** *for a sequent $\Gamma \to \Delta$ is a finite* **S4***-frame (W, R) such that W consists of $\mathcal{S}\mathbf{S4}$-saturated sequents w with $a(w), \ s(w) \subseteq (\Gamma \cup \Delta)^{\mathcal{S}\mathbf{S4}*}$ and*

1. *$\Gamma \subseteq a(w_0)$ and $\Delta \subseteq s(w_0)$ for some $w_0 \in W$,*
2. *if $\Box A \in s(w)$ then there exists some $w' \in W$ with wRw' and $A \in s(w')$,*
3. *if wRw' and $\Box A \in a(w)$ then $A \in a(w')$.*

For the readability's sake, we call $\mathcal{S}\mathbf{S4}$-model graphs just model graphs. Second of all, it reminds to be defined R. For $w, w' \in W$, where w' is an immediate successor of w, if the occurrences of $\Box$-formulas in $a(w)$ and that of $a(w')$ are the same, then put wRw' and $w'Rw$, otherwise wRw'. Finally, replace R with the reflexive and transitive closure of R and we can obtain a model graph (W, R).

Once we construct a model graph, we can obtain an **S4**-model immediately by giving a valuation such that for any $w \in W$, $w \models p \iff p \in a(w)$.

Lemma 2.2 (Satisfiability Lemma) *If (W, R) is a model graph for $\Gamma \to \Delta$ then there exists an* **S4***-model $(W, R, \models)$ such that $w \not\models \Gamma_* \supset \Delta^*$ for some $w \in W$.*

3 Reduction of counter-models

We can observe that some counter-models from model graph by the above construction include redundant possible worlds. In this section, we discuss an approach to obtain reduced model graphs.

Is it known that transitive frames (W, R) form cluster-trees. Formally, we first define an equivalence class relation $\sim$ on W. For every $w, w' \in W$, we write $w \sim w'$ if $w = w'$ or (wRw' and $w'Rw$). The equivalence classes with respect to $\sim$ are called *clusters*. The cluster containing a possible world w is denoted by $C(w)$. Then a transitive frame $(W/_\sim, R/_\sim)$ with respect to $\sim$ forms a cluster-tree, where $W/_\sim = \{C(w) \mid w \in W\}$, and $C(w)\ R/_\sim\ C(w') \iff wRw'$.

Here, we call $\Box$-formulas occurring in the left hand-side of sequents, *valid $\Box$-formulas*. Once a valid $\Box$-formula occurs in a sequent in proof of $\mathcal{S}\mathbf{S4}$, it also occurs in all of the above sequents. Taking it into account, when we observe a model graph constructed based on $\mathcal{S}\mathbf{S4}$, we can seen that the upper clusters from the root of the cluster-tree we visit, the more occurrences of valid $\Box$-formulas we have. In addition, the occurrences of valid $\Box$-formulas in all sequents(possible worlds) contained in each cluster are the same. From the observation, we have an approach to reduce the number of clusters in modal graphs. Suppose that a sequent $\Gamma \to \Delta$ unprovable in $\mathcal{S}\mathbf{S4}$ is given. We say that $\Box A_1, \cdots, \Box A_n \in Sub(\Gamma \cup \Delta)$ $(n \geq 0)$ are *left-valid* $\Box$-formulas of $\Gamma \to \Delta$, if $\Box A_1, \cdots, \Box A_n$ occur in some cluster of a model graph for $\Gamma \to \Delta$ and $\{\Box A_1, \cdots, \Box A_n\} \cup \Gamma \to \Delta$ is still unprovable in $\mathcal{S}\mathbf{S4}$. We always take the maximal set of left-valid $\Box$-formulas of $\Gamma \to \Delta$. Our approach is to put the maximal set of left-valid $\Box$-formulas of $\Gamma \to \Delta$ in the root of the cluster-tree. Then we can easily see the following theorem:

Theorem 3.1 *Let (W_0, R_0) be a model graph for $\Gamma \to \Delta$ and $\Box\Lambda$ be the maximal set of left-valid $\Box$-formulas of $\Gamma \to \Delta$. Also, let $(W_1, R_1, \models_1)$ be a counter-model for $((\Box\Lambda)_* \wedge \Gamma_*) \supset \Delta^*$, where (W_1, R_1) is a mode graph. Then,*

- *if, for $w \in W_1$, $w \not\models_1 ((\Box\Lambda)_* \wedge \Gamma_*) \supset \Delta^*$, then $w \not\models_1 \Gamma_* \supset \Delta^*$,*
- *$|W_1/_{\sim_1}| \leq |W_0/_{\sim_0}|$, where $\sim_1$ and $\sim_0$ are the equivalence class relations on W_1 and W_0, respectively.*

However, it is hard to find the maximal set of left-valid $\Box$-formulas of $\Gamma \to \Delta$ because not all valid $\Box$-formulas are always left-valid $\Box$-formulas. For the purpose of facilitating finding them, we incorporate the following static rule into $\mathcal{S}\mathbf{S4}$:

$$\frac{\Gamma \to \Delta, \blacksquare A, A\ \langle \Box\Pi \mid \Box A, \Box\Sigma\rangle \qquad A, \Gamma \to \Delta, \blacksquare A\ \langle \Box\Pi \mid \Box\Sigma\rangle}{\Gamma \to \Delta, \Box A\ \langle \Box\Pi \mid \Box\Sigma\rangle}\ (cut)_R$$

In the $(cut)_R$ rule, the principal formula $\Box A$ of the $(cut)_R$ is marked with $\blacksquare$ after its application so that $(cut)_R$ would never be re-applied to the same $\Box A$. We call the sequent system $\mathcal{S}\mathbf{S4}$ with $(cut)_R$, $\mathcal{S}\mathbf{S4}_{(cut)_R}$, where the $(\Box)_s$ and $(\Box)_t$ rules are meant to be applicable even if $\blacksquare$-formulas occur in the right-hand side of the lower sequents. Since applications the (T) and $(cut)_R$ rules can bring formulas within the scope of $\Box$ to the current possible world, we can find the maximal set of its left-valid $\Box$-formulas throughout the saturation step of $\mathcal{S}\mathbf{S4}_{(cut)_R}$.

In the following example, two counter-examples for $\Box\neg\Box p$ are shown, where the left one is constructed by $\mathcal{S}\mathbf{S4}$, while the right one is done by $\mathcal{S}\mathbf{S4}_{(cut)_R}$:

$$\dfrac{\dfrac{\dfrac{p, \blacksquare p \to \langle\emptyset|\Box\neg\Box p\rangle}{\Box p \to \langle\emptyset|\Box\neg\Box p\rangle}}{\to \neg\Box p\langle\emptyset|\Box\neg\Box p\rangle}}{\to \Box\neg\Box p\langle\emptyset|\emptyset\rangle}\ (\Box)_s \qquad \dfrac{\dfrac{\dfrac{p, \blacksquare p \to \blacksquare\neg\Box p\langle\emptyset|\Box\neg\Box p\rangle}{\Box p \to \blacksquare\neg\Box p\langle\emptyset|\Box\neg\Box p\rangle}}{\to \blacksquare\neg\Box p, \neg\Box p\langle\emptyset|\Box\neg\Box p\rangle} \qquad \vdots}{\to \Box\neg\Box p\langle\emptyset|\emptyset\rangle}\ (cut)_R$$

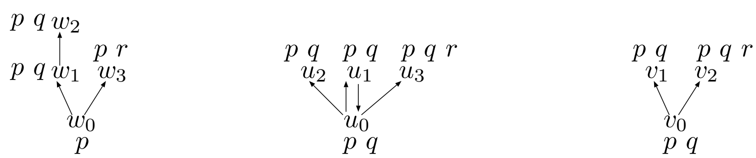
$$\begin{array}{c} p \\ w_1 \\ \uparrow \\ w_0 \end{array} \qquad\qquad\qquad \begin{array}{c} p \\ u_0 \end{array}$$

$$\begin{aligned} w_1 &= p, \blacksquare p, \Box p \to \neg\Box p \\ w_0 &= \to \Box\neg\Box p \end{aligned} \qquad\qquad u_0 = p, \blacksquare p, \Box p \to \neg\Box p, \blacksquare\neg\Box p, \Box\neg\Box p$$

Figure 2: Counter-models for $\Box\neg\Box p$

In the Figure 2, the possible worlds w_0 and w_1 are generated from the left failed proof, while u_0 is generated from the right one. The arrows denote the accessibility relations, though the reflexive ones are omitted. Each propositional variable shown at every possible world is true at the world, for example in the left counter-model, p is true at w_1, but it is not true at w_0. Neither w_0 nor u_0 makes $\Box\neg\Box p$ true. The maximal set of left-valid formulas of $\rightarrow \Box\neg\Box p$ is the set $\{\Box p\}$. In the right failed proof, we can see that $\Box p$ occurs at the left-hand side of the sequent $\Box p \rightarrow \blacksquare\neg\Box p\ \langle\emptyset \mid \Box\neg\Box p\rangle$ after the consecutive applications of the $(cut)_R$ and $(\rightarrow \neg)$ rules. In the left counter-model, there are 2 cluster and 2 possible worlds, while in the right one, 1 cluster and 1 possible world, therefore, the number of clusters is reduced. Since the right counter-example has no redundant possible world, we obtained a really reduced counter-model for $\Box\neg\Box p$ with $\mathcal{S}\mathbf{S4}_{(cut)_R}$.

However, we can not always obtain a really reduced counter-model even if the $(cut)_R$ is applied as shown in the following example:



$$\begin{aligned} a(w_3)^{\Box} &= \{\Box p,\ \Box r\} & a(u_3)^{\Box} &= \{\Box p,\ \Box q,\ \Box r\} \\ a(w_2)^{\Box} &= \{\Box p,\ \Box q,\ \Box\neg r\} & a(u_2)^{\Box} &= \{\Box p,\ \Box q,\ \Box\neg r\} \\ a(w_1)^{\Box} &= \{\Box p,\ \Box q\} & a(u_1)^{\Box} &= \{\Box p,\ \Box q\} \\ a(w_0)^{\Box} &= \{\Box p\} & a(u_0)^{\Box} &= \{\Box p,\ \Box q\} \end{aligned}$$

Figure 3: Counter-models for $\Box p \supset (\Box(\neg\Box q \vee \Box\neg\Box\neg r) \vee \Box\neg\Box r)$

In Figure 3, three counter-models for $\Box p \supset (\Box(\neg\Box q \vee \Box\neg\Box\neg r) \vee \Box\neg\Box r)$ (denoted by A below) are shown, where the left one is constructed by $\mathcal{S}\mathbf{S4}$, while the middle one is done by $\mathcal{S}\mathbf{S4}_{(cut)_R}$. The right one will be explained later. For briefness's sake, only valid $\Box$-formulas of every possible world are shown in the above, where for a possible world w, $a(w)^{\Box}$ denotes the set of the valid $\Box$-formulas of w. Neither w_0 nor u_0 makes A true. We can see that there are 4 clusters and 4 possible worlds in the left counter-model, while 3 cluster and 4 possible worlds in the middle one. Since the maximal set of left-valid $\Box$-formulas of $\rightarrow A$ is $\{\Box q\}$, $\Box q$ occurs at every world in the middle counter-model.

Although the number of clusters is reduced, the number of possible worlds still leaves unchanged. The middle counter-model includes a redundant world u_1. In fact, there is a counter-model as shown in the right in which there are 3 clusters and 3 possible worlds, and $v_0 \not\models A$. Here, we note that the right counter-model is not a model graph, that is every possible world does not correspond to any sequent. In constructing the middle counter-model, or rather model graph by $\mathcal{S}\mathbf{S4}_{(cut)_R}$, the world u_1 has to be generated so that the second condition of model graph can be satisfied. Therefore, the right counter-model can not be constructed from model graph by $\mathcal{S}\mathbf{S4}_{(cut)_R}$. From this case, we can see that the structure of model graph can be an obstacle to obtain a reduced counter-model.

References

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