Contentwise Complexity of Inferences based on Epistemic Logics of Shallow Depths in Game Theoretical Situations*

— An Introduction for MLG —

Mamoru Kaneko† and Nobu-Yuki Suzuki‡

Introduction

This extended abstract is to introduce a logical study of game theoretical problem with a focus on *bounded rationality* in game theory. This interdisciplinary field has many aspects, which can be investigated from various research lines. Here, we will take one specific approach which has been developed in Kaneko-Suzuki [3, 4, 5], and will report some new concepts and results along that approach.

Since game theory intends to treat behavior and decisions of people in social contexts, it naturally includes limitations of people' s cognitive abilities, which is so-called "bounded rationality". In fact, "bounded rationality" is not a single notion, but has many different aspects. In the literature of game theory and economics, however, the problem is not well specified. The present game theory does not have enough vocabulary in its mathematical formulation to talk about bounded aspects of cognitive abilities of players. The approach initiated in Kaneko-Suzuki [3, 4, 5] has enough vocabulary for it. In this article, particularly, we will introduce a measure of logical inferences required for some decision making in a game. More generally, it provides a measure of how large a mathematical proof is.

In particular, we will introduce the *contentwise complexity* measure, which measures a size of a proof or a size of required inferences. This will help us study decision making of a player with limitations on logical abilities.

In §1, we present our basic motivation of our reseach line. In §2, we briefly explain what a game means in this article. Game theoretic motivation of "bounded rationality" is shown in §3. Our main technical feature, *contentwise complexity measure*, is given in §4. This tool is properly defined in a system of intuitionistic epistemic (multi-modal) logic IG_{EF}. We will give concluding remarks in §5.

1 Basic Motivation of Our Research

The game theory may be described as a scientific (or, mathematically sophisticated) study of behavior of people in iteractive situations. Those people are called *players* of a game, and game theorists typically assume that players are all "rational". On the other hand, logic may be described as a scientific study of reasoning and inferences of rational human beings. We focus on the deductive reasoning as an important facet of rationality, and link two disciplines with the word "rationality". In particular, *epistemic* logics deal with epistemic notions such as beliefs and knowledge. Given a game with multiple players, we have an interactive situation in which those players think about other players' beliefs and/or knowledges as well as their own. Our present approach is a logical investigation of epistemic reasonigs and inferences of players in such a game-theoretical iteractive situation.

Although we take the logical approach and borrow basic notions from epistemic logics, our epistemic-logical approach is not simple applications of old results and techniques in the field of logic. Considerations of game theoretical phenomena in terms of epistemic logic supply genuinely new problems in logic itself. Our interactive studies between logic and game theory must be beneficial to both fields. Some of such interactive discussions can be found in Kaneko [2] as a drama.

---

*The authors are partially supported by Grant-in-Aids for Scientific Research No.16340022, Japan Society for the Promotion of Science.

†Institute of Policy and Planning Sciences, University of Tsukuba, e-mail: kaneko@shako.sk.tsukuba.ac.jp

‡Department of Mathematics, Faculty of Science, Shizuoka University, e-mail: smnsuzu@ipc.shizuoka.ac.jp

2 Games and Players

In this section, we briefly explain what a *game* means in this article. We will illustrate some basic concepts by giving examples. Note that this section is not intended to be an introduction of game theory. Readers who intened to learn basics of game theory are recommended to consult good text books e.g. [7].

Let us take a look at Tabel 1.1 and Table 1.2 which describe two games g^1 and g^2.

Table 1.1: $g^1 = (g^1_1, g^1_2)$

	s_{21}	s_{22}
s_{11}	$(5,5)$	$(1,6)$
s_{12}	$(6,1)$	$(3,3)$

Table 1.2: $g^2 = (g^2_1, g^2_2)$

	s_{21}	s_{22}
s_{11}	$(5,5)$	$(1,2)$
s_{12}	$(6,1)$	$(3,3)$

These g^1 and g^2 are a two-person games, i.e. we have *players* 1 and 2. Each player $i = 1, 2$ has two *pure strategies* s_{i1}, s_{i2}, and chooses simultaneously one from them. The entries of Tables are pairs of *payoffs* to the players, e.g., if 1 and 2 choose s_{12} and s_{21}, they would receive 6 and 1, respectively. This is expressed also by *payoff functions* $g^i = (g^i_1, g^i_2)$: $g^i_1(\mathsf{s}_{12}, \mathsf{s}_{21}) = 6$ and $g^i_2(\mathsf{s}_{12}, \mathsf{s}_{21}) = 1$.

In g^1, the second strategy, s_{12}, for player 1 gives a better payoff whatever player 2 chooses, and the symmetric argument holds for player 2. In this sense, the second strategy for each player is called a *dominant* strategy. If both players have the *decison criterion DC*: "Choose a dominant strategy", then both players can make their decison. Following DC, player i has only to check that the second strategy is dominant by making use of the information on ones own payoff g^1_i. This is done by two comparisons: $5 < 6$ and $1 < 3$. Then each player infers that DC recommends s_{i2}.

On the other hand, in g^2, player 2 has no dominant strategy, while player 1 has the same dominant strategy. Thus, player 2 cannot make his/her decision by making use of the criterion DC any more. One possible way to have a decision for player 2 is to have a prediction about what player 1 would choose. That is, player 2 would choose s_{22} maximizing his/her payoff under the prediction that player 1 would choose the dominant strategy s_{12}. This criterion is called a *prediction-decision criterion PDC*. Following PDC, player 2 has to think about player 1's mind and player 1's second strategy is dominant, i.e., player 2 have to check $5 < 6$ and $1 < 3$. Next player 2 has to predict player 1' choise, s_{12} with a prediction criterion that player 1 uses DC, and has to check that s_{22} gives the best response to s_{12}.

Thus, players can make their decision in both cases, if they have enough informaton, good criteria, and *rationality*. An important difference between these two cases is that in g^2 player 2 needs *epistemic* inference, i.e., player 2 has to think about player 1's mind, while in g^1 none of them can make their decision without epistemic inferences.

3 Bounded Rationality in Game Theory

As mentioned in §2, the players of g^1 and g^2 can make a decision (choose a strategy), if they are rational. Hence, the assumption that all players are rational can be regarded as reasonable for gametheoretical considerations.[1] However, it is common that most people are, and could be, only partly rational. Let us explain one aspect of such limitation on rationality by giving examples.

Table 3.1: g^3

s_1	s_2	s_3
2	3	1

Table 3.2: g^4

s_1	s_2	s_3	s_4	s_5	s_6	$\dots$	s_{2005}
2	3	1	2	1	2	$\dots$	1

These games g^3 and g^4 are one-person games. In both games, player can make a decision by finding a strategy which gives the largest payoff. Here the decision criterion is simply to maximize the payoff. However, we can see that the decison making in g^4 "costs" more than that in g^3 does. The difference is the cost of inference, i.e., how much inferences are required for the decision making for the player. If a player's resources (time, effort, etc.) for decision making process (inference) is limited, then the player can make decision in g^3 but cannot in g^4.

Our contemplexity measure is applied to each of these problems to measure the required inferences for decision making in each of those problems. The point is that the structures of these games are the

[1] In this line, many game theorists have worked on generalization of the structure of games under the same assumption of the ratoinality of players. Superfacially, they retaine the same assumption. But, in fact, the more the structure of games gets, the more *hyperrational* players are supposed to be. See [2, Act 1].

same and simple. The difference that we want to evaluate is depending on the content of games. In g^3, the player needs at least two comparisons to conclude that $\mathbf{s}_2$ gives the largest payoff, while in g^4 the player needs 2004 comparisons.

4 Contentwise Complexity as a Cost of Inferences

Now we introduse our system IG_{EF}, which is an epistemic (multi-modal) propositional calculus. We formulate our logical system IG_{EF} as sequent calculus in the Gentzen-style [2]. We introduced GL_{EF}, with which we developed a theory for intrapersonal epistemic inferences. The subscripts E and F are called *descriptive* and *inferential epistemic* structures, which are constraints on interpersonal epistemic introspections. The first E constrains the description of formulae or sequents just syntactically, but F constrains how deep a player thinks about other players' minds, in which sense the latter is more important. Our system IG_{EF} is obtained from GL_{EF} by substituting intuitionistic logic as the base logic for classical logic. This substitution is crucial for the development of the theory of the contentwise complexity measure, though the definition of the measure is possible independent of a base logic.

Most important technical features of IG_{EF} (and GL_{EF} as well) are:

1. introducing thought sequents,
2. separating descriptive and inferential epistemic structures,
3. separating epistemic and non-epistemic rules.

All these points are related to each other. Here we briefly explain 1 and 3. For the 2nd point and interralaton between these three points, see [3, 4, 5] for details.

Our propositional language consists of the usual propositional language for sequent calculi plus *epistemic operators* $B_1, B_2, \ldots, B_n$, where $N = \{1, 2, \ldots, n\}$ is the set of all players. The intended meaning of an expression $B_i(A)$ is that player i *believes that* A. Let $e = (i_1, i_2, \ldots, i_m)$ be a finite (possibly empty) sequence of players, Γ, Θ finite sets of formulae. Using auxiliary symbols [and], we introduce a new expression $B_e[\Gamma \to \Theta] := B_{i_1}B_{i_2}\cdots B_{i_m}[\Gamma \to \Theta]$, which we call a *thought sequent*. By $B_{i_1}B_{i_2}\cdots B_{i_m}[\Gamma \to \Theta]$, we express the idea that player i_m the mind of i_{m-1} ... in the mind of i_1 conducts logical reasoning and believes that $\Gamma \to \Theta$. Note that we are working with intuitionistic logic as our base logic. Hence we put a restriction $|\Theta| \leq 1$ in $B_e[\Gamma \to \Theta]$, where $|\Theta|$ is the cardinality of Θ.

In this article, the 3rd point is crucial. The non-epistemic logical reasoning of the innermost player i_m is governed by intuitionistic logic **LJ**. Non-epistemic rules are generally characterized as follows:

Non-Epistemic Rules: If $\dfrac{S_1\ (S_2)}{S}$ is an **LJ**-rule, then $\dfrac{B_e[S_1]\ (B_e[S_2])}{B_e[S]}$ is a IG_{EF}-rule.

That is the innermost player i_m is assumed to be capable of conducting (non-epistemic) logical reasonings described by **LJ**. Thus, the outermost $B_e[\cdots]$ is kept unchanged. On the other hand, epistemic reasoning is described in the following way:

Epistemic Rule (Distribution Rule): $\dfrac{B_eB_i[\Gamma \to \Theta]}{B_e[B_i(\Gamma) \to B_i(\Theta)]}\ (B_i \to B_i)$, subject to $|\Theta| \leq 1$.

Note that IG_{EF} has only one epistemic rule, and that any application of Epistemic Rule must leave its fingerprint on the outer B_e of thought sequents. Hence we can inpose constraints on epistemic inferences by controling e in B_e. See [3, 4, 5] for details.

Now we define the contentwise complexity measure η. Given a proof P in IG_{EF}, we define the *contentwise complexity measure* $\eta(P)$ of P to be the number of occurrences of initial sequents of P. Then we define the contentwise complexity $\eta(B_e[\Gamma \to \Theta])$ of a thought sequent $B_e[\Gamma \to \Theta]$ by

$$\eta(B_e[\Gamma \to \Theta]) = \begin{cases} \min\{\eta(P); P \text{ is a cut-free proof of } B_e[\Gamma \to \Theta]\} & \text{if } B_e[\Gamma \to \Theta] \text{ is provable,} \\ \infty & \text{otherwise.} \end{cases}$$

The contentwise complexity for a given statement is the minimum number of indispensable contents included in the statemane to prove itself. Note that we ask a *minimum* proof of a given statement, that

[2] The cut-elimination theorem holds for IG_{EF}. The cut-elimination theorem is crucial for the development of our present reseach. We can establish Kripke-style semantics as well.

is, the *best* case for the given instance, not the "worst." It measures the contents of a statement from the viewpoint of inferences. If the contentwise complexity of a statement surpasses a given limitation of the logical ability of a player, then the player cannot conclude the statement, even if the statement is correct and theoretically provable.

Let us calculate η of the statements of feasibility of decision making in g^3 and g^4. After somewhat complicated discussion, we have (η in g^3) = 2 and (η in g^4) = 2004. Then we can say that the "cost" of decision making in g^4 is greater than that in g^3. The η captures, at least, qualitative aspects in these examples. (See [6] for details).

5 Concluding Remarks

We introduced the contentwise complexity measure η defined on the framework of an epistemic logic based on intuitionistic logic IG_{EF}. The η captures, at least, qualitative aspects in examples. We will apply the η to measure the required inferences for decision making in a game. There are abundant of examples, in which the complexity values given by η give insights to our study of game theoretical decision making.

There are some open questions, conceptual and technical ones. The question must be *How well does η capture the "complexity of inferences" required from the viewpoint of logic as well as game theory?* First, we should text our mathematical definition of in rather simple examples. Indeed, we considered various relatively simple examples in [6]. Then we go further to investigate more conceptual and technical properties of it. In this manner, we will evaluate the contentwise complexity measure.

The first technical question must be: *Is there any language and/or system dependence?* This question depends strongly on the previous conceptual question. Namely, the way of mathematical expresseion of games and structures of games itself is the point of this question.

Since η is defined in an epistemic logic IG_{EF}, the η must reflect the interactive nature of intrapersonal inferences and interpersonal epistemic introspection. That is, interpersonal introspections would decrease contentwise complexity, while in other cases, it would be more complex. We explain it in the following example. (See [6] for details.)

Table 5.1: g^5

		s_{21}	s_{22}	s_{23}	$\cdots$	$\mathsf{s}_{2\,10}$
s_{11}	200	0	11	10	$\cdots$	3
s_{12}	600	2	3	2	$\cdots$	-5

In g^5, player 2's payoffs are strongly affected by player 1's choice, while player 1's payoffs are not. Note that player 2 has a dominant strategy. Hence player 2 can make decision using DC. The η of this decision making process is 9+9=18. Player 2 can make decision by using PDC as well. In this case, η is much smaller: 1+1+9=11. However, this 1+1 is caused by epistemic inference. At present, we cannot evaluate this epistemic "cost." Here we have one question: *How to evaluate the cost (complexity) of epistemic inferences?* This question is both conceptual and technical.

References

[1] Kaneko, M., *Epistemic logics and their game theoretical applications: Introduction*, **Economic Theory 19** (2002), 7–62.

[2] Kaneko, M., **Game theory and mutual misunderstanding**, Studies in Economic Theory, 21. Springer-Verlag, Berlin, 2005.

[3] Kaneko, M. and N.-Y. Suzuki, *Bounded interpersonal inferences and decision making*, **Economic Theory** 19 (2002), 63–103.

[4] Kaneko, M. and N.-Y. Suzuki, *Epistemic logic of shallow depths and game-theoretical applications*, **Advances in Modal Logic Vol.3** (2002), F. Wolter et al eds. World Scientific Publishing Co. London. 279–298.

[5] Kaneko, M. and N.-Y. Suzuki, *Epistemic models of shallow depths and game theoretical decision making: Horticulture*, **Journal of Symbolic Logic** 68 (2003), 163–186.

[6] Kaneko, M. and N.-Y. Suzuki, *Contentwise Complexity of Inferences in Epistemic Logics of Shallow Depths I: General Development*, Preprint.

[7] Myerson, R. B., **Game Theory**, Harvad University Press, Cambridge, 1991.