GL-provability of \(-\)-formulas in \Grz

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Here we discuss a relation between the provability of Grzegorczyk logic \Grz and the provability of provability logic \GL. It was defined a function \(f\) satisfying, for any formula \(A, A \in \Grz\) if and only if \(f(A) \in \GL\) (cf. Boolos [Boo93] and Goldblatt [Gol78]). While we define a function \(g\) satisfying, for any formula \(A\) having no propositional variable, \(g(A) \in \Grz\) if and only if \(A \in \GL\).

1 Introduction

We use lower case Latin letters \(p, q, \cdots\) for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and \(-\) (contradiction) by using logical connectives \(\wedge\) (conjunction), \(\vee\) (disjunction), \(\supset\) (implication) and \(\Box\) (necessitation).

Definition 1.1. The depth \(d(A)\) of a formula \(A\) is defined inductively as follows:
(1) \(d(D) = 0\), for an atomic formula \(D\),
(2) \(d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}\),
(3) \(d(\Box B) = d(B) + 1\).

Let \(D\) be an atomic formula in \{\(p, \bot\}\}. By \(S(D)\), we mean the set of formulas constructed from \(D\) by using \(\wedge, \vee, \supset\) and \(\Box\). We put \(S^n(D) = \{B \in S(D) \mid d(B) \leq n\}\).

By \Grz, we mean the smallest set of formulas containing all the tautologies and the axioms
\[K : \Box(A \supset B) \supset (\Box A \supset \Box B),\]
\[T : \Box A \supset A,\]
\[grz : \Box(\Box(A \supset \Box A) \supset A) \supset \Box A \quad \text{(Grzegorczyk axiom)},\]
and closed under modus ponens and necessitation. By \GL, we mean the smallest set of formulas containing all the tautologies, \(K\) and the axiom
\[L : \Box(\Box A \supset A) \supset \Box A \quad \text{(Löb’s axiom)},\]
and closed under modus ponens and necessitation.

Definition 1.2. A list \(F_0, F_1, \cdots\) of formulas are defined inductively as follows:
(1) \(F_0 = p,\)
(2) \(F_{k+1} = F_k \supset \Box F_k.\)

Definition 1.3. A list \(g_0, g_1, \cdots\) of functions from \(S(\bot)\) to \(S(p)\) are defined inductively as follows:
(1) \(g_0(\bot) = \Box F_1,\)
(2) \(g_i(B \wedge C) = g_i(B) \wedge g_i(C),\)
(3) \(g_i(B \vee C) = g_i(B) \vee g_i(C),\)
(4) \(g_i(B \supset C) = g_i(B) \supset g_i(C),\)
(5) \(g_i(\Box B) = \Box g_{i+1}(B).\)

The function \(g_0\) transforms the formula \(\Box(\Box(\bot \supset \bot) \supset \bot)\), a instance of the axiom \(L\), into a formula
\[g_0(\Box(\Box(\bot \supset \bot) \supset \bot)) = \Box(\Box(\Box(F_1 \supset \Box F_1) \supset \Box F_1) \supset \Box \Box F_1).\]
Here we note that the image is similar to \( \Box (\Box (F_1 \supset \Box F_1) \supset F_1) \supset \Box F_1 \), an instance of the axiom \( grz \), and that the image and the instance are equivalent in \( Grz \).

The main result is

**Theorem 1.4.** For any formula \( A \in S(\bot), \)

\[
A \in GL \text{ if and only if } g_i(A) \in Grz.
\]

To prove the theorem, we use properties of the structures \( (S^n(p)/ \equiv_{Grz}, \leq_{Grz}) \) and \( (S^n(\bot)/ \equiv_{GL}, \leq_{GL}) \), where for \( L \in \{Grz, GL\}, \)

\[
A \equiv_L B \text{ if and only if } (A \supset B) \land (B \supset A) \in L,
\]

\[
[A] \leq_L [B] \text{ if and only if } B \supset A \in L.
\]

In the next section, we construct a representative of each equivalent class of the above two structures following [Boo93] and [Sas04]. In section 3, we show an outline of a proof of the theorem using the lemmas in section 2.

## 2 Construction of representatives

Here we construct a representative of each equivalent class in the quotient sets \( S^n(p)/ \equiv_{Grz} \) and \( S^n(\bot)/ \equiv_{GL} \). It is known, however, two structures \( (S^n(p)/ \equiv_{Grz}, \leq_{Grz}) \) and \( (S^n(\bot)/ \equiv_{GL}, \leq_{GL}) \) are boolean(cf. Chagrov and Zakharyaschev [CZ97]). So, we have only to construct representatives of generators of these two booleans. Representatives of generators of the structure for \( Grz \) was given in [Sas04] and those for \( GL \), we can refer [Boo93].

**Definition 2.1.** For a formula \( A, \Box^n A \) \((n = 0, 1, \cdots)\) are defined inductively as follows:

1. \( \Box^0 A = A \),
2. \( \Box^{k+1} A = \Box \Box^k A \).

**Definition 2.2.** The sets \( G_n \) \((n = 0, 1, 2, \cdots)\) of formulas are defined as follows:

\[
G_0 = \{F_0\}, \\
G_1 = \{F_0, F_1\}, \\
G_{k+2} = \{F_{k+1}, F_{k+2}, \Box F_{k+1} \supset \Box F_k, \cdots, \Box F_1 \supset \Box F_0\}
\]

**Lemma 2.3.**

1. \( S^n(p)/ \equiv_{Grz} = \{[\bigwedge_{A \in S} A] | S \subseteq G_n\} \).
2. For subsets \( S_1 \) and \( S_2 \) of \( G_n\),
   
   (2.1) \( S_1 \subseteq S_2 \) if and only if \([\bigwedge_{A \in S_1} A] \leq_{Grz} [\bigwedge_{A \in S_2} A]\),
   
   (2.2) \( S_1 = S_2 \) if and only if \([\bigwedge_{A \in S_1} A] = [\bigwedge_{A \in S_2} A]\).

**Definition 2.4.** The sets \( G_n^* \) \((n = 0, 1, 2, \cdots)\) of formulas are defined as follows:

\[
G_0^* = \{\bot\}, \\
G_{k+1}^* = \{\Box^{k+1} \bot, \Box^{k+1} \bot \supset \Box^{k} \bot, \cdots, \bot \supset \bot\}
\]

**Lemma 2.5.**

1. \( S^n(\bot)/ \equiv_{GL} = \{[\bigwedge_{A \in S} A] | S \subseteq G_n^*\} \).
2. For subsets \( S_1 \) and \( S_2 \) of \( G_n^*\),
   
   (2.1) \( S_1 \subseteq S_2 \) if and only if \([\bigwedge_{A \in S_1} A] \leq_{GL} [\bigwedge_{A \in S_2} A]\),
   
   (2.2) \( S_1 = S_2 \) if and only if \([\bigwedge_{A \in S_1} A] = [\bigwedge_{A \in S_2} A]\).
(2.2) \( S_1 = S_2 \) if and only if \( \bigcap_{A \in S_1} A = \bigcap_{A \in S_2} A \).

3 An outline of a proof

Here we give an outline of a proof of Theorem 1.4. We define functions \( h_i \) and show two lemmas.

**Definition 3.1.** For \( S \in \mathcal{P}(G^*_n) \), we put

\[
h_i(S) = \{ F_{n+i} \mid \Box^n \bot \in S \} \cup \{ F_{n+i+1} \mid \Box^n \bot \in S \} \cup \bigcup_{k=1}^{n} \{ \Box F_k \Box F_{k+1} \mid \Box^k \bot \in S \}.
\]

**Lemma 3.2.** Let \( S \) and \( S_1 \) be subsets of \( G^*_n \). Then

1. \( h_i(S) \subseteq G_{n+i+1} \),
2. \( \bigwedge_{B \in S} g_i(B) \equiv_{Grz} \bigwedge_{B \in h_i(S)} B \),
3. if \( S \neq S_1 \) implies \( \bigwedge_{B \in h_i(S)} B \neq_{Grz} \bigwedge_{B \in h_i(S_1)} B \).

**Lemma 3.3.** Let \( A \) be a formula in \( S^n(\bot) \) and let \( S \) be a subset of \( G^*_n \). Then for any \( i \),

\[
A \equiv_{GL} \bigwedge_{B \in S} B \text{ if and only if } g_i(A) \equiv_{Grz} \bigwedge_{B \in S} g_i(B).
\]

Proof. We use an induction on \( A \).

Basis (\( A = \bot \)): It is not hard to see that

\[
A = \bot \equiv_{GL} (\Box^n \bot) \land (\Box^n \bot \lor \Box^{n-1} \bot) \land \cdots \land (\Box \bot \lor \bot) = \bigwedge_{B \in G^*_n} B.
\]

and

\[
g_i(A) = g_i(\bot) = \Box F_i \equiv_{Grz} \Box F_{n+i} \land (\Box F_{n+i} \lor \Box F_{n+i-1}) \land \cdots \land (\Box F_i \lor \Box F_{i+1}) \equiv_{Grz} \bigwedge_{C \in h_i(G^*_n)} C.
\]

So, using Lemma 3.2, if \( S = G^*_n \), then we have both of \( A \equiv_{GL} \bigwedge_{B \in S} B \) and \( g_i(A) \equiv_{Grz} \bigwedge_{B \in S} g_i(B) \); if not, neither.

Induction step (\( A \neq \bot \)): We only show the case that \( A = \Box A_1 \): By Lemma 2.3, there exists a subset \( S_1 \) of \( G_{n-1}^* \) such that

\[
A_1 \equiv_{GL} \bigwedge_{B \in S_1} B.
\]

Using the induction hypothesis, we have for any \( k \),

\[
g_k(A_1) \equiv_{Grz} \bigwedge_{B \in S_1} g_k(B).
\]

If \( S_1 \subseteq \{ \Box^{n-1} \bot \} \), then the lemma is not so difficult. Here we suppose that \( S_1 \not\subseteq \{ \Box^{n-1} \bot \} \). Then we have \( S_1 - \{ \Box^{n-1} \bot \} \neq \emptyset \). Since \( S_1 \subseteq G_{n-1}^* \), we have \( \emptyset \neq S_1 - \{ \Box^{n-1} \bot \} \subseteq \{ \Box^{n-1} \bot \lor \Box^{n-1} \bot \lor \cdots \lor \bot \lor \bot \} \). So, there exists the minimum \( k = \min\{ \ell \mid \Box^{n-1} \bot \lor \Box^{n-1} \bot \lor \cdots \lor \bot \lor \bot \} \). Hence we have

\[
A = \Box A_1 \equiv_{GL} \Box \bigwedge_{B \in S_1} B \equiv_{GL} \Box B \equiv_{GL} \bigwedge_{C \in \{ \Box B \mid B \in S_1 \}} C
\]
\[ \equiv_{GL} \quad \bigwedge_{C \in \{0^n \mid \text{n-1} \leq i \leq n \}} C \]

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and

\[ g_i(A) = g_i(\square A_1) = \square g_{i+1}(A_1) \equiv_{Grz} \bigwedge_{B \in S_1} g_{i+1}(B) \equiv_{Grz} \bigwedge_{B \in S_1} \square g_{i+1}(B) \]

\[ \equiv_{Grz} \quad \bigwedge_{C \in \{0^n \mid \text{n-1} \leq i \leq n \}} C \]

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So, by Lemma 3.2, if \( S = \{ \square^n \perp, \square^n \perp \supset \square^{n-1} \perp, \ldots, \square^{k+1} \perp \supset \square^k \perp \} \), then we have both of \( A \equiv_{GL} \bigwedge_{B \in S} B \)

and \( g_i(A) \equiv_{Grz} \bigwedge_{B \in S} g_i(B) \); if not, neither.

Considering the case that \( S = \emptyset \) in Lemma 3.3, we obtain Theorem 1.4.

References


