# **GL**-provability of $\bot$-formulas in **Grz**

Katsumi Sasaki
sasaki@ms.nanzan-u.ac.jp
Department of Mathematical Sciences,
Faculty of Mathematical Sciences and Information Engineering, Nanzan University

Here we discuss a relation between the provability of Grzegorczyk logic **Grz** and the provability of provability logic **GL**. It was defined a function $f$ satisfying, for any formula $A$, $A \in$ **Grz** if and only if $f(A) \in$ **GL** (cf. Boolos [Boo93] and Goldblatt [Gol78]). While we define a function $g$ satisfying, for any formula $A$ having no propositional variable, $g(A) \in$ **Grz** if and only if $A \in$ **GL**.

## 1 Introduction

We use lower case Latin letters $p, q, \cdots$ for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\bot$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation).

**Definition 1.1.** The depth $d(A)$ of a formula $A$ is defined inductively as follows:
(1) $d(D) = 0$, for an atomic formula $D$,
(2) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
(3) $d(\Box B) = d(B) + 1$.

Let $D$ be an atomic formula in $\{p, \bot\}$. By $\mathbf{S}(D)$, we mean the set of formulas constructed from $D$ by using $\wedge$, $\vee$, $\supset$ and $\Box$. We put $\mathbf{S}^n(D) = \{B \in \mathbf{S}(D) \mid d(B) \leq n\}$.

By **Grz**, we mean the smallest set of formulas containing all the tautologies and the axioms
$K : \Box(A \supset B) \supset (\Box A \supset \Box B)$,
$T : \Box A \supset A$,
$grz : \Box(\Box(A \supset \Box A) \supset A) \supset \Box A$ (Grzegorczyk axiom),
and closed under modus ponens and necessitation. By **GL**, we mean the smallest set of formulas containing all the tautologies, $K$ and the axiom
$L : \Box(\Box A \supset A) \supset \Box A$ (Löb's axiom),
and closed under modus ponens and necessitation.

**Definition 1.2.** A list $F_0, F_1, \cdots$ of formulas are defined inductively as follows:
(1) $F_0 = p$,
(2) $F_{k+1} = F_k \supset \Box F_k$.

**Definition 1.3.** A list $g_0, g_1, \cdots$ of functions from $\mathbf{S}(\bot)$ to $\mathbf{S}(p)$ are defined inductively as follows:
(1) $g_i(\bot) = \Box F_i$,
(2) $g_i(B \wedge C) = g_i(B) \wedge g_i(C)$,
(3) $g_i(B \vee C) = g_i(B) \vee g_i(C)$,
(4) $g_i(B \supset C) = g_i(B) \supset g_i(C)$,
(5) $g_i(\Box B) = \Box g_{i+1}(B)$.

The function $g_0$ transforms the formula $\Box(\Box\bot \supset \bot) \supset \Box\bot$, a instance of the axiom $L$, into a formula

$$g_0(\Box(\Box\bot \supset \bot) \supset \Box\bot) = \Box(\Box\Box(F_1 \supset \Box F_1) \supset \Box F_1) \supset \Box\Box F_1).$$

Here we note that the image is similar to $\Box(\Box(F_1 \supset \Box F_1) \supset F_1) \supset \Box F_1$, an instance of the axiom $grz$, and that the image and the instance are equivalent in **Grz**.

The main result is

**Theorem 1.4.** For any formula $A \in \mathbf{S}(\bot)$,

$$A \in \mathbf{GL} \text{ if and only if } g_i(A) \in \mathbf{Grz}.$$

To prove the theorem, we use properties of the structures $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}} \rangle$ and $\langle \mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}} \rangle$, where for $\mathbf{L} \in \{\mathbf{Grz}, \mathbf{GL}\}$,
$A \equiv_{\mathbf{L}} B$ if and only if $(A \supset B) \wedge (B \supset A) \in \mathbf{L}$,
$[A] \leq_{\mathbf{L}} [B]$ if and only if $B \supset A \in \mathbf{L}$.

In the next section, we construct a representative of each equivalent class of the above two structures following [Boo93] and [Sas04]. In section 3, we show an outline of a proof of the theorem using the lemmas in section 2.

## 2 Construction of representatives

Here we construct a representative of each equivalent class in the quotient sets $\mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}$ and $\mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}}$. It is known, however, two structures $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}} \rangle$ and $\langle \mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}} \rangle$ are boolean(cf. Chagrov and Zakharyaschev [CZ97]). So, we have only to construct representatives of generators of these two booleans. Representatives of generators of the structure for **Grz** was given in [Sas04] and those for **GL**, we can refer [Boo93].

**Definition 2.1.** For a formula $A$, $\Box^n A$ $(n = 0, 1, \cdots)$ are defined inductively as follows:
(1)$\Box^0 A = A$,
(2)$\Box^{k+1} A = \Box\Box^k A$.

**Definition 2.2.** The sets $\mathbf{G}_n$ $(n = 0, 1, 2, \cdots)$ of formulas are defined as follows:
$\mathbf{G}_0 = \{F_0\}$,
$\mathbf{G}_1 = \{F_0, F_1\}$,
$\mathbf{G}_{k+2} = \{F_{k+1}, F_{k+2}, \Box F_{k+1} \supset \Box F_k, \cdots, \Box F_1 \supset \Box F_0\}$

**Lemma 2.3.**
(1) $\mathbf{S}^n(p)/\equiv_{\mathbf{Grz}} = \{[\bigwedge_{A \in \mathbf{S}} A] \mid \mathbf{S} \subseteq \mathbf{G}_n\}$.
(2) *For subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbf{G}_n$,
(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge_{A \in \mathbf{S}_1} A] \leq_{\mathbf{Grz}} [\bigwedge_{A \in \mathbf{S}_2} A]$,
(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge_{A \in \mathbf{S}_1} A] = [\bigwedge_{A \in \mathbf{S}_2} A]$.

**Definition 2.4.** The sets $\mathbf{G}^*_n$ $(n = 0, 1, 2, \cdots)$ of formulas are defined as follows:
$\mathbf{G}^*_0 = \{\bot\}$,
$\mathbf{G}^*_{k+1} = \{\Box^{k+1}\bot, \Box^{k+1}\bot \supset \Box^k\bot, \cdots, \Box\bot \supset \bot\}$

**Lemma 2.5.**
(1) $\mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}} = \{[\bigwedge_{A \in \mathbf{S}} A] \mid \mathbf{S} \subseteq \mathbf{G}^*_n\}$.
(2) *For subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbf{G}^*_n$,
(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge_{A \in \mathbf{S}_1} A] \leq_{\mathbf{GL}} [\bigwedge_{A \in \mathbf{S}_2} A]$,

(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge_{A\in\mathbf{S}_1} A] = [\bigwedge_{A\in\mathbf{S}_2} A]$.

# 3 An outline of a proof

Here we give an outline of a proof of Theorem 1.4. We define functions $h_i$ and show two lemmas.

**Definition 3.1.** For $\mathbf{S} \in \mathcal{P}(\mathbf{G}_n^*)$, we put

$$h_i(\mathbf{S}) = \{F_{n+i} \mid \Box^n\bot \in \mathbf{S}\} \cup \{F_{n+i+1} \mid \Box^n\bot \in \mathbf{S}\} \cup \bigcup_{k=1}^{n}\{\Box F_{k+i} \supset \Box F_{k+i-1} \mid \Box^k\bot \supset \Box^{k-1}\bot \in \mathbf{S}\}.$$

**Lemma 3.2.** *Let* $\mathbf{S}$ *and* $\mathbf{S}_1$ *be subsets of* $\mathbf{G}_n^*$. *then*
(1)$h_i(\mathbf{S}) \subseteq \mathbf{G}_{n+i+1}$,
(2) $\bigwedge_{B\in\mathbf{S}} g_i(B) \equiv_{\mathbf{Grz}} \bigwedge_{B\in h_i(\mathbf{S})} B$,
(3)$\mathbf{S} \neq \mathbf{S}_1$ *implies* $\bigwedge_{B\in h_i(\mathbf{S})} B \not\equiv_{\mathbf{Grz}} \bigwedge_{B\in h_i(\mathbf{S}_1)} B$.

**Lemma 3.3.** *Let* $A$ *be a formula in* $\mathbf{S}^n(\bot)$ *and let* $\mathbf{S}$ *be a subset of* $\mathbf{G}_n^*$. *Then for any* $i$,

$$A \equiv_{\mathbf{GL}} \bigwedge_{B\in\mathbf{S}} B \text{ if and only if } g_i(A) \equiv_{\mathbf{Grz}} \bigwedge_{B\in\mathbf{S}} g_i(B).$$

Proof. We use an induction on $A$.
Basis($A = \bot$): It is not hard to see that

$$A = \bot \equiv_{\mathbf{GL}} (\Box^n\bot) \wedge (\Box^n\bot \supset \Box^{n-1}\bot) \wedge \cdots \wedge (\Box\bot \supset \bot) = \bigwedge_{B\in\mathbf{G}_n^*} B.$$

and

$$g_i(A) = g_i(\bot) = \Box F_i \equiv_{\mathbf{Grz}} \Box F_{n+i} \wedge (\Box F_{n+i} \supset \Box F_{n+i-1}) \wedge \cdots \wedge (\Box F_{i+1} \supset \Box F_i)$$
$$\equiv_{\mathbf{Grz}} F_{n+i} \wedge F_{n+i+1} \wedge (\Box F_{n+i} \supset \Box F_{n+i-1}) \wedge \cdots \wedge (\Box F_{i+1} \supset \Box F_i) \equiv_{\mathbf{Grz}} \bigwedge_{C\in h_i(\mathbf{G}_n^*)} C.$$

So, using Lemma 3.2, if $\mathbf{S} = \mathbf{G}_n^*$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge_{B\in\mathbf{S}} B$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge_{B\in\mathbf{S}} g_i(B)$; if not, neither.

Induction step($A \neq \bot$): We only show the case that $A = \Box A_1$: By Lemma 2.3, there exists a subset $\mathbf{S}_1$ of $\mathbf{G}_{n-1}^*$ such that

$$A_1 \equiv_{\mathbf{GL}} \bigwedge_{B\in\mathbf{S}_1} B.$$

Using the induction hypothesis, we have for any $k$,

$$g_k(A_1) \equiv_{\mathbf{Grz}} \bigwedge_{B\in\mathbf{S}_1} g_k(B).$$

If $\mathbf{S}_1 \subseteq \{\Box^{n-1}\bot\}$, then the lemma is not so difficult. Here we suppose that $\mathbf{S}_1 \not\subseteq \{\Box^{n-1}\bot\}$. Then we have $\mathbf{S}_1 - \{\Box^{n-1}\bot\} \neq \emptyset$. Since $\mathbf{S}_1 \subseteq \mathbf{G}_{n-1}^*$, we have $\emptyset \neq \mathbf{S}_1 - \{\Box^{n-1}\bot\} \subseteq \{\Box^{n-1}\bot \supset \Box^{n-1}\bot, \cdots, \Box\bot \supset \bot\}$. So, there exists the minimum $k = \min\{\ell \mid \Box^\ell\bot \supset \Box^{\ell-1} \in \mathbf{S}_1\}$. Hence we have

$$A = \Box A_1 \equiv_{\mathbf{GL}} \Box \bigwedge_{B\in\mathbf{S}_1} B \equiv_{\mathbf{GL}} \bigwedge_{B\in\mathbf{S}_1} \Box B \equiv_{\mathbf{GL}} \bigwedge_{C\in\{\Box B \mid B\in\mathbf{S}_1\}} C$$

$$\equiv_{\mathbf{GL}} \bigwedge_{C\in\{\Box(\Box^{n-1}\bot)|\Box^{n-1}\bot\in\mathbf{S}_1\}\cup\{\Box(\Box^{n-1}\bot\supset\Box^{n-2}\bot)|\Box^{n-1}\bot\supset\Box^{n-2}\bot\in\mathbf{S}_1\}\cup\cdots\cup\{\Box(\Box\bot\supset\bot)|\Box\bot\supset\bot\in\mathbf{S}_1\}} C$$

$$\equiv_{\mathbf{GL}} \bigwedge_{C\in\{\Box^n\bot|\Box^{n-1}\bot\in\mathbf{S}_1\}\cup\{\Box^{n-1}\bot|\Box^{n-1}\bot\supset\Box^{n-2}\bot\in\mathbf{S}_1\}\cup\cdots\cup\{\Box\bot|\Box\bot\supset\bot\in\mathbf{S}_1\}} C$$

$$\equiv_{\mathbf{GL}} \bigwedge_{C\in\{\Box^n\bot|\Box^{n-1}\bot\in\mathbf{S}_1\}\cup\{\Box^{n-1}\bot|\Box^{n-1}\bot\supset\Box^{n-2}\bot\in\mathbf{S}_1\}\cup\cdots\cup\{\Box^k\bot|\Box^k\bot\supset\Box^{k-1}\bot\in\mathbf{S}_1\}} C$$

$$\equiv_{\mathbf{GL}} \bigwedge_{C\in\{\Box^k\bot\}} C \equiv_{\mathbf{GL}} \bigwedge_{C\in\{\Box^n\bot,\Box^n\bot\supset\Box^{n-1}\bot,\cdots,\Box^{k+1}\bot\supset\Box^k\bot\}} C$$

and

$$g_i(A) = g_i(\Box A_1) = \Box g_{i+1}(A_1) \equiv_{\mathbf{Grz}} \Box \bigwedge_{B\in\mathbf{S}_1} g_{i+1}(B) \equiv_{\mathbf{Grz}} \bigwedge_{B\in\mathbf{S}_1} \Box g_{i+1}(B)$$

$$\equiv_{\mathbf{Grz}} \bigwedge_{C\in\{\Box g_{i+1}(\Box^{n-1}\bot)|\Box^{n-1}\bot\in\mathbf{S}_1\}\cup\{\Box g_{i+1}(\Box^{n-1}\bot\supset\Box^{n-2}\bot)|\Box^{n-1}\bot\supset\Box^{n-2}\bot\in\mathbf{S}_1\}\cup\cdots\cup\{\Box g_{i+1}(\Box\bot\supset\bot)|\Box\bot\supset\bot\in\mathbf{S}_1\}} C$$

$$\equiv_{\mathbf{Grz}} \bigwedge_{C\in\{\Box\Box^n F_{n+i}|\Box^{n-1}\bot\in\mathbf{S}_1\}\cup\{\Box(\Box^n F_{n+i}\supset\Box^{n-1}F_{n+i-1})|\Box^{n-1}\bot\supset\Box^{n-2}\bot\in\mathbf{S}_1\}\cup\cdots\cup\{\Box(\Box^2 F_{i+2}\supset\Box F_{i+1})|\Box\bot\supset\bot\in\mathbf{S}_1\}} C$$

$$\equiv_{\mathbf{Grz}} \bigwedge_{C\in\{\Box F_{n+i}|\Box^{n-1}\bot\in\mathbf{S}_1\}\cup\{\Box F_{n+i-1}|\Box^{n-1}\bot\supset\Box^{n-2}\bot\in\mathbf{S}_1\}\cup\cdots\cup\{\Box F_{i+1}|\Box\bot\supset\bot\in\mathbf{S}_1\}} C$$

$$\equiv_{\mathbf{Grz}} \bigwedge_{C\in\{\Box F_{n+i}|\Box^{n-1}\bot\in\mathbf{S}_1\}\cup\{\Box F_{n+i-1}|\Box^{n-1}\bot\supset\Box^{n-2}\bot\in\mathbf{S}_1\}\cup\cdots\cup\{\Box F_{k+i}|\Box^k\bot\supset\Box^{k-1}\bot\in\mathbf{S}_1\}} C$$

$$\equiv_{\mathbf{Grz}} \bigwedge_{C\in\{\Box F_{k+i}\}} C \equiv_{\mathbf{Grz}} \bigwedge_{C\in\{F_{n+i},F_{n+i+1},\Box F_{n+i}\supset\Box F_{n+i-1},\cdots,\Box F_{k+i+1}\supset\Box F_{k+i}\}} C$$

$$\equiv_{\mathbf{Grz}} \bigwedge_{C\in h_i(\{\Box^n\bot,\Box^n\bot\supset\Box^{n-1}\bot,\cdots,\Box^{k+1}\bot\supset\Box^k\bot\})} C.$$

So, by Lemma 3.2, if $\mathbf{S} = \{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\}$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge_{B\in\mathbf{S}} B$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge_{B\in\mathbf{S}} g_i(B)$; if not, neither. ⊣

Considering the case that $\mathbf{S} = \emptyset$ in Lemma 3.3, we obtain Theorem 1.4.

# References


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