Independent Axiom Systems of Minimal formulas for Classical Logic

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In this note, we shall prove the independency of the system BCIK'W' for the implicational fragment of intuitionistic logic. Then we will pose a problem on the Axiomatization for the implicational fragment of classical logic.

The systems in this note are Hilbert type systems. All has two inference rules Modus Ponens and Substitution. Modus Ponens is that $\alpha \supset \beta, \alpha \vdash \beta$.

We begin with remembering that the system in Matsumoto[1] is independent (the proof is also in [1]). The system in Matsumoto has the following three axioms:

$$\begin{array}{lcl} K & : & p \supset q \supset p, \\ S & : & (p \supset q \supset r) \supset (p \supset q) \supset p \supset r, \\ A3 & : & (\neg p \supset \neg q) \supset (\neg p \supset q) \supset p. \end{array}$$

The following three valued model shows the independency of Axiom K.

$p \backslash q$	0	1	2	p	$\neg p$
0	0	2	2	0	1
1	2	2	0	1	1
2	0	0	0	2	0

Designated value $= 0$

The above table shows that $2 \supset 1 = 0$, for example. In any model, the rule of Substitution is valid. In this model, the rule of Modus Ponens is valid and Axioms S and A3 are valid. Let $p = 1$ and $q = 2$. Then $K = 2$.

The following three valued model shows the independency of Axiom S.

$p \backslash q$	0	1	2	p	$\neg p$
0	0	2	1	0	1
1	0	2	0	1	0
2	0	0	0	2	1

Designated value $= 0$

Let $p = 0$, $q = 0$ and $r = 1$. Then $S = 2$.

The following two valued model shows the independency of Axiom A3.

$p \backslash q$	0	1	p	$\neg p$
0	1	1	0	0
1	0	1	1	1

Designated value $= 1$

Let $p = 0$ and $q = 0$. Then $A3 = 0$.

Next, we will confirm that the system KSP for the implicational fragment of classical logic. The system has the following three axioms:

$$\begin{array}{lcl} K & : & p \supset q \supset p, \\ S & : & (p \supset q \supset r) \supset (p \supset q) \supset p \supset r, \\ P & : & ((p \supset q) \supset p) \supset p. \end{array}$$

The following three valued model shows the independency of Axiom K.

$p\backslash q$	0	1	2
0	1	1	1
1	0	1	1
2	0	0	0

Designated values $= 1, 2$

Let $p = 1$ and $q = 2$. Then $K = 0$.

The following four valued model shows the independency of Axiom S.

$p\backslash q$	0	1	2	3
0	0	1	2	1
1	0	0	2	0
2	0	1	0	0
3	0	0	0	0

Designated value $= 0$

Let $p = 0$, $q = 2$ and $r = 3$. Then $S = 1$.

Godel's three valued logic shows the independency of Axiom P. So, we now confirm that the system KSP is independent.

All of formulas K, S, A3 and P are Classical-logic-minimal. A formula α is called a *trivial substitution instance* of β iff α is a substitution instance of β and β is a substitution instance of α. A formula is *L-minimal* iff it is provable in L and it is not nontrivial substitution instance of another formula provable in L. In this note, Classical-Logic-minimal is simply called by *minimal*.

1 The system BCIK'W' for intuitionistic implicational logic

We want to get a system for classical logic which has as many axioms as possible, Because we can get many substructual logics by taking some formulas (of axioms of the system) as axioms. For the implicational fragment of intuitionistic logic, we can get a satisfactory system that is the system BCIK'W'. The system BCIK'W' has the following five axioms:

$$\begin{array}{lcl}
B & : & (q \supset r) \supset (p \supset q) \supset p \supset r, \\
C & : & (p \supset q \supset r) \supset q \supset p \supset r, \\
I & : & p \supset p, \\
K' & : & (p \supset q) \supset p \supset r \supset q, \\
W' & : & (p \supset p \supset p \supset q) \supset p \supset q.
\end{array}$$

All of formulas B, C, I, K' and W' are minimal. We will show the indendence of the system.

The following four valued model shows the independency of Axiom B.

$p\backslash q$	0	1	2	3
0	0	1	2	1
1	0	0	2	0
2	0	0	0	0
3	0	0	0	0

Designated value $= 0$

Let $p = 0$, $q = 3$ and $r = 2$. Then $B = 2$.

The following three valued model shows the independency of Axiom C.

$p\backslash q$	0	1	2
0	0	1	1
1	0	0	2
2	0	0	0

Designated value $= 0$

Let $p = 1$, $q = 0$ and $r = 2$. Then $C = 1$.

The following three valued model shows the independency of Axiom I.

$p\backslash q$	0	1	2
0	0	2	2
1	0	2	2
2	0	0	0

Designated value = 0

Let $p = 1$. Then $I = 2$.

The following two valued model shows the independency of Axiom K'.

$p\backslash q$	0	1
0	1	0
1	0	1

Designated value = 1

Let $p = 1$, $q = 1$ and $r = 0$. Then $K' = 0$.

The following three valued model shows the independency of Axiom W'.

$p\backslash q$	0	1	2
0	0	1	2
1	0	0	1
2	0	0	0

Designated value = 0

Let $p = 1$ and $q = 2$. Then $W' = 1$.

The above ensures the indepnndency of the system BCIK'W'.

2 The system BCIK'W'P for classical implicational logic

Now we consider the system BCIK'W'P which is obtained from the system BCIK'W' by adding a formula P as a axiom. The system BCIK'W'P is the implicational fragment of classical logic. K is proved from BCIP. The proof is B(BP)(BC(CI)) [$\lambda xya.axy$]. So K' is proved from BCIP. The system BCIK'W'P is not independent. W is also proved from BCIP. The proof is obtained from $\lambda xay.a(xy)$. ['x' and 'y' are λ-variables and 'a' is a ρ-variable. The translation for ρ-variables is $(\lambda a.M)^* = P(\lambda z.[z/a]M)^*$.] So W' is proved frpm BCIP and the system BCIP is a system of the implicational fragment of classical logic(I do not know whether it is independent or not).

PROBLEM 2.1 If possible, find an independent Axiom System of minimal formulas for classical implicational Logic which is obtained from the system BCIK'W' by adding formulas as axioms.

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References

[1] Kazuo Matsumoto. *Suuri Ronrigaku.* Kyouritsu, enlarged edition, 1971. In Japanese.

[2] John Slaney. *FINDER: Finite Domain Enumerator.* Center for Information Science Research, Australian National Univ., 3.0 edition, 2005. Available online: http://arp.anu.edu.au/ftp/FINDER/.