Extensions of Glivenko's theorem to non-commutative substructural logics

Extended abstract

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1 Glivenko's theorem and substructural logics

Glivenko's theorem states that a formula ϕ is provable in classical propositional logic iff its double negation is provable in intuitionistic logic, in symbols $\vdash_{\mathbf{Cl}} \phi$ iff $\vdash_{\mathbf{Int}} \neg\neg\phi$. Nevertheless, $\vdash_{\mathbf{Int}}$ is not the only relation that has this property relative to $\vdash_{\mathbf{Cl}}$, nor is $\vdash_{\mathbf{Cl}}$ the only relation that admits a double negation interpretation in another logic. For example, it is shown in [1] that, for every formula ϕ, $\vdash_{\mathbf{Cl}} \phi$ iff $\vdash_{\mathbf{SBL}} \neg\neg\phi$, and that, for every formula ϕ, $\vdash_{\mathbf{Lu}} \phi$ iff $\vdash_{\mathbf{BL}} \neg\neg\phi$, where **SBL** is the extension of Hajek basic logic **BL** by the axiom $\psi \wedge \neg\psi = \bot$ and **Lu** is Łukasiewicz infinite valued logic. All the logics discussed so far are substructural logics. In what follows, we discuss pairs of substructural logics for which a double negation translation (in one of four different forms) holds. The contents of this extended abstract are part of joint work, [2] and [3], with H. Ono.

We assume familiarity with the Gentzen-style sequent calculus **FL** over sequents in the language $\{\wedge, \vee, \cdot, /, \backslash, 1, 0\}$; for the definition, see for example [2]. Because of the lack of the rule of exchange in **GL**, the language includes two implications $\backslash$ and $/$. This gives rise to two negation connectives $\sim\phi = \phi\backslash 0$ and $-\phi = 0/\phi$ that will be involved in the definition of a Glivenko property between two logics.

If $\Phi \cup \{\psi\}$ is a set of formulas and the sequent $\Rightarrow \psi$ is provable in **GL** from assumptions $\{\Rightarrow \phi \,|\, \phi \in \Phi\}$, then we write $\Phi \vdash_{\mathbf{FL}} \psi$. If $\emptyset \vdash_{\mathbf{FL}} \psi$, then we say that ϕ is *provable* in **FL**. We abuse notation and denote the set of all provable formulas in **FL** also by **FL**. A set of formulas is said to be a *substructural logic*, if it is closed under substitution and under $\vdash_{\mathbf{FL}}$. The next theorem shows that modus ponens (mp_ℓ) is not enough for capturing deducibility in **FL**.

Theorem 1.1. *A set of formulas* **L** *is a substructural logic iff it is closed under substitution and under the following rules*

(*fl*)	$\mathbf{FL} \subseteq \mathbf{L}$.
(mp_ℓ)	*If* $\phi, \phi\backslash\psi \in \mathbf{L}$, *then* $\psi \in \mathbf{L}$.
(*adju*)	*If* $\phi \in \mathbf{L}$, *then* $\phi \wedge 1 \in \mathbf{L}$.
(*pn*)	*If* $\phi \in \mathbf{L}$, *then* $\psi\backslash\phi\psi, \psi\phi/\psi \in \mathbf{L}$.

Given a substructural logic **L** we define $\Phi \vdash_{\mathbf{L}} \psi$ iff $\Phi \cup \mathbf{L} \vdash_{\mathbf{FL}} \psi$.

A *(pointed) residuated lattice*, is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid and multiplication is residuated with respect to the order by the division operations $\backslash$, $/$; i.e., for all $a, b, c \in A$, $a \cdot b \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a\backslash c$. Note that 0 is an arbitrary element of A. The class of all residuated lattices forms a variety, which we denote by $\mathcal{RL}$.

If $\mathcal{K}$ is a class of algebras, $E = \{s_i \approx t_i \,|\, i \in I\}$ is a set of equations and $s \approx t$ is an equation, we define $E \models_{\mathcal{K}} s \approx t$, iff, for all $\mathbf{A} \in \mathcal{K}$ and every assignment $\bar{a}$ in $\mathbf{A}$ for the variables $\bar{x}$, if $\mathbf{A} \models s_i(\bar{a}) = t_i(\bar{a})$, for all $i \in I$, then $\mathbf{A} \models s(\bar{a}) = t(\bar{a})$.

For every class $\mathcal{K}$ of pointed residuated lattices and for every set Φ of formulas, let $\mathbf{L}(\mathcal{K}) = \{\phi \,|\, \mathcal{K} \models 1 \leq \phi\}$ and $\mathsf{V}(\Phi) = \mathrm{Mod}(\{1 \leq \phi \,|\, \phi \in \Phi\})$. Moreover, if Σ is a set of formulas and E is a set of equations, we define the set of equations $Eq(\Sigma) = \{1 \leq \phi \,|\, \phi \in \Sigma\}$, and the set of formulas $Fm(E) = \{t\backslash s \wedge s\backslash t \,|\, (t \approx s) \in E\}$.

Theorem 1.2.

1. *For every $\mathcal{K} \subseteq \mathcal{RL}$, $\mathbf{L}(\mathcal{K})$ is a substructural logic and for every set of formulas Φ, $\mathsf{V}(\Phi)$ is a subvariety of $\mathcal{RL}$. Moreover, the maps $\mathbf{L} : \mathbf{S}(\mathcal{RL}) \to \mathbf{SL}$ and $\mathsf{V} : \mathbf{SL} \to \mathbf{S}(\mathcal{RL})$ are mutually inverse, dual lattice isomorphisms between the lattice $\mathbf{S}(\mathcal{RL})$ of varieties of residuated lattices and the lattice $\mathbf{SL}$ of substructural logics.*

2. *If a substructural logic $\mathbf{L}$ is axiomatized by a set of formulas Φ, then the variety $\mathsf{V}(\mathbf{L})$ is axiomatized by the set of equations $Eq(\Phi)$. Also, if a subvariety $\mathcal{V}$ of $\mathcal{RL}$ is axiomatized by a set of equations E, then the substructural logic $\mathbf{L}(\mathcal{V})$ is axiomatized by the set of formulas $Fm(E)$.*

3. *If $\Sigma \cup \{\phi\}$ is a set of formulas and $\mathbf{L}$ is a substructural logic, then*

$$\Sigma \vdash_{\mathbf{L}} \phi \text{ iff } Eq(\Sigma) \models_{\mathsf{V}(\mathbf{L})} 1 \leq \phi; \text{ also } \phi \dashv\vdash_{\mathbf{L}} 1\backslash\phi \wedge \phi\backslash 1.$$

4. *If $E \cup \{t \approx s\}$ is a set of equations and $\mathcal{V}$ is a subvariety of $\mathcal{RL}$, then*

$$E \models_{\mathcal{V}} t \approx s \text{ iff } Fm(E) \vdash_{\mathbf{L}(\mathcal{V})} t\backslash s \wedge s\backslash t; \text{ also } s \approx t \;=\!\mid\models_{\mathcal{V}} s\backslash t \wedge t\backslash s \approx 1.$$

An *iterated conjugate* is a composition of polynomials of the form $\lambda_a(x) = a\backslash xa \wedge 1$ and $\rho_b(x) = bx/b \wedge 1$, for various values of a and b. For example, $\gamma(x) = a\backslash(b\backslash(cx/c \wedge 1)b \wedge 1)a \wedge 1$ is an iterated conjugate. The following theorem is a weak version of the classical deduction theorem and is called *parameterized local deduction theorem*. We make use of it in the proofs of subsequent theorems.

Theorem 1.3. *If $\Sigma \cup \Delta \cup \{\phi\}$ is a set of formulas and $\mathbf{L}$ is a substructural logic, then $\Sigma, \Delta \vdash_{\mathbf{L}} \phi$ iff $\Sigma \vdash_{\mathbf{L}} (\prod_{i=1}^{n} \gamma_i(\psi_i))\backslash\phi$, for some non-negative integer n, iterated conjugates γ_i and $\psi_i \in \Delta$, $i \leq n$.*

2 Glivenko properties and Glivenko equivalence

Let $\mathbf{L}$ and $\mathbf{K}$ be substructural logics. We consider the following properties for $\mathbf{K}$ relative to $\mathbf{L}$.

- Glivenko property: For all formulas ϕ, $\vdash_{\mathbf{L}} \phi$ iff $\vdash_{\mathbf{K}} -\!\sim\!\phi$ iff $\vdash_{\mathbf{K}} \sim\!-\phi$.
- Deductive Glivenko property: For all sets of formulas $\Sigma \cup \{\phi\}$ (for $-\!\sim\!\Sigma = \{-\!\sim\!\sigma \mid \sigma \in \Sigma\}$),

$$\Sigma \vdash_{\mathbf{L}} \phi \text{ iff } -\!\sim\!\Sigma \vdash_{\mathbf{K}} -\!\sim\!\phi \text{ iff } \sim\!-\Sigma \vdash_{\mathbf{K}} \sim\!-\phi.$$

- Equational Glivenko property: For all equations $s \approx t$.

$$\models_{\mathsf{V}(\mathbf{L})} s \approx t \text{ iff } \models_{\mathsf{V}(\mathbf{K})} -\!\sim\! s \approx -\!\sim\! t \text{ iff } \models_{\mathsf{V}(\mathbf{K})} \sim\!- s \approx \sim\!- t$$

- Deductive equational Glivenko property: For all sets of equations $E \cup \{s \approx t\}$ (we define $-\!\sim\! E = \{-\!\sim\! u \approx -\!\sim\! v \mid (u \approx v) \in E\}$),

$$E \models_{\mathsf{V}(\mathbf{L})} s \approx t \text{ iff } -\!\sim\! E \models_{\mathsf{V}(\mathbf{K})} -\!\sim\! s \approx -\!\sim\! t \text{ iff } \sim\!- E \models_{\mathsf{V}(\mathbf{K})} \sim\!- s \approx \sim\!- t,$$

It will follow from Theorems 3.1, 3.2 and 3.3 that the last two properties are equivalent and they imply the second property, which in turn implies the first.

We say that the substructural logics $\mathbf{L}$ and $\mathbf{K}$ are *Glivenko equivalent* if for all formulas ϕ,

$$\vdash_{\mathbf{K}} \sim\!\phi \text{ iff } \vdash_{\mathbf{L}} \sim\!\phi.$$

Obviously, Glivenko equivalence is an equivalence relation on the set of all substructural logics. The following theorem shows that Glivenko equivalence is a notion that is stable and related to all four Glivenko properties.

Theorem 2.1. *Let* $\mathbf{L}$ *and* $\mathbf{K}$ *be substructural logics. The following statements are equivalent.*

1. $\mathbf{L}$ and $\mathbf{K}$ are Glivenko equivalent
2. $\vdash_{\mathbf{K}} -\phi$ iff $\vdash_{\mathbf{L}} -\phi$
3. $\Phi \vdash_{\mathbf{K}} \sim\phi$ iff $\Phi \vdash_{\mathbf{L}} \sim\phi$
4. $\sim\Phi \vdash_{\mathbf{K}} \sim\phi$ iff $\sim\Phi \vdash_{\mathbf{L}} \sim\phi$
5. $\vdash_{\mathsf{V}(\mathbf{K})} \sim\phi \approx \sim\psi$ iff $\vdash_{\mathsf{V}(\mathbf{L})} \sim\phi \approx \sim\psi$
6. $E \vdash_{\mathsf{V}(\mathbf{K})} \sim\phi \approx \sim\psi$ iff $E \vdash_{\mathsf{V}(\mathbf{L})} \sim\phi \approx \sim\psi$
7. $\sim E \vdash_{\mathsf{V}(\mathbf{K})} \sim\phi \approx \sim\psi$ iff $\sim E \vdash_{\mathsf{V}(\mathbf{L})} \sim\phi \approx \sim\psi$

It can be shown that each Glivenko equivalence class is convex. The next theorem shows that it is actually a bounded interval in the lattice $\mathbf{SL}$ of all substructural logics.

Let $\mathbf{L}$ be a substructural logic and let Γ denote the set of all iterated conjugates. We define the logics

$$\mathbf{G}(\mathbf{L}) = \mathbf{FL} + \{-\sim\phi \mid \phi \in \mathbf{L}\} \quad \text{and} \quad \mathbf{M}(\mathbf{L}) = \mathbf{FL} + \{\phi \mid -\sim\gamma(\phi) \in \mathbf{L}, \text{ for every } \gamma \in \Gamma\}.$$

Theorem 2.2. *For every substructural logic* $\mathbf{L}$*, the logic* $\mathbf{G}(\mathbf{L})$ *is the smallest and* $\mathbf{M}(\mathbf{L})$ *is the greatest element of the Glivenko equivalence class of* $\mathbf{L}$*.*

Proposition 2.3. *If an involutive logic* $\mathbf{L}$ *is axiomatized by a set of formulas* Φ*, then* $\mathbf{G}(\mathbf{L})$ *is axiomatized by the formulas* $\{\sim-\phi \mid \phi \in \Phi\} \cup \{(\sim(\phi \star \psi))/(\sim(-\sim\phi \star -\sim\psi))\}$*, where* $\star \in \{\wedge, \backslash, /, \cdot\}$*.*

3 Involutiveness and correspondence with Glivenko properties

We define three degrees of involutiveness for a substructural logic $\mathbf{L}$ and show that these correspond to the Glivenko properties.

- $\mathbf{L}$ is called *involutive*, if, for every ϕ, $\vdash_{\mathbf{L}} (\sim-\phi)\backslash\phi$ and $\vdash_{\mathbf{L}} (-\sim\phi)\backslash\phi$.
- $\mathbf{L}$ is *weakly involutive*, if, for every ϕ, $(\sim-\phi) \vdash_{\mathbf{L}} \phi$ and $(-\sim\phi) \vdash_{\mathbf{L}} \phi$.
- $\mathbf{L}$ is *Glivenko involutive*, if, for every ϕ, $\vdash_{\mathbf{L}} (\sim-\phi)$ implies $\vdash_{\mathbf{L}} \phi$, and $\vdash_{\mathbf{L}} (-\sim\phi)$ implies $\vdash_{\mathbf{L}} \phi$.

Clearly, involutiveness is the strongest and Glivenko involutiveness is the weakest among the three properties. The three notions are distinct, as the logic $\mathbf{FL}_e$ is Glivenko involutive, but not weakly involutive, and $\mathbf{FL}_e + (\sim\sim p)^2 \to p$ is weakly involutive, but not involutive.

Theorem 3.1. *If* $\mathbf{L}$ *and* $\mathbf{K}$ *are substructural logics, then the following are equivalent.*

1. *The Glivenko property holds for* $\mathbf{K}$ *relative to* $\mathbf{L}$*.*
2. $\mathbf{K}$ *and* $\mathbf{L}$ *are Glivenko equivalent and* $\mathbf{L}$ *is Glivenko involutive.*
3. $\mathbf{L} = \mathbf{M}(\mathbf{K})$ *and* $\mathbf{M}(\mathbf{K})$ *is Glivenko involutive.*

Theorem 3.2. *If* $\mathbf{L}$ *and* $\mathbf{K}$ *are substructural logics and* $\Phi \cup \{\psi\}$ *are formulas, then the following are equivalent.*

1. *The deductive Glivenko property holds for* $\mathbf{K}$ *relative to* $\mathbf{L}$*.*
2. $\Phi \vdash_{\mathbf{L}} \psi$ *iff* $\Phi \vdash_{\mathbf{K}} \sim-\psi$ *iff* $\Phi \vdash_{\mathbf{K}} -\sim\psi$*.*
3. $\mathbf{K}$ *and* $\mathbf{L}$ *are Glivenko equivalent and* $\mathbf{L}$ *is weakly involutive.*
4. $\mathbf{L} = \mathbf{M}(\mathbf{K})$ *and* $\mathbf{M}(\mathbf{K})$ *is weakly involutive.*

Theorem 3.3. *Let* $\mathbf{L}$ *and* $\mathbf{K}$ *be substructural logics Then, the following statements are equivalent.*

1. *The equational Glivenko property holds for* $\mathbf{K}$ *relative to* $\mathbf{L}$.
2. *The deductive equational Glivenko property holds for* $\mathbf{K}$ *relative to* $\mathbf{L}$.
3. $\mathbf{K}$ *and* $\mathbf{L}$ *are Glivenko equivalent and* $\mathbf{L}$ *is involutive.*
4. $\mathbf{L} = \mathbf{M}(\mathbf{K})$ *and* $\mathbf{M}(\mathbf{K})$ *is involutive.*

It follows from the discussion on involutiveness, that the Glivenko property holds for $\mathbf{FL}_e$, but the deductive Glivenko property fails; also the deductive Glivenko property holds for the logic $\mathbf{FL}_e + (\sim\sim p)^2 \to p$, but the equational Glivenko property fails. For involutive logics, though, all three properties become equivalent; i.e., if $\mathbf{L}$ is involutive, then some property holds for a logic $\mathbf{K}$ relative to $\mathbf{L}$ iff all properties hold for $\mathbf{K}$ relative to $\mathbf{L}$.

Let $\mathbf{In}(\mathbf{L}) = \mathbf{L} + \{(\sim - \phi)\backslash\phi, (-\sim\phi)\backslash\phi\}$ and $\mathbf{G} = \mathbf{G}(\mathbf{In}(\mathbf{FL}))$. The following theorem shows that either all logics in a Glivenko equivalence class contain $\mathbf{G}$ or none does. The theorem provides a characterization for the case when a Glivenko property holds for an logic relative to an involutive logic.

Theorem 3.4. *The following are equivalent.*

1. $\mathbf{L}$ *is an extension of* $\mathbf{G}$.
2. $\mathbf{G}(\mathbf{L})$ *is an extension of* $\mathbf{G}$.
3. $\mathbf{M}(\mathbf{L})$ *is an extension of* $\mathbf{G}$.
4. $\mathbf{M}(\mathbf{L})$ *is an extension of* $\mathbf{In}(\mathbf{FL})$.
5. $\mathbf{M}(\mathbf{L}) = \mathbf{In}(\mathbf{L})$.
6. *The (equational) Glivenko property holds for* $\mathbf{L}$ *relative to* $\mathbf{In}(\mathbf{L})$.
7. *The equational Glivenko property holds for* $\mathbf{L}$ *relative to some logic.*

4 The case of classical logic

Theorem 4.1. $\mathbf{G}(\mathbf{Cl})$ *is axiomatized by the following formulas.*

$$\sim(\phi \wedge \psi) \leftrightarrow \sim(\phi\psi), \qquad \sim(\phi\backslash\psi) \leftrightarrow \sim(-\phi \vee \psi), \qquad -(\phi\backslash\psi) \leftrightarrow -(\sim\phi \vee \psi),$$
$$\sim(\phi\backslash\psi) \leftrightarrow \sim(-\sim\phi\backslash -\sim\psi) \quad \text{and} \quad \sim(\phi/\psi) \leftrightarrow \sim(-\sim\phi/-\sim\psi).$$

Corollary 4.2. *The logic* $\mathbf{G}(\mathbf{Cl})$ *does not admit any structural rule (exchange, contraction, weakening).*

Proof. For every bounded residuated lattice $\mathbf{A}$, consider the residuated lattice $\mathbf{A}'$, that is obtained by appending to $\mathbf{A}$ a new bottom element $\perp$ and setting $0 = \perp$. It is easy to see that $\mathbf{A}' \in \mathsf{V}(\mathbf{G}(\mathbf{Cl}))$, but $\mathbf{A}'$ is neither commutative, nor contractive, nor integral. □

Corollary 4.3. *The logic* $\mathbf{G}(\mathbf{Cl})+\mathbf{FL_{ew}}$ *is axiomatized relative to* $\mathbf{FL_{ew}}$ *by the formula:* $\sim\sim(\sim\sim\phi \to \phi)$ *and either one of the formulas*

$$\sim(\phi \wedge \psi) \leftrightarrow \sim(\phi\psi), \qquad \sim(\phi \wedge \sim\phi) \text{ and } \qquad \sim(\phi^2) \leftrightarrow \sim\phi.$$

Corollary 4.4. *The (equational) Glivenko property holds for* $\mathbf{Int}$ *relative to* $\mathbf{Cl}$.

References

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