Problems on axiomatization of intermediate propositional logics

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1 First problem (see [1] for detail)

Definitions

- Formulas (denoted by $A, B, X, Y, \ldots$) are constructed from propositional variables (denoted by $a, b, x, y, \ldots$) and $\to$, $\wedge$, $\vee$, $\bot$.
- $\mathrm{Var}(A)$ = the set of propositional variables appearing in A.
- $A \preceq B \iff \exists\theta(A^\theta = B)$. ($\theta$ is a substitution of formulas for propositional variables.)
- A is a *nonminimal tautology* $\iff \exists A_0$:tautology ($A_0 \preceq A$ and $A \not\preceq A_0$).
- **INT** = intuitionistic logic.
- **INT**$[A]$ = **INT** + axiom schema A.

Komori's first problem in [3] (modified version)

Prove or disprove:
If **INT**$[A]$ is a proper intermediate logic (i.e., strictly stronger than intuitionistic logic, and strictly weaker than classical logic), then A is a nonminimal tautology.

Example
$A = (a{\to}b)\vee(b{\to}a)$, $A_0 = (a{\to}b)\vee(c{\to}a)$.

Sequent calculus $\mathbf{LK_m}$ ($\Gamma, \Delta, \ldots$ are multisets of formulas)

Initial sequents $\quad a \Rightarrow a \qquad \bot \Rightarrow$

Inference rules

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Sigma}{A{\to}B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ ({\to}\mathrm{L})^{(1)} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A{\to}B}\ ({\to}\mathrm{R})$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A{\wedge}B, \Gamma \Rightarrow \Delta}\ ({\wedge}\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma, B}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, A{\wedge}B}\ ({\wedge}\mathrm{R})^{(1)}$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Pi \Rightarrow \Sigma}{A{\vee}B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ ({\vee}\mathrm{L})^{(1)} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A{\vee}B}\ ({\vee}\mathrm{R})$$

$$\frac{\Gamma \Rightarrow \Delta}{a, \Gamma \Rightarrow \Delta}\ (\text{weakening L})^{(2)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a}\ (\text{weakening R})^{(2)}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{C, \Gamma^\theta \Rightarrow \Delta^\theta}\ (\text{contraction L})^{(3)} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma^\theta \Rightarrow \Delta^\theta, C}\ (\text{contraction R})^{(3)}$$

proviso:

(1) $\mathrm{Var}(A, \Gamma, \Delta) \cap \mathrm{Var}(B, \Pi, \Sigma) = \emptyset$.

(2) $a \notin \mathrm{Var}(\Gamma, \Delta)$.

(3) $A^\theta = B^\theta = C$, θ is a most general unifier for $\langle A, B\rangle$, and $\mathrm{Var}(C) \cap \mathrm{Var}(A, B, \Gamma, \Delta) = \emptyset$.

Fact

$\forall A : \text{tautology}, \exists A_0 \preceq A, (\mathbf{LK_m} \vdash (\Rightarrow A_0))$.

Problem 1

Prove or disprove:
If $\mathbf{LK_m} \vdash (\Rightarrow A)$, then either

- $\mathbf{INT} \vdash A$, or
- $\mathbf{INT} \vdash A^\theta \rightarrow (x \vee (x \rightarrow y))$ for some substitution θ.

Fact

(Positive proof of Problem 1) $\Longrightarrow$ (positive proof of Komori's first problem).

Sequent calculus $\mathbf{LK_{ma}}$ is obtained from $\mathbf{LK_m}$ by the restrictions:

- "$\bot$" does not occur.
- Contractions are admitted only for propositional variables:

$$\frac{a, b, \Gamma \Rightarrow \Delta}{c, \Gamma^\theta \Rightarrow \Delta^\theta} \text{(contraction L)} \qquad \frac{\Gamma \Rightarrow \Delta, a, b}{\Gamma^\theta \Rightarrow \Delta^\theta, c} \text{(contraction R)}$$

  where $\theta = [a := c, b := c]$, and $c \notin \mathrm{Var}(a, b, \Gamma, \Delta)$.

Theorem — partial proof of Problem 1

If $\mathbf{LK_{ma}} \vdash (\Rightarrow A)$, then either

- $\mathbf{INT} \vdash A$, or
- $\mathbf{INT} \vdash A^\theta \rightarrow (x \vee (x \rightarrow y))$ for some substitution θ.

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2 Second problem (see [2] for detail)

Definitions

- "$\forall$" = universal quantifier for propositional variables.
- $A[a := B]$ = the formula obtained from A by replacing all the free occurrences of a by B.
- "B is *not* substitutable for a in A" $\Longleftrightarrow$ $\exists b$(b occurs free in B, and some free occurrence of a is in a scope of $\forall b$ in A).

Sequent calculus LJ2 is obtained from propositional **LJ** by adding the inference rules:

$$\frac{A[x := B], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C} \ (\forall \mathrm{L}) \text{ where } B \text{ is substitutable for } x \text{ in } A$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \forall x A} \ (\forall \mathrm{R}) \text{ where } x \text{ does not occur free in } \Gamma$$

Problem 2

Prove or disprove:
If A and B are $\{\rightarrow\}$-formulas, and $\mathrm{Var}(A) = \{a_1, a_2, \ldots, a_n\}$, then the following two conditions are equivalent:

- **LJ2** $\vdash \Big((\forall a_1 \forall a_2 \cdots \forall a_n A) \Rightarrow B\Big)$.
- **INT**$[A] \vdash_{\{\rightarrow\}} B$.

($\vdash_{\{\rightarrow\}}$ = provability in $\{\rightarrow\}$-fragment.)

Komori's second problem in [3]

Prove or disprove:
If A_1, A_2 and B are $\{\rightarrow\}$-formulas, then the following two conditions are equivalent:

- **INT**$[A_1 \vee A_2] \vdash_{\{\rightarrow, \vee\}} B$.
- **INT**$[(A_1 \rightarrow x) \rightarrow (A_2 \rightarrow x) \rightarrow x] \vdash_{\{\rightarrow\}} B$.

Fact

(Positive proof of Problem 2) $\Longrightarrow$ (positive proof of Komori's second problem).

Another form of Problem 2

Prove or disprove:
If A and B are $\{\rightarrow\}$-formulas, and $\mathrm{Var}(A) = \{a_1, a_2, \ldots, a_n\}$, then the following two conditions are equivalent:

- **LJ2** $\vdash \Big((\forall a_1 \forall a_2 \cdots \forall a_n A) \Rightarrow B\Big)$.
- **LJ** $\vdash \Big(A[\vec{a} := \vec{X}], A[\vec{a} := \vec{X'}], \ldots, A[\vec{a} := \vec{X''}] \Rightarrow B\Big)$, for some $\{\rightarrow\}$-formulas $\vec{X}, \vec{X'}, \ldots, \vec{X''}$.

($[\vec{a} := \vec{X}]$ = simultaneous substitution $[a_1 := X_1, a_2 := X_2, \ldots, a_n := X_n]$.)

Without the condition "$\mathrm{Var}(A) = \{a_1, a_2, \ldots, a_n\}$", the statement does not hold:

Counterexample

LJ2 $\vdash \Big(\forall a((a \rightarrow p) \rightarrow (a \rightarrow q) \rightarrow r) \Rightarrow r\Big)$.

LJ $\nvdash \Big((X \rightarrow p) \rightarrow (X \rightarrow q) \rightarrow r, (X' \rightarrow p) \rightarrow (X' \rightarrow q) \rightarrow r, \ldots, (X'' \rightarrow p) \rightarrow (X'' \rightarrow q) \rightarrow r \Rightarrow r\Big)$, for any $\{\rightarrow\}$-formulas $X, X', \ldots, X''$.

Using "$\wedge$" in $\vec{X}, \vec{X'}, \ldots, \vec{X''}$, the statement hold:

Theorem

If A and B are $\{\rightarrow, \wedge\}$-formulas, then the following two conditions are equivalent:

- **LJ2** $\vdash \Big((\forall a_1 \forall a_2 \cdots \forall a_n A) \Rightarrow B\Big)$.
- **LJ** $\vdash \Big(A[\vec{a} := \vec{X}], A[\vec{a} := \vec{X'}], \ldots, A[\vec{a} := \vec{X''}] \Rightarrow B\Big)$, for some $\{\rightarrow, \wedge\}$-formulas $\vec{X}, \vec{X'}, \ldots, \vec{X''}$.

References

[1] 鹿島亮,「中間命題論理の公理の非一般性について」, `http://www.is.titech.ac.jp/~kashima/manuscript/05Jul.pdf`

[2] 鹿島亮,「直観主義二階命題論理について」, `http://www.is.titech.ac.jp/~kashima/manuscript/05Nov.pdf`

[3] 古森雄一,「古典論理で極小な論理式で公理化される論理についての問題など」, 日本数学会 2005 年度秋季総合分科会 数学基礎論および歴史分科会.