Hybrid logic with Pure and Sahlqvist axioms

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abstract: In this paper, we propose expanding the hybrid language with "Succecer operater" and prove more general completeness theorem "pure and Sahlqvist completeness theorem".

1 Introduction

Hybrid logic comes with a general completeness resalt: Every extension with pure axioms of the basic hybrid logic with [Neme] and [Paste] rules (we denote $\mathbf{K}_h^+$) is complete [2]. Pure axiom is formula not containing arbitrary proposition variabls. Pure axiom corresponds to first order frame condition and quite expressive, for instance, $i \to \neg\Diamond i$ defines the class of irreflexive frames. Recentry ten Cate et al. proved another general completeness theorem: Every extension with modal Sahlqvist axioms of the basic hybrid logic is complete [1]. And they proved negative resalt in same paper: Two general completeness theorem can not combine, i.e. there is a pure formula λ and a Sahlqvist formula ϕ such that the hybrid logic $\mathbf{K}_h^+\{\lambda, \phi\}$ is imcomplete for the frame class defined by $\lambda \wedge \phi$. This second resalt is not hopeful because there exists some pure formula which there exists no Sahlqvist formula corresponds it and there exists some Sahlqvist formula which there exists no pure formula corresponds it. In this paper, we propose a new extension of hybrid langage "Succecer operater" and prove combinning completeness theorem in the sence denoted above.

2 Preliminaries

The language $\mathcal{HL}(@, S)$ is defined using (i) the set of propositional variables: $\mathsf{Prop} = \{p_n | n \in \omega\}$, (ii) the set of nominals: $\mathsf{Nom} = \{i_n | n \in \omega\}$, (iii) the propositional connectives: $\neg, \vee$, (iv) the modal operaters: $\Diamond, @_t\,(t \in \mathsf{Term})$, and (v) the Succecer operater: S. Where Term is a set of all t which is inductively defined by $t ::= i \,|\, S(t)\,(i \in \mathsf{Nom})$, namely $S(S(\ldots S(i)\ldots))$ is term. Let i is nominal and $S(S(\ldots S(i)\ldots))$ is n-times itaration of Succecer oerater S, we abbreviate it $S^n(i)$. we define the formulas of the hybrid language $\mathcal{HL}(@, S)$ to be

$$\phi ::= p \,|\, t \,|\, \neg\phi \,|\, \phi \vee \psi \,|\, \Diamond\phi \,|\, @_t\phi$$

where $p \in \mathsf{Prop}$ and $t \in \mathsf{Term}$.

Definition 2.1. *Let $\mathfrak{F} = (W, R)$ is frame in usual sence, then $\mathfrak{M} = (\mathfrak{F}, V)$ is model if $V : \mathsf{Prop} \cup \mathsf{Nom} \to \mathcal{P}(W)$ such that all nominal i, $|V(i)| = 1$. For any model $\mathfrak{M}$, any state $w \in W$ and any formula ϕ of $\mathcal{HL}(@, S)$, the relation $\Vdash$ is defined inductively as follow:*

- $\mathfrak{M}, w \Vdash i \Leftrightarrow w \in V(i) \Leftrightarrow \{w\} = V(i)$
- $\mathfrak{M}, w \Vdash S^n(i) \Leftrightarrow R^n vw$ *where v is a denotation of nominal i*
- $\mathfrak{M}, w \Vdash @_{S^n(i)}\phi \Leftrightarrow$ *for all $v \in W$, if $R^n uv$ then $\mathfrak{M}, v \Vdash \phi$, where u is denotation of nominal i.*

And other cace is defined as usual way. Modal general frame $\mathfrak{g} = (\mathfrak{F}, \mathbb{A})$ is hybrid general frame if $\mathbb{A}$ satisfy following two condition: (i) *There exists $w \in W$ such that $\{w\} \in \mathbb{A}$.* (ii) *For every state $w \in W$,* *if* $\{w\} \in \mathbb{A}$ *then* $R^n[\{w\}] \in \mathbb{A}$*, where for* $X \subseteq W$*,* $R[X] := \{w \mid \exists x \in X Rxw\}$*. First clause* (i) *needs for denotation of nominal* i*, and second clause* (ii) *needs for closure condition of Succecer oerater* S*.*

Definition 2.2 (Sahlqvist formula)**.** *A boxed atom is a formula of the form* $\Box\Box\ldots\Box p$*. A Sahlqvist antecedent is formula built from* $\top$*,* $\bot$*, negative formula and boxed atom, using only* $\wedge, \vee$ *and* $\Diamond$*. A Sahlqvist implication* ϕ *is formula of the form* $\phi \equiv \psi \to \chi$ *where* ψ *is a Sahlqvist antesedent, and* χ *is a positive formula. Sahlqvist formula is formula built from Sahlqvist implication using* $\wedge, \Diamond, \Box$ *freely and* $\vee$ *if two Sahlqvist formulas share no propositional variables.*

Definition 2.3. *A hybrid general frame* $\mathfrak{g} = (\mathfrak{F}, \mathbb{A})$ *is ample if for every state* $w \in \mathfrak{F}$ *and* $n \in \omega$*,* $R^n[\{w\}] \in \mathbb{A}$*. A formula* ϕ *is ample-persistence if for every ample general frame* $\mathfrak{g} = (\mathfrak{F}, \mathbb{A})$*,* $\mathfrak{g} \Vdash \phi$ iff $\mathfrak{F} \Vdash \phi$*.*

Following lemma plays central role in proof of completeness theorem of Sahlqvist part [3].

Lemma 2.4. *Every Sahlqvist formula* ϕ *is ample persistence.*

As in basic hybrid logic, a model is called *named* iff every state is the denotation of some nominal. Substitution σ uniformly replaces nominals by nominals and proposition letters by arbitrary formulas. We need following lemma in proof of completeness theorem of pure part (see [2] Lemma 7.22).

Lemma 2.5. *Let* $\mathfrak{M} = (\mathfrak{F}, V)$ *be a named model and* λ *is pure formula, if* $\mathfrak{M} \Vdash \lambda^\sigma$ *where* σ *is arbitrary substitution, then* $\mathfrak{F} \Vdash \lambda$

3 Completeness

we axiomatisate $\mathbf{K}^+_{hs}$ as follow.

Axioms	
(CT)	all classical tautologies
($K_\Box$)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
($K_@$)	$\vdash @_i(p \to q) \to (@_i p \to @_i q)$
(Selfdual)	$\vdash @_i p \leftrightarrow \neg @_i \neg p$
(Intro)	$\vdash i \wedge p \to @_i p$
(Ref)	$\vdash @_i i$
(Nom)	$\vdash @_i j \wedge @_j p \to @_i p$
(Agree)	$\vdash @_i @_j p \leftrightarrow @_j p$
(Back)	$\vdash \Diamond @_i p \to @_i p$
($Suc_\Box$)	$\vdash @_{S^n(i)} p \leftrightarrow @_i \Box^n p$
($Suc_\Diamond$)	$\vdash @_i S^n(j) \leftrightarrow @_j \Diamond^n i$
Rules	
[MP]	If $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$
[Sub]	If $\vdash \phi$, then $\vdash \phi^\sigma$
[$Gen_\Box$]	If $\vdash \phi$, then $\vdash \Box\phi$
[$Gen_@$]	If $\vdash \phi$ then $\vdash @_i \phi$
[Name]	If $\vdash i \to \theta$ and i does not occur in θ, then $\vdash \theta$
[Paste]	If $\vdash @_i \Diamond j \wedge @_j \phi \to \theta$ and $i \neq j, j$ does not occur in ϕ or θ, then $\vdash @_i \Diamond \phi \to \theta$

Remark that rule [Sub] allows us to replace nominals by nominals.

Lemma 3.1 (Extended Lidenbaum Lemma)**.** *Let* Ω' *be a (countably) infinite new collection of nominals. And let* $\mathcal{HL}'(@, S)$ *be the language obtained by adding these new nominals to* $\mathcal{HL}(@, S)$*. Then every* $\mathbf{K}^+_{hs}$*-consistent set* Γ *of language* $\mathcal{HL}(@, S)$ *can be extended to* $\mathbf{K}^+_{hs}$-MCS Γ^+ *of language* $\mathcal{HL}'(@, S)$ *which satisfy following two condition*:

- *(Named) : There exists nominal* k *such that* $k \in \Gamma^+$*.*
- *(Pasted) : For all formula* ϕ *and for all nominal* i*, if* $@_i \Diamond \phi \in \Gamma^+$ *then there exists nominal* j *such that* $@_i \Diamond j \wedge @_j \phi \in \Gamma^+$*.*

Proof. see [2], lemme 7.25. $\square$

Lemma 3.2. *Let Γ^+ be a $\mathbf{K}_{hs}^+$-MCS that satisfes condition denoted lemma 3.1. For all nominal i, let Δ_i be $\{\phi \,|\, @_i\phi \in \Gamma^+\}$ And We define "canonical relation" R by $R\Delta_i\Delta_j$ iff for all ϕ, if $\phi \in \Delta_j$, then $\Diamond\phi \in \Delta_i$. Then :*

(i). For all nominal i, Δ_i is a $\mathbf{K}_{hs}^+$-MCS that contains i.

(ii). For all nominal i and j, if $i \in \Delta_j$ then $\Delta_i = \Delta_j$.

(iii). For all nominal i and j, $@_j\phi \in \Delta_j$ iff $@_j \in \Gamma^+$.

(iv). For all nominak k,If $k \in \Gamma^+$, Then $\Delta_k = \Gamma^+$.

(v). For all nominal i and formula ϕ, if $\Diamond\phi \in \Delta_i$, then there exists nominal j such that $\phi \in \Delta_j$ and $R\Delta_i\Delta_j$.

(vi). For all nominal i, j and $n \in \omega$, $R^n\Delta_i\Delta_j$ iff $S^n(i) \in \Delta_j$

Proof. (i) $\sim$ (v) see [2] lemma7.24. and 7.27.
(vi) If $S^n(i) \in \Delta_j$, then $@_jS^n(i) \in \Gamma^+$. By (Suc$_\Diamond$), $@_i\Diamond^n j \in \Gamma^+$, so $\Diamond^n j \in \Delta_i$. From (ii) and (v), we have $R^n\Delta_i\Delta_j$. Conversely if $R^n\Delta_i\Delta_j$, then $\Diamond^n j \in \Delta_i$, so $@_i\Diamond^n j \in \Gamma^+$. By (Suc$_\Diamond$), $@_jS^n(i) \in \Gamma^+$, and we have $S^n(i) \in \Delta_j$. $\square$

Theorem 3.3 (Completeness)**.** *Let Λ is a set of pure formulas and Σ is a set of Sahlqvist formulas. Then $\mathbf{K}_{hs}^+(\Lambda \cup \Sigma)$ is strongly complete for the frame class defined by $\Lambda \cup \Sigma$.*

Proof. Let Γ be a arbitrary $\mathbf{K}_{hs}^+(\Lambda \cup \Sigma)$-consistent set, then we can extend it $\mathbf{K}_{hs}^+(\Lambda \cup \Sigma)$-MCS Γ^+ satisfing condition (*Named*) and (*Pasted*). We construct model $\mathfrak{M} = (W, R, V)$ from Γ^+ as follows:

- $W = \{\Delta_i \,|\, i \in \mathsf{Nom}\}$ where $\Delta_i = \{\psi \,|\, @_i\psi \in \Gamma^+\}$
- $R\Delta_i\Delta_j$ iff for all formula ψ, if $\psi \in \Delta_j$, then $\Diamond\psi \in \Delta_i$
- $V(a) = \{\Delta_i \,|\, a \in \Delta_i\}$ where $a \in \mathsf{Prop} \cap \mathsf{Nom}$

Then we can prove by induction on ψ following "Truth lemma"

$$\mathfrak{M}, \Delta_i \Vdash \psi \quad \text{iff} \quad \psi \in \Delta_i$$

Indeed, if $\psi \equiv S^n(i)$, then we can prove Truth lemma from Lemma 3.2 (vi). And if $\psi \equiv @_{S^n(i)}\chi$, then we can prove from Lemma 3.2 (i), (v) and (Suc$_\Box$). Another cases are proved in the same way. By (*Named*) condition and Truth lemma, we have $\mathfrak{M}, \Gamma^+ \Vdash \Gamma$. Model $\mathfrak{M}$ is named, so by Lemma 2.5, underlying frame $\mathfrak{F}$ validate set of pure formula Λ. And general frame $\mathfrak{g} = (\mathfrak{F}, \{V(\psi) \,|\, \psi \text{ is a formula}\})$ is ample, so by Lemma 2.4, $\mathfrak{F}$ validate set of Sahlqvist formula Σ. Therefore $\mathfrak{F}$ validate $\Lambda \cup \Sigma$. $\square$

References

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