On Evans's vague object

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1 Introduction

Gareth Evans proved that if two objects are indeterminately equal then they are different in reality [Ev78]. He defined vague objects as having vague identity statement: a is a vague object if there exists an object b such that $a = b$ is of indeterminate truth value. Let us assume *there can be vague objects in the world*; we call this Evans's Vagueness Assumption (**EVA**). Let a, b be vague objects, then

(I)	$\bigtriangledown(a = b)$,	i.e. $a = b$ is indeterminate (assumption),
(II)	$\lambda x[\bigtriangledown(a = x)]_b$,	i.e. b is indeterminately equal to a (from (I)),
(III)	$\neg \bigtriangledown (a = a)$,	i.e. $a = a$ is determinate,
(IV)	$\neg\lambda x[\bigtriangledown(a = x)]_a$,	i.e. a is not indeterminately equal to a (from (III)),
(V)	$a \neq b$,	i.e. a is not equal to b (from (II) and (IV)).

We note that $\bigtriangledown\varphi$ means that the truth value of φ is indeterminate. He insisted that this contradicts the assumption that there can be vague objects. Therefore, he seems to conclude **EVA** does not hold.

Some philosophers have agreed with his conclusion. For example, Brian Garrett denied the possibility of vague identity and vague object [Ga91]. However, many articles have been published against Evans's conclusion. One typical approach is to analyze his proof within many-valued logic. For example, Jack Copeland tried to prove that the derivation of (V) from (I) is not valid within fuzzy logic without mentioning **EVA** [Co95]. However, it has been objected that 'the writers who adopt this strategy rarely provide much argument for the need for a many-valued logic' [KS97, 55]. Another approach takes the modal point of view. For example, Ken Akiba defined a vague object as a transworld object. He distinguished identity relation from coincidence relation on identity in Evans's proof, and then tried to show that Evans's proof holds true only for the case of identity statement [Ak00].

In this paper, we introduce the defense of both **EVA** and Evans's derivation from (I) to (V) [YI06]: We show the consistency between Evans's proof and the existence of vague objects within classical logic. We formalize Evans's proof in a set theory without the axiom of extensionality, and we define a set to be vague if it violates extensionality with respect to some other set. There exist models of set theory where the axiom of extensionality does not hold, so this shows that there can be vague objects.

Technically speaking, Evans's proof seems to have three implicit assumptions as follows:

(i) For every a, $a = a$ has definite truth value ($\neg \bigtriangledown (a = a)$),

(ii) *the Diversity of the Dissimilar* (**DD**) : if object a has a property that b lacks, then you can infer $a \neq b$,

(iii) $\vdash \varphi$ implies $\vdash \bigtriangleup\varphi$ (as the generalization law in $\mathbf{S_5}$-modal logic).

For more details, see [KS97]. We note that $\bigtriangleup\varphi$ means that the truth value of φ is determinate. (ii) is used to infer (V) from (II) and (IV). Now, $\bigtriangleup(a \neq b)$ is inferred from (V) and (iii). For duality[1], $\neg \bigtriangledown (a = b)$ is inferred from $\bigtriangleup(a \neq b)$. This contradicts (I). But, we can disregard (iii). Indeed, (i) and (ii) are

[1] Duality between $\bigtriangleup$ and $\bigtriangledown$ is valid: $\neg \bigtriangleup \neg\varphi \leftrightarrow \bigtriangledown\varphi$.

necessary to derive (V) from (I), but (iii) has nothing to do with the derivation itself. Furthermore, from a viewpoint of the logic of knowledge, (iii) seems to be doubtable. Then, if we do not admit (iii), what Evans proved is merely that *the vague identity statement (I) implies (V)*. We call '$\triangledown(a = b) \to a \neq b$' Evans Conditional (**EC**) as in [Co95]. We show that our definition of indeterminate equality satisfies **EC**, and that both **EVA** and **EC** are consistent when we employ a set theory without the axiom of extensionality.

2 Formalizing Evans's proof in set theory

In this section, we attempt to formalize Evans's proof in set theory. First we claim that, among other properties, extension is worth being focused on when we consider a vague object. In fact, philosophical discussions about vagueness often begin with explaining or sometimes defining it in terms of extension [KS97]. Now, one of the simplest frameworks to consider extension is set theory, so we employ set theory in this paper. The key to formalize Evans's proof in set theory is to interpret his word "*indeterminate*". There are many ways to interpret it. For example, it is interpreted as "its truth value is neither 0 nor 1" in many-valued logic, or it is represented by using a modal operator in modal logic. However, we regard the truth value of any formula as determinate, and we add neither a new predicate nor operator which represent indeterminacy. We interpret '$a = b$ is indeterminate' as some set-theoretic property, namely *the axiom of extensionality is violated for a and b*, which is definable by membership relation. Here we only introduce the outline of the formalization: For the detailed discussion and the philosophical justification, see [YI06].

In Evans's proof, two kinds of relation are used: Leibniz equality relation and vague equality relation. The confusion of these relations seems to make Evans's proof paradoxical, so it is important to distinguish them. The famous relations are as follows:

Leibniz equality $x = y$ iff $(\forall z)[(z \in x \leftrightarrow z \in y) \,\&\, (x \in z \leftrightarrow y \in z)]$,

Extensional equality $x =_{\text{ext}} y$ iff $(\forall z)[z \in x \leftrightarrow z \in y]$.

Of course $x = y \to x =_{\text{ext}} y$ holds. Since $(\forall z)[x \in z \leftrightarrow y \in z] \to (\forall z)[z \in x \leftrightarrow z \in y]$ holds[2], the definition of Leibniz equality is usually written as $x = y$ iff $(\forall z)[x \in z \leftrightarrow y \in z]$. The Leibniz law $a = b \to (\varphi(a) \leftrightarrow \varphi(b))$ surely holds for Leibniz equality, however there is no guarantee that it holds for extensional equality. It is necessary to consider the axiom of extensionality when we think about identity relations: The axiom of extensionality guarantees that, for any set x and y, $x =_{\text{ext}} y \to x = y$.

As we see, **EC** itself does not imply a contradiction. However, if we assume $a \neq b \to \triangle(a \neq b)$, it implies a contradiction. Let us weaken this not to imply a contradiction (it is enough to strengthen the premise "$a \neq b$"). Since extensional equality is a relationship weaker than Leibniz equality (i.e. $a \neq_{\text{ext}} b \to a \neq b$), we weaken this as follows:

$$a \neq_{ext} b \to \triangle(a \neq b)$$

Take a contraposition of this, we have $\triangledown(a = b) \to a =_{\text{ext}} b$. By this and **EC**, we can conclude the following:

$$\triangledown(a = b) \to a =_{\text{ext}} b \,\&\, a \neq b \tag{1}$$

It shows that, whenever $a = b$ is indefinite, the axiom of extensionality is violated.

So, if we employ (1), we can formalize Evans's proof in set theory as follows:

(I') $\triangledown(a = b)$,

(I") $a =_{ext} b \,\&\, a \neq b$ (from (1)),

(II') $b \in X$ where X is such that $(\forall x)[x \in X \leftrightarrow x \in Y \,\&\, a =_{ext} x \,\&\, a \neq x]$ and Y is such that $(\forall x)[x \in Y \leftrightarrow x = a \vee x = b]$ (from (I')),

[2] By definition, when $a \neq_{\text{ext}} b$, there is a set c such that $c \in a \,\&\, c \notin b$ or vice versa. So the axiom of separation and the axiom of power set guarantee that, a set D such that $(\forall x)[x \in D \leftrightarrow (\forall y)[y \in x \to y \in a] \,\&\, c \in x]$ exists, and it distinguishes a and b; $a \in D$ and $b \notin D$ holds. We remind that such a way of distinction is the same as Evans's proof.

(III') $\neg(a =_{ext} a \,\&\, a \neq a)$,

(IV') $a \notin X$ (from (III')),

(V') $a \neq b$ (from **DDS**)

This shows that **EC** is always valid in our set theory. It does not imply a contradiction (for the proof, see [YI06]). This is because the theory does not have any principle which derives $\triangle(a = b)$ from (V').

3 Models of EVA and EC

As far as (1) is concerned, we take notice of the violation of the axiom of extensionality. The axiom of extensionality can be seen as a representation of precision since any set is determined precisely by its members. In this sense, the violation of the axiom of extensionality represents some aspect of vague object. This means that the converse of (1) holds in set theory. So we regard such violation of this axiom as a representation of vagueness here.

$$\bigtriangledown(a = b) \leftrightarrow a =_{\text{ext}} b \,\&\, a \neq b \tag{2}$$

As we saw, vague object is defined by using vague identity, i.e. a is a vague object if and only if $(\exists x) \bigtriangledown (a = x)$. So we call a vague object when this axiom is violated. More precisely,

Definition 1 *a is a vague object iff the axiom of extensionality is violated for a, i.e.*

$$(\exists x)[a =_{ext} x \,\&\, a \neq x]$$

Assuming Definition 1, **EVA** implies a contradiction only when we assume the axiom of extensionality. Otherwise, $\bigtriangledown(a = b)$ implies $a \neq b$ without implying $\triangle(a \neq b)$. So (2) and Definition 1 show that any model of set theory in which the negation of the axiom of extensionality holds is a model of **EVA** and **EC**. There exist many such models, so this fact proves consistency of our definition.

We can easily generalize Definition 1: The violation of the axiom of extensionality represents vagueness not only within classical logic but also within a greater variety of logics. So, within any logic, we insist that set theory without the axiom of extensionality is required to represent vague object. Conversely, many such set theories have been proposed by now.

Traditionally, this has been studied within intuitionistic logic; one of the most famous results is due to Harvey Friedman [Fr73]. V.N.Grisǐn showed that the comprehension principle alone does not imply Russell paradox within Grisǐn logic, which is classical logic minus the contraction rule [Gr82]. He also showed that the comprehension principle and the axiom of extensionality are incompatible within Grisǐn's logic: the axiom of extensionality implies the contraction rule (so this implies Russell paradox) in set theory with the comprehension principle within Grisǐn logic.

Peter Hajek and Zuzana Hanikova developed Fuzzy Set Theory **FST** [HH03], within the framework of fuzzy logic with operator $\triangle$ which means 'determinately true', i.e. the truth value of $\triangle\varphi$ is 1 if the value of φ is already 1; otherwise, the truth value of φ is less than 1, then $\triangle\varphi$ takes value 0 (in BL-chains). It is in the style of **ZF**, and it seems to be an attempt to axiomatize our intuition of fuzzy set. In **FST**, the axiom of extensionality cannot be valid. This is because that Leibniz equality becomes crisp (i.e. its truth value is 0 or 1) nevertheless the truth value of extensional equality can be indeterminate. For the proof, see [HH03, §4]. Here, the axiom of extensionality holds for any crisp set, but it might be violated for some fuzzy set: The truth value of Leibniz equality and that of extensional equality might be different for some fuzzy set. So **FST** can only have the weakened version of the axiom of extensionality: $x = y$ iff $\triangle(x \subseteq y) \& \triangle (y \subseteq x)$. Such violation of the axiom has been regarded as merely introduced for technical reasons, however this seems to suggest that such violation is a necessary feature of fuzzy set implicitly connoted by our intuition of fuzziness itself. In this sense, Definition 1 can be regarded as an isolation of some aspect of fuzziness so that we can represent it even within classical logic.

As for modal logic, Jan Krajicek developed the Modal Set Theory **MST** [Kr87] [Kr88]. It has an operator $\Box$ which represents 'to be knowable', and it is an axiomatization of a set theory based on a modal version of the comprehension axioms as the only non-logical axioms. Unfortunately the consistency of

MST is still an open problem. It is worthy of special mention that **MST** disproves the axiom of extensionality: Therefore such the similarity, with Grisǐn's and with ours, is worthy of attention. These theories seem to give an example of vague object in the sense of Definition 1.

4 Conclusion

In this paper, we examined Evans's proof from a set theoretic viewpoint. We defined vague objects as objects for which the axiom of extensionality does not hold within classical logic, so we could construct a model of **EVA** and **EC**. This means that the assumption that *there can be vague objects in the world* itself does not imply a contradiction nevertheless Evans's proof is still valid. Namely, if you accept our definition of vague objects, you can conclude that there can be vague object in the world.

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