The Finite Embeddability Property of the Variety of Modal Residuated Lattices

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1 Introduction

We consider the finite embeddability property(FEP) of the variety of modal residuated lattices introduced in [3]. The property is a stronger version of the well-known finite model property and implies the decidability of the universal theory of the class while the FMP implies the decidability of the equational theory. Most of the FEP proved so far is on the modality-free algebras. Our concern here is residuated lattices augmented with an S4-like modality.

Definition 1 A octuple $\mathbf{M} = \langle M, \wedge, \vee, \cdot, \rightarrow, 0, 1, \Box \rangle$ is a modal residuated lattice[1] if

- $\langle M, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice with the greatest element 1 and the least element 0;
- $\langle M, \cdot, 1 \rangle$ is a commutative monoid;
- For any $x, y \in M$, $x \cdot y \leqslant z$ iff $x \leqslant y \rightarrow z$.;
- $\Box$ satisfies the following:
  - $1 \leqslant \Box 1$;
  - $\Box x \cdot \Box y \leqslant \Box(x \cdot y)$;
  - $\Box x \leqslant x$;
  - $\Box x \leqslant \Box\Box x$;
  - if $x \leqslant y$, then $\Box x \leqslant \Box y$.

2 The finite embeddability property

A class $\mathcal{K}$ of algebras is said to have the finite embeddability property if every finite partial subalgebra of a member of $\mathcal{K}$ can be embedded into a finite member of $\mathcal{K}$.

Formally, a partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ is an algebra of the same similarity type as $\mathbf{A}$ and each function $f_i^{\mathbf{B}}$ is defined as

$$f_i^{\mathbf{B}}(b_1, \ldots, b_k) = \begin{cases} f_i^{\mathbf{A}}(b_1, \ldots, b_k) & \text{if } f_i^{\mathbf{A}}(b_1, \ldots, b_k) \in B \\ \textit{undefined} & \text{otherwise.} \end{cases}$$

A partial subalgebra $\mathbf{B}$ is embedded into A if there is an injective function h from B to A such that if $f(b_1, \ldots, b_k)$ is defined, then

$$h(f^{\mathbf{B}}(b_1, \ldots, b_k)) = f^{\mathbf{A}}(h(b_1), \ldots, h(b_k))$$

holds for any $b_1, \ldots, b_k \in B$.

We briefly list the definitions of the FMP and of its variations. Here $\mathcal{K}_F$ denotes the class of the finite members of $\mathcal{K}$.

[1] The residuated lattice part of this algebra is what is called a "commutative integral" residuated lattice in recent literature.

Definition 2 Let $\mathcal{K}$ be a class of algebras. We say $\mathcal{K}$ has

1. the finite model property(FMP) if $\mathcal{K}_F \models s = t$ implies $\mathcal{K} \models s = t$ for any identity $s = t$,
2. the strong FMP(SFMP) if $\mathcal{K}_F \models \sigma$ implies $\mathcal{K} \models \sigma$ for any quasi-identity σ,
3. the universal FMP(UFMP) if $\mathcal{K}_F \models \varphi$ implies $\mathcal{K} \models \varphi$ for any universal sentence φ.

Clearly the FEP implies the UFMP, the UFMP implies the SFMP, and the SFMP implies the FMP[2]. Each version of the FMP implies some kind of decidability with increasing strength.

Proposition 3 *Let $\mathcal{K}$ be a finitely axiomatizable class of algebras. If $\mathcal{K}$ has the FMP(SFMP, UFMP), then its equational (resp. quasi-equational, universal) theory is decidable.*

3 Main result

A natural extension (for the modal operator) of the method used by Blok and van Alten gives the FEP of the variety of modal residuated lattices. We repeat the necessary construction from Blok and van Alten [2].

Let $\mathbf{B}$ be a finite partial subalgebra of a modal residuated lattice $\mathbf{A}$. We define M to be the (base set of) submonoid generated by B.

We construct the underlying set. For each $a \in M$ and $b \in B$, define

$$\begin{aligned} (a \rightsquigarrow b] &= \{c \in M : ac \leqslant b\} \\ &= \{c \in M : c \leqslant a \to b\} \end{aligned}$$

Each $(a \rightsquigarrow b]$ is a downward closed set, for if $d \leqslant c \in (a \rightsquigarrow b]$ then $ad \leqslant ac$ and $ac \leqslant b$, so that $ad \leqslant b$, i.e., $d \in (a \rightsquigarrow b]$. Now set

$$\overline{D} = \{(a \rightsquigarrow b] : a \in M \text{ and } b \in B\} (\subseteq \mathcal{P}(M))$$

and then define D as

$$D = \{\cap \mathcal{X} : \mathcal{X} \subseteq \overline{D}\} (\subseteq \mathcal{P}(M))$$

Each element of D is a downward closed subset of M and D is closed under intersection. Note that $M \in D$.

Let us define the closure operator C on $\mathcal{P}(M)$ associated with $\overline{D}$; define C as follows:

$$C(X) = \bigcap \{Y \in \overline{D} : X \subseteq Y\}$$

Note that $C(X)$ is the smallest element of D that contains X. Thus the following proposition holds.

Proposition 4 *If $X \subseteq M$ and $Y \in D$, then $X \subseteq Y$ implies $C(X) \subseteq Y$.*

Next we define an algebra whose underlying set is D. For $X, Y \subseteq M$ and $a \in M$, put $XY = \{ab : a \in X$ and $b \in Y\}$ and $Xa = X\{a\}$.

Let us define an algebra $\mathbf{D} = \langle D, \wedge^{\mathbf{D}}, \vee^{\mathbf{D}}, \cdot^{\mathbf{D}}, \to^{\mathbf{D}}, 0^{\mathbf{D}}, 1^{\mathbf{D}}, \Box^{\mathbf{D}} \rangle$. For $X, Y \subseteq D$, $X_i \in D$ $(i \in I)$ define[3]

$$\begin{aligned} \bigwedge_{i \in I}^{\mathbf{D}} X_i &= \bigcap_{i \in I} X_i \\ \bigvee_{i \in I}^{\mathbf{D}} X_i &= C(\bigcup_{i \in I} Y) \\ X \cdot^{\mathbf{D}} Y &= C(XY) \\ X \to^{\mathbf{D}} Y &= \{a \in M : Xa \subseteq Y\} \end{aligned}$$

[2]Proof of FEP $\to$ UFMP. Suppose $\mathcal{K} \not\models \varphi$, where $\mathcal{K}$ has the FEP and φ is universal. Restrict the algebra refuting φ to the relevant elements in φ yields a partial algebra. Here we use the FEP to get the finite algebra which refutes φ

[3]The definition of modality follows [1].

$$\begin{array}{rcl} 0^{\mathbf{D}} & = & \bigcap \overline{D} \\ 1^{\mathbf{D}} & = & M \\ \Box^{\mathbf{D}} X & = & C(\{a \in X | a = \Box a\}) \end{array}$$

We show that $\Box^{\mathbf{D}} X$ satisfies the requirement: it is S4-like. It is easy to see $\Box X \subseteq \Box Y$ if $X \subseteq Y$.

For $\Box M = M$, trivially $\Box^{\mathbf{D}} M \subseteq M$. For the converse, note $\{a \in M | a = \Box a\}$ contains 1 since $\Box 1 = 1$. Then the downward closure of it contains all the elements of M.

We next show $\Box X \cdot \Box Y \subseteq \Box(X \cdot Y)$. We write X° for $\{a | a = \Box a\}$ and omit the superscript $\mathbf{D}$ for brevity.

$$\begin{array}{rcl} \Box X \cdot^{\mathbf{D}} \Box Y & = & C(C(X^\circ)C(Y^\circ)) \\ & \subseteq & C(C(X^\circ Y^\circ)) \\ & = & C(X^\circ Y^\circ) \\ & = & C(\{ab | a \in X \& a = \Box a \& b \in Y \& b = \Box b\}) \\ & \subseteq & C(\{ab \in C(XY) | ab = \Box(ab)\}) \\ & \subseteq & C(\{c \in C(XY) | c = \Box c\}) = \Box(X \cdot^{\mathbf{D}} Y) \end{array}$$

The transition from 4th to 5th line holds because $\{a \cdot b | a \in X \& a = \Box a \& b \in Y \& b = \Box b\} \subseteq \{ab \in C(XY) | ab = \Box(ab)\}$. In fact, consider ab in the former set. Then

$$ab = \Box a \Box b \leqslant \Box(ab).$$

Hence ab is in the latter set.

We turn to prove $\Box X \subseteq X$ for any $X \in D$. It is easy to see $X^\circ \subseteq X$. Then proposition 4 provides the inclusion.

Lastly we want $\Box X \subseteq \Box\Box X$. Let $a \in X^\circ$. Then $a \in C(X^\circ)$ and $a = \Box a$.Thus we have

$$X^\circ \subseteq \{a \in C(X^\circ) | a = \Box a\},$$

from which we can infer by the property of closure operator

$$\Box X = C(X^\circ) \subseteq C(\{a \in C(X^\circ) | a = \Box a\}) = \Box\Box X,$$

as desired.

Next we define an embedding from $\mathbf{B}$ into $\mathbf{D}$. Just as in [2], we define the embedding h is as $h(a) = (a] = \{b \in M | b \leqslant a\}$. We show this mapping preserves $\Box$.

We want to show, for $\Box a \in B$, $(\Box a] = \Box^{\mathbf{D}}(a] = C(\{b \in (a] | b = \Box b\}) = C(\{b \leqslant a | b = \Box b\})$. To prove the right-to-left inclusion, take $b \leqslant a$ with $b = \Box b$. By monotonicity of the original $\Box$, we have $b = \Box b \leqslant \Box a$. This shows $\{b \leqslant a | b = \Box b\} \subseteq (\Box a]$. Since $(\Box a]$ is in D, $C(\{b \leqslant a | b = \Box b\}) \subseteq (\Box a]$.

For the left-to-right inclusion, take an arbitrary $(c \rightsquigarrow d] \in \overline{D}$ that includes $\{b \leqslant a | b = \Box b\}$. We want $(\Box a] \subseteq (c \rightsquigarrow d]$. Note that $\Box a = \Box\Box a$ and that $\Box a \leqslant a$, whence[4] $\Box a$ is in $\{b \leqslant a | b = \Box b\}$. Therefore $(\Box a] \subseteq (c \rightsquigarrow d]$ because $(c \rightsquigarrow d]$ is downward closed. Thus $(\Box a] \subseteq C(\{b \leqslant a | b = \Box b\})$.

Now we got the algebra $\mathbf{D}$ into which we can embed $\mathbf{B}$. The last thing we have to do is to prove the finiteness of the constructed algebra. Since we start the generated submonoid M, the proof is exactly the same as in [2].

We can conclude:

Theorem 5 *The variety of modal residuated lattices has the FEP.*

Corollary 6 *The universal theory of modal residuated lattices is decidable.*

[4]More importantly $\Box a \in M$ because we only consider $\Box a$ that is in B from the beginning.

4 Other results

What if we consider other modal logics K or $\mathsf{K4}$ based on FL_{ew}? In this section we add one more axiom $\mathtt{dist}_\Box : \Box(ab) \leftrightarrow \Box a \Box b$. The FEP of the variety corresponding to $\mathsf{KT}_{\mathsf{FL}_{ew}} + \mathtt{dist}_\Box$ can be shown with slight modifications. Others remain unanswered at this moment.

Here we only mention how to modify the above proof. First we take M to be closed under $\Box$ and $\cdot$, instead of $\cdot$ alone. Namely,

$$M = \langle \Box, \cdot \rangle\text{-subalgebra generated by } B.$$

In addition we change $\Box^{\mathbf{D}}$ as

$$\Box^{\mathbf{D}} X = C(\{\Box a | a \in X\}).$$

The last thing to prove is the free $\langle \Box, \cdot \rangle$-algebra generated by k elements is well-quasi-ordered.

Lemma 7 *The free $\langle \Box, \cdot \rangle$-subalgebra satisfying $\Box(a \cdot b) = \Box a \cdot \Box b$ and $\Box x \leqslant x$ is well-quasi-ordered.*

Proof. We only outline the proof. First note that $x \geqslant \Box x \geqslant \Box^2 x \geqslant \cdots$. For arbitrary element $u \in M$, we show there are at most finitely many elements in M that are greater than u or are incomparable to u. The former is clear because to be greater means to have lower exponent. For the latter, observe that an element of the free algebra is of the following *normal form*

$$x_1^{n_1^0} x_2^{n_2^0} \cdots x_k^{n_k^0} \Box x_1^{n_1^1} \cdots \Box x_k^{n_k^1} \Box^2 x_1^{n_1^2} \cdots \Box^2 x_k^{n_k^2} \cdots \Box^m x_1^{n_1^m} \cdots \Box^m x_k^{n_k^m}, m \in \mathbb{N}$$

Now consider the normal form of u. When we think of generating elements of M, more and more boxes are necessary. But at some point the elements that have enough boxes becomes comparable to u due to the note above. Hence the number of elements in M incomparable to u is finite. ❏

Since the free algebra is not residuated, the finiteness argument requires a small modification. But we omit it here. The preceding lemma gives

Theorem 8 *The variety corresponding to $\mathsf{KT}_{\mathsf{FL}_{ew}} + \mathtt{dist}_\Box$ has the FEP*

Note that the above method does not work for $\mathsf{K}_{\mathsf{FL}_{ew}}$ or $\mathsf{K4}_{\mathsf{FL}_{ew}}$ as long as we take $\langle \Box, \cdot \rangle$-subalgebra. It is because the free algebra has an infinite antichain if we don't have $\mathsf{T} : \Box a \to a$.

In conclusion, we showed Blok and van Alten's method works naturally for the finite embeddability property of the variety of modal residuated lattices, which corresponds to the $\mathsf{S4}$-like modal logic based on FL_{ew}. Furthermore we considered a KT-like modal logic with additional axiom $\mathtt{dist}_\Box$. It is not obvious whether closing M under $\Box$ is really necessary.

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References

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