Modalities in substructural logics — a preliminary report

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What is a natural way of introducing *modalities* in substructural logics? There are several papers (e.g. by K. Došen, G. Restall) which discussed modalities in substructural logics, but choices of modal axioms for substructural logics in them look sometimes ad hoc even for the case of the substructural analogue of the normal classical modal logic **K**. For, two axiom systems which determine a same classical modal logic are not always equivalent when they lack some of structural rules.

Here, we propose some of basic normal modal logics over the substructural logic **FL** using sequent formulations, and give a justification from an algebraic point of view. At the end, we show an extended form of Gödel translation. We assume familiarity with basic results on substructural logics over **FL** (full Lambek calculus) and their relations to residuated lattices. Recall here that an algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \backslash, /, 1 \rangle$ is a *residuated lattice* (RL), if it satisfies the following:

1. $\langle A, \vee, \wedge \rangle$ is a lattice,
2. $\langle A, \cdot, 1 \rangle$ is a monoid with the unit 1,
3. for all $x, y, z \in A$, $x \cdot y \leq z$ iff $y \leq x \backslash z$ iff $x \leq z/y$.

When we add 0 as a type and interpret it by a fixed element of a given RL, it is called a *FL-algebra* or a *pointed* residuated lattice. The element 0 is used for defining two kinds of *negations* $\sim x = x \backslash 0$ and $-x = 0/x$. In the following, we consider always pointed RLs and call them simply RLs.

Adding structural rules is reflected by additional algebraic conditions on RLs. For instance, exchange rule is expressed by the commutativity of the monoid operation $\cdot$. In commutative RLs, $x \backslash y = y/x$ holds, and either of them is usually denoted by $x \to y$. Also, contraction rule, left and right weakening rules can be expressed by $x \leq x^2$ (*square-increasingness*), $1 = \top$ (*integrality*) and $0 = \bot$, respectively. The class of RLs and all of subclasses with some of the above conditions are equationally definable, and thus each of them forms a *variety*. For more information, see e.g. [2, 4]. This is a joint work with N. Galatos.

1 Modalities in substructural logics and modal residuated lattices

As shown in [2], sequent formulations can describe well how *fusion* behaves. Thus, to get a proper form of modal axioms, it seems suggestive to consider rules for $\Box$ in sequent systems. Let us consider the following rule K for $\Box$ over the sequent system **FL**:

$$\frac{\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha} : K$$

In Hilbert-style system, this rule can be expressed by the following axioms and rule.

- 1. $\Box 1$ 2. $(\Box\alpha \cdot \Box\beta) \backslash \Box(\alpha \cdot \beta)$,
- from $\alpha \backslash \beta$, infer $\Box\alpha \backslash \Box\beta$ (monotonicity).

Similarly, axioms T and 4 are given as follows.

- $\Box\alpha\backslash\alpha$,
- $\Box\alpha\backslash\Box\Box\alpha$.

Thus, Hilbert-style system for our substructural analogue of modal logic **S4** over **FL** ($\mathbf{S4}_{FL}$, in symbol) is given by adding these four axioms and the monotonicity rule for $\Box$ to the axiom system **FL**. These axioms and rule are expressed by the following algebraic conditions on RLs.

- (M1) $\Box x \leq x$
- (M2) $\Box x \leq \Box\Box x$
- (M3) $x \leq y$ implies $\Box x \leq \Box y$
- (M4) $\Box x \cdot \Box y \leq \Box(x \cdot y)$
- (M5) $1 \leq \Box 1$

If a unary operation $\Box$ satisfies conditions from (M1) to (M3), it is called an *interior* (or, a *coclosure*) operation. Also, an interior operation satisfies moreover (M4), it is called a (quantic) *conucleus* in the context of *quantales*, i.e. lattice complete RLs without assuming the existence of the unit 1 (see [5]). For any operation $\Box$ on a RL, satisfying the monotonicity (M3), the following two conditions are shown to be equivalent:

- for all x, y, $\Box x \cdot \Box y \leq \Box(x \cdot y)$,
- for all x, y, $\Box(x\backslash y) \leq \Box x\backslash\Box y$.

Now let us define a *modal residuated lattice* to be an algebra $\langle \mathbf{A}, \Box\rangle$, where $\mathbf{A}$ is a pointed RL and $\Box$ is a conucleus satisfying the necessitation (M5). Note that the condition (M3) on the monotonicity of $\Box$ can be replaced by the inequation $\Box(x \wedge y) \leq \Box x$. Hence the class of modal RLs forms also a variety.

For a given modal RL $\langle \mathbf{A}, \Box\rangle$, let $A_\Box = \{\Box x; x \in A\}$. We can show that $A_\Box$ is closed under $\vee$ and $\cdot$. Then, it is easy to see the following, where $\wedge_*, \backslash_*$, and $/_*$ are defined respectively as $x \wedge_* y = \Box(x \wedge y), x\backslash_* y = \Box(x\backslash y)$ and $x/_* y = \Box(x/y)$.

Lemma 1 *For each modal RL* $\langle \mathbf{A}, \Box\rangle$, $\mathbf{A}_\Box = \langle A_\Box, \vee, \wedge_*, \cdot, \backslash_*, /_*, 1, \Box 0\rangle$ *forms a pointed RL.*

Note that the identity map Id on a RL is a conucleus satisfying (M5). Hence $\langle \mathbf{A}, Id\rangle$ is a modal RL, and moreover the induced RL $\mathbf{A}_{Id}$ is equal to $\mathbf{A}$. A conucleus $\Box$ is *localic* if it satisfies $\Box(x \wedge y) = \Box x \cdot \Box y$. Then, for any conucleus $\Box$ satisfying (M5), the following are equivalent:

1. $\Box$ is localic,
2. in $\mathbf{A}_\Box$, the element 1 is the greatest, and the monoid operation is both commutative and square-increasing.

Hence, for any localic conucleus $\Box$ satisfying both (M5) and $\Box 0 \leq \Box x$ for all x, $\mathbf{A}_\Box$ forms a Heyting algebra. Note that Girard's exponential ! can be regarded as a localic conucleus from an algebraic point of view.

2 Gödel translation extended

Let us define a translation D of non-modal formulas into modal formulas, inductively as follows.

1. $D(p) = \Box p$ for every propositional variable or constant p,
2. $D(\phi * \psi) = D(\phi) * D(\psi)$ for $* \in \{\vee, \cdot\}$,
3. $D(\phi * \psi) = \Box(D(\phi) * D(\psi))$, otherwise.

Then, we can show the following.

Lemma 2 *For each modal RL $\langle \mathbf{A}, \Box \rangle$ and for each non-modal formula ϕ, $D(\phi)$ is valid in $\langle \mathbf{A}, \Box \rangle$ iff ϕ is valid in the RL $\mathbf{A}_\Box$.*

For each modal substructural logic $\mathbf{M}$ over $\mathbf{S4}_{FL}$, define the substructural logic $\rho\mathbf{M}$ over $\mathbf{FL}$ by

$$\rho\mathbf{M} = \mathbf{FL} + \{\phi \mid D(\phi) \in \mathbf{M}\} \ .$$

Note that $\rho(\mathbf{S4}_{FL}) = \mathbf{FL}$. The following theorem is a generalization of a result by Maksimova and Rybakov [3].

Theorem 3
For every modal substructural logic $\mathbf{M}$ over $\mathbf{S4}_{FL}$ and for every non-modal formula ϕ, $\vdash_{\rho\mathbf{M}} \phi$ iff $\vdash_{\mathbf{M}} D(\phi)$.

Following the idea by Dummett and Lemmon [1], for every substructural logic $\mathbf{L}$ over $\mathbf{FL}$, we can define the modal substructural logic $\tau\mathbf{L}$ by

$$\tau\mathbf{L} = \mathbf{S4}_{FL} \oplus \{D(\phi) \mid \phi \in \mathbf{L}\}.$$

Theorem 4
For every substructural logic $\mathbf{L}$ over $\mathbf{FL}$, $\mathbf{L} = \rho\tau\mathbf{L}$, and $\tau\mathbf{L}$ is the smallest modal companion *of $\mathbf{L}$, i.e. it is the smallest logic in $\rho^{-1}(\mathbf{L})$.*

As a particular case, consider modal companions of **Int**. As is well-known, **S4** is a modal companion of intuitionistic logic **Int** and the logic $\mathbf{S4} \oplus Grz$ is the greatest modal companion. Using the above results, we can show that the smallest modal companion $\tau\mathbf{Int}$ is axiomatized as $\mathbf{S4}_{FL} \oplus \{\Box\alpha\backslash(\Box\alpha)^2, \Box\beta\backslash 1, \Box 0\backslash\Box\gamma\}$. By using the fact that the set of modal companions of a give logic is convex, $\mathbf{ILL} + \{\Box 0 \to \bot\}$ and **IntS4** are modal companions of **Int**.

Using Zorn's lemma, we can show that $\rho^{-1}(\mathbf{L})$ has always a maximal logic for each substructural logic $\mathbf{L}$. It will be an interesting problem to see whether $\rho^{-1}(\mathbf{L})$ has always a single maximal (hence the greatest) logic.

References

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