Predicate Bimodal Logics with Irreflexive Modality

Katsuhiko SANO
ksano@sings.jp
Department of Philosophy and History of Science,
Graduate School of Letters, Kyoto University, Sakyo, Kyoto, 606-8501, Japan.

In this paper, we discuss predicate extensions of bimodal logics with irreflexive modality [5]. We show that our intended predicate extension without Barcan formulas: $(\forall x)\Box A \supset \Box(\forall x)A$, etc., is complete with respect to the class of expanding Kripke frames. Suzuki, however, showed that the usual construction of the canonical model fails [6]. In order to avoid this difficulty, we modify the construction of the canonical model by the idea from [4].

1 Preliminaries

$\equiv$ denotes the equivalence as a symbol string. $\alpha[\beta/x]$ denotes the results of the simultaneous substitution of β for all occurrences of x in α, where α and β are symbol strings.

The *bimodal predicate language* $\mathcal{L}$ is based on the following alphabet: (i) a countably infinite list of *free variables*: a, b, ..., (ii) a countable infinite list of *bound variables*: x, y, ..., (iii) *individual constants*: $\mathbf{c}$, $\mathbf{d}$, ..., (iv) *predicate symbols*: $\mathbf{P}$, $\mathbf{Q}$, ..., (v) the *Boolean logical connectives*: $\sim$, $\supset$, (vi) the *universal quantifier*: $\forall$, (vi) the *necessity operators*: $\Box$, $\blacksquare$. Note that our language does not contain function symbols and the equality symbol for simplicity. Free variables and individual constants are called *terms* (written: $t, s, \dots$). *Formulas* (written: A, B, ...) and closed formulas of $\mathcal{L}$ are defined as usual. Γ, Δ, ... denote sets of formulas.

A *Kripke frame* is a quadruple $\mathfrak{F} = \langle W, R, S, D \rangle$ that satisfies the following conditions (*inclusion requirements*): for any $w, w' \in W$, $wRw' \implies D(w) \subset D(w')$ and $wSw' \implies D(w) \subset D(w')$, where W is a non-empty set of states, R, $S \subset W \times W$, D is a function that assigns $w \in W$ a non-empty set $D(w)$.

Let us introduce some classes of Kripke frames. We denote the class of all Kripke frame by K_0. We define K_1 as the class of all Kripke frames $\mathfrak{F} = \langle W, R, S, D \rangle$ that satisfy $S = (R \cap \neq)$, i.e., $wSw' \iff (wRw'$ and $w \neq w')$ for any $w, w' \in W$.

A valuation V on a Kripke frame $\mathfrak{F}$ is defined as follows: $\mathbf{c}^V \in \bigcup_{w \in W} D(w)$ for an individual constant $\mathbf{c}$; $\mathbf{P}^V_w \subseteq D(w)^n$ for an n-ary predicate symbol $\mathbf{P}$. In the case where $n = 0$, we define $\mathbf{P}^V_w \in \{\mathsf{true}, \mathsf{false}\}$. A pair $\langle \mathfrak{F}, V \rangle$ of a Kripke frame $\mathfrak{F}$ and a valuation V on $\mathfrak{F}$ is called a *Kripke model* (written: $\mathfrak{M}$).

Consider a Kripke model $\mathfrak{M} = \langle W, R, S, D, V \rangle$. We expand the language $\mathcal{L}$ to the language $\mathcal{L}[D]$ that contains *name constants* $\underline{d}$ for any $d \in \bigcup_{w \in W} D(w)$ other than the alphabets of $\mathcal{L}$. Then, we define $\underline{d}^V = d$ for any name constant $\underline{d}$ of $\mathfrak{M}$ (rigorously, here $\underline{d}^V$ must be $\underline{d}^{V_D}$ where V_D is the expansion of V by D).

A formula X of $\mathcal{L}[D]$ *admits the interpretation with respect to w in* $\mathfrak{M} = \langle W, R, S, D, V \rangle$ if for any individual constants $\mathbf{c}$ that occur in X, $\mathbf{c}^V$ belongs to $D(w)$. For a Kripke model $\mathfrak{M} = \langle W, R, S, D, V \rangle$, $w \in W$, and a closed formula X of $\mathcal{L}[D]$ that admits the interpretation with respect to w, a *satisfaction relation* $\mathfrak{M}, w \Vdash X$ (read: 'X is true at w of $\mathfrak{M}$') is defined as follows:

$$\mathfrak{M}, w \Vdash \mathbf{P}(t_1, \dots, t_n) \iff \langle t_1^V, \dots, t_n^V \rangle \in \mathbf{P}^V_w;$$
$$\mathfrak{M}, w \Vdash \sim A \iff \mathfrak{M}, w \nVdash A;$$
$$\mathfrak{M}, w \Vdash A \supset B \iff \mathfrak{M}, w \nVdash A \text{ or } \mathfrak{M}, w \Vdash B;$$
$$\mathfrak{M}, w \Vdash (\forall x) A \iff \mathfrak{M}, w \Vdash A[\underline{d}/x] \text{ for any } d \in D(w);$$
$$\mathfrak{M}, w \Vdash \Box A \iff [wRw' \text{ implies } \mathfrak{M}, x \Vdash A] \text{ for any } w' \in W;$$
$$\mathfrak{M}, w \Vdash \blacksquare A \iff [wSw' \text{ implies } \mathfrak{M}, x \Vdash A] \text{ for any } w' \in W.$$

Definition 1 (Validity)**.** Let $\mathfrak{F}$ be a Kripke frame, A a formula of $\mathcal{L}$, and $a_1, \dots, a_n$ all free variables that occur in A. A is *valid in* $\mathfrak{F}$ (written: $\mathfrak{F} \Vdash A$) if for any valuation V on $\mathfrak{F}$, any w in $\mathfrak{F}$, and any $d_1, \dots, d_n \in D(w)$, if A admits the interpretation with respect to w, then $\langle \mathfrak{F}, V \rangle, w \Vdash A[\underline{d_1}/a_1] \cdots [\underline{d_n}/a_n]$.

Let F be a class of Kripke frame and A a formula of $\mathcal{L}$. A is *valid in* F (written: $\mathsf{F} \Vdash A$) if $\mathfrak{F} \Vdash A$ for any $\mathfrak{F} \in \mathsf{F}$.

Definition 2. *Hilbert Calculus* $\mathbf{K}_{\Box\blacksquare}$ consists of the following set of axiom schemata and rules:

$(A1)$	$A \supset (B \supset A)$	($\Box$-rule)	From A, we may infer $\Box A$
$(A2)$	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	($\blacksquare$1)	$\blacksquare(A \supset B) \supset (\blacksquare A \supset \blacksquare B)$
$(A3)$	$(\sim A \supset \sim B) \supset (B \supset A)$	($\blacksquare$-rule)	From A, we may infer $\blacksquare A$
(MP)	From $A \supset B$ and A, we may infer B	$(P1)$	$(\forall x) A \supset A[t/x]$
($\Box$1)	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	($\forall$-rule)	From $A \supset B[a/x]$, we may infer $A \supset (\forall x) B$ $(*)$

where $(*)$ means the condition that free variable a does not occur in A or B. *Hilbert Calculus* **Ks** consists of the above all schemata, rules, (M1) $\Box A \supset \blacksquare A$, and (M2) $A \wedge \blacksquare A \supset \Box A$. $\vdash$ is defined as usual.

Here we give a derivation of one important theorem (used later): **Ks** $\vdash \blacksquare A \wedge \sim \Box A \supset (\Box B \supset B)$.

1.	$\vdash (B \supset A) \wedge \blacksquare(B \supset A) \supset \Box(B \supset A)$	(M2)
2.	$\vdash \blacksquare A \supset \blacksquare(B \supset A)$	($\blacksquare$1), ($\blacksquare$-rule)
3.	$\vdash (B \supset A) \wedge \blacksquare A \supset \Box(B \supset A)$	1, 2, PC
4.	$\vdash (B \supset A) \wedge \blacksquare A \supset (\Box B \supset \Box A)$	3, ($\Box$1), PC
5.	$\vdash (B \supset A) \wedge \blacksquare A \wedge \Box B \supset \Box A$	4, PC
6.	$\vdash \blacksquare A \wedge \Box B \supset \Box A \vee \sim (B \supset A)$	5, PC
7.	$\vdash \blacksquare A \wedge \Box B \supset \Box A \vee B$	6, PC
8.	$\vdash \blacksquare A \wedge \sim \Box A \supset (\Box B \supset B)$	7, PC

It is known that $\mathbf{K}_{\Box\blacksquare}$ is sound and complete with respect to K_0, i.e., $\mathsf{K}_0 \Vdash A \Longleftrightarrow \mathbf{K}_{\Box\blacksquare} \vdash A$ for any A(for unimodal case, see [2, ch.15]). And, it is easy to check that **Ks** is sound with respect to $\mathsf{K}_1 = \{\langle W, R, S, D \rangle \mid S = (R \cap \neq)\}$. From now on, we focus on the following question: is **Ks** complete with respect to K_1? That is, does $\mathsf{K}_1 \Vdash A$ imply **Ks** $\vdash A$ for any A? We will give a positive answer to this question(Theorem 14).

2 Bulldozing in Predicate Modal Logics

In [5], we use the notion of p-morphism to prove completeness for propositional part of **Ks**. Let us introduce the appropriate notion of p-morphism matched to our predicate setting (for intuitionistic predicate logic, see [3]).

Definition 3 (p-Morphism Pair)**.** Let $\mathfrak{F} = \langle W, R, S, D \rangle$ and $\mathfrak{F}' = \langle W', R', S', D' \rangle$ be Kripke frames. Let $f : W \to W'$ and $g : \bigcup_{w \in W} D(w) \to \bigcup_{w' \in W'} D'(w')$ be mappings. $\langle f, g \rangle$ is a *p-morphism pair* if f and g satisfy:

xRy implies $f(x)R'f(y)$. (R-forth)

$f(x)R'y'$ implies that there exists $y \in W$ such that $f(y) = y'$ and xRy. (R-back)

(S-forth) and (S-back) conditions defined similarly.

$g[D(w)] = D'(f(w))$.

$\mathfrak{F}'$ is called a *p-morphic image* of $\mathfrak{F}$ (written: $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$) if there exists a p-morphism pair $\langle f, g \rangle$ such that f is a surjective mapping.

Proposition 4. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be Kripke frames with $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$. Then, for any A of $\mathcal{L}$, $\mathfrak{F} \Vdash A$ implies $\mathfrak{F}' \Vdash A$.*

Next, we define the notion of $\blacksquare$-realizer and $\blacksquare$-realization, which is a simple generalization of bulldozing [1, ch 4.5] used in [5].

Definition 5 ($\blacksquare$-realizer)**.** Let $\mathfrak{F}$ be a Kripke frame and $\mathfrak{G}$ in K_1. A pair $\langle f, g \rangle$ is called a $\blacksquare$-*realizer* if $\langle f, g \rangle$ is a p-morphism pair from $\mathfrak{G}$ to $\mathfrak{F}$ and f is surjective. $\mathfrak{G}$ is *a* $\blacksquare$-*realization of* $\mathfrak{F}$ if there exists a $\blacksquare$-realizer $\langle f, g \rangle$ from $\mathfrak{G}$ onto $\mathfrak{F}$.

A similar construction to [5, Theorem 4] gives us the following proposition.

Proposition 6 (Existence of $\blacksquare$-realization)**.** *Let $\mathfrak{F} = \langle W, R, S, D \rangle$ be a Kripke frame that satisfies $(R \cap \neq) \subset S$ and $S \subset R$. Then, there exists a $\mathfrak{G} \in \mathsf{K}_1$ such that $\mathfrak{G}$ is a $\blacksquare$-realization of $\mathfrak{F}$.*

3 Difficulties in Proving Completeness of Ks

In this section, we construct the canonical bimodal model $\mathfrak{M}$ of **Ks** according to [2, ch.15] and give a bit different but direct proof of Suzuki's theorem [6], which says that $(R\cap \neq) \subset S$ fails in $\mathfrak{M}$ in spite of **Ks** $\vdash$ (M2).

Let us introduce some basic notions. A set Γ of formulas has $\forall$-*property* if for any schema $(\forall x) A$, there exists a free variable a such that $A[a/x] \supset (\forall x) A \in \Gamma$. We define the notions of *consistent*, *maximal*, and *maximal consistent set* (MCS) as usual.

We call $\mathcal{L}'$ *an extended language of* $\mathcal{L}$ *by free variables* if $\mathcal{L}'$ has the same parameters, bound variables, and logical connectives as $\mathcal{L}$ and contains all the free variables of $\mathcal{L}$. For an extended languages $\mathcal{L}'$ of $\mathcal{L}$ by free variables, $V(\mathcal{L}')$ denotes the set of free variables.

For extended languages $\mathcal{L}_1$, $\mathcal{L}_2$ of $\mathcal{L}$ by free variables, $\mathcal{L}_1$ is ω*-proper sublanguage of* $\mathcal{L}_2$ (written: $\mathcal{L}_1 \sqsubset \mathcal{L}_2$ or $\mathcal{L}_2 \sqsupset \mathcal{L}_1$) if $V(\mathcal{L}_1) \subset V(\mathcal{L}_2)$ and $|V(\mathcal{L}_2) \setminus V(\mathcal{L}_1)| \geq \omega$.

Definition 7 ([2]). Fix a language $\mathcal{L}^+$ of which $\mathcal{L}$ is a ω-proper sublanguage. Then, the canonical model of **Ks** is defined as follows:

$$W = \{\, w \subset \mathcal{L}^+ \mid (\exists \mathcal{L}_w) [\mathcal{L} \sqsubset \mathcal{L}_w \sqsubset \mathcal{L}^+ \text{ and } w \text{ is MCS of } \mathcal{L}_w \text{ that enjoys } \forall\text{-property}] \,\}$$
$$wRw' \iff \{\, A \text{ of } \mathcal{L}_w \mid \Box A \in w \,\} \subset w'$$
$$wS\,w' \iff \{\, A \text{ of } \mathcal{L}_w \mid \blacksquare A \in w \,\} \subset w'$$
$$D(w) = \{\, t \text{ of } \mathcal{L}_w \mid t \text{ is term} \,\}$$
$$\mathbf{c}^V = \mathbf{c} \in \bigcup\nolimits_{w\in W} D(w) \text{ for any constants } \mathbf{c}$$
$$\langle\, t_1, \dots, t_n \,\rangle \in \mathbf{P}^V_w \iff \mathbf{P}(t_1, \dots, t_n) \in w, \text{ for any predicate symbols } \mathbf{P}.$$

Fact 8 ([2]). *Suppose that $\mathcal{L}_1$ is ω-proper sublanguage of $\mathcal{L}_2$. Let Λ be a consistent set of formulas of $\mathcal{L}_1$. Then, there exists a MCS Δ of $\mathcal{L}_2$ such that $\Lambda \subseteq \Delta$ and Δ has $\forall$-property.*

Fact 9 ([2]). *Suppose that $\mathcal{L} \sqsubset \mathcal{L}_w \sqsubset \mathcal{L}^+$ and w is MCS of $\mathcal{L}_w$. And suppose that $\sim \blacksquare A \in w$. Then, there is a MCS $w' \in W$ with $\forall$-property in a language $\mathcal{L}_{w'} \sqsubset \mathcal{L}^+$ such that $\mathcal{L}_w \sqsubset \mathcal{L}_{w'}$ and $\{\sim A\} \cup \{\, B \text{ of } \mathcal{L}_w \mid \blacksquare B \in w \,\} \subset w'$.*

However, the canonical model of **Ks** does not work well because of the following theorem.

Theorem 10 ([6]). *$(R\cap \neq) \subset S$ fails in the canonical model of* **Ks***.*

Proof. Let $\mathbf{P} \in \mathcal{L}$ be 0-ary predicate symbol. Let us consider $\Gamma = \{\sim \Box\mathbf{P}, \blacksquare\mathbf{P}\}$. Take the following frame in K_1: $W = \{0\}$, $R = \{\langle 0, 0 \rangle\}$, $S = \emptyset$, $D(0) = \{a\}$, $\mathbf{P}^V_0 = \mathsf{false}$. It is easy to check that for all element of Γ is true at 0, i.e., Γ is satisfiable in K_1. It follows from the soundness of Ks with respect to K_1 that Γ is consistent.

Then, we can construct MCS w of $\mathcal{L}_w \sqsupset \mathcal{L}$ that enjoys $\forall$-property and satisfies $\Gamma \subset w$ and $\mathcal{L}_w \sqsubset \mathcal{L}^+$. From **Ks** $\vdash (\blacksquare\mathbf{P} \wedge \sim \Box\mathbf{P})) \supset (\Box B \supset B)$, we deduce that $\Box A \in w$ implies $A \in w$ for any A of $\mathcal{L}_w$.

Take $\mathcal{L}_{w'}$ with $\mathcal{L} \sqsubset \mathcal{L}_w \sqsubset \mathcal{L}_{w'} \sqsubset \mathcal{L}^+$. Since w is consistent, we can construct MCS w' of $\mathcal{L}_{w'}$ that satisfies $w \subset w'$ and enjoys $\forall$-property. Note that $w \neq w'$ and $w' \in W$. From $w \subset w'$, we can deduce that $\Box A \in w$ implies $A \in w'$ for any A of $\mathcal{L}_w$. Thus, we conclude that wRw' but $w \neq w'$.

We prove that $wS\,w'$ fails. Suppose for contradiction that $wS\,w'$ holds. It follows from $\blacksquare\mathbf{P} \in w$ that $\mathbf{P} \in w'$ by the definition of S. By $\Gamma = \{\sim \Box\mathbf{P}, \blacksquare\mathbf{P}\} \subset w \subset w'$, however, we have $\sim \Box\mathbf{P}, \blacksquare\mathbf{P} \in w'$. From $\mathbf{P} \in w'$ and $\blacksquare\mathbf{P} \in w'$, we have $\Box\mathbf{P} \in w'$ by **Ks** $\vdash$ (M2), which contradicts to $\sim \Box\mathbf{P} \in w'$. Thus, we conclude that $wS\,w'$ fails. QED

4 Kripke Completeness

Lemma 11. *Suppose that $\mathcal{L} \sqsubset \mathcal{L}_w \sqsubset \mathcal{L}^+$ and w is MCS of $\mathcal{L}_w$. And, suppose that $\sim \Box A \in w$. Then, (1) there is a MCS $w' \in W$ with $\forall$-property in a language $\mathcal{L}_{w'} \sqsubset \mathcal{L}^+$ such that $\mathcal{L}_w \sqsubset \mathcal{L}_{w'}$ and $\{\sim A\} \cup \{\, B \text{ of } \mathcal{L}_w \mid \Box B \in w \,\} \subset w'$, or (2) $\{\sim A\} \cup \{\, B \text{ of } \mathcal{L}_w \mid \Box B \in w \,\} \subseteq w$.*

Proof. Suppose that $\sim \Box A \in w$. Note that A is a formula of $\mathcal{L}_w$. (Case 1) Assume that $\sim \blacksquare A \in w$. Clearly, (1) holds. (Case 2) Assume that $\blacksquare A \in w$. It follows from **Ks** $\vdash A \wedge \blacksquare A \supset \Box A$ that $A \supset \Box A \in w$. Since $\sim \Box A \in w$, we have $\sim A \in w$. Let B be a formula of $\mathcal{L}_w$ with $\Box B \in w$. It follows from $\blacksquare A$, $\sim \Box A \in w$ and **Ks** $\vdash \blacksquare A \wedge \sim \Box A \supset (\Box B \supset B)$ that $\Box B \supset B \in w$. From $\Box B \in w$, we conclude that $B \in w$. QED

Definition 12. We define W, S, D, and V in the same way as Definition 7. We define the relation $\boldsymbol{R}$ as follows:

$$w\boldsymbol{R}w' \iff [w = w' \text{ and } \{\, A \text{ of } \mathcal{L}_w \mid \Box A \in w \,\} \subseteq w] \text{ or } wSw'.$$

Then, $\langle\, W, \boldsymbol{R}, S, D, V \,\rangle$ is called *the modified canonical model of* **Ks**.

Remark that inclusion requirements are satisfied and $S \subset \boldsymbol{R}$ and $(\boldsymbol{R} \cap \neq) \subset S$ hold trivially. Note that the modified canonical model may contain S-reflexive states.

Lemma 13 (Truth Lemma)**.** *Let* $\mathfrak{M}$ *be the modified canonical model of* **Ks***. Then, for any* w *in* $\mathfrak{M}$ *and any* A *of* $\mathcal{L}_w$*,* $\mathfrak{M}, w \Vdash A[\underline{\overrightarrow{a_i}}/\overrightarrow{a_i}] \iff A \in w$*, where* $\overrightarrow{a_i}$ *are all free variables of* $\mathcal{L}_w$ *that occurs in* A*.*

Note that a_i belongs to $D(w)$ and A admits the interpretation with respect to w in $\mathfrak{M}$.

Proof. Prove the equivalence by induction on the length of A. It suffices to show the case where A is $\Box C$.

($\Longleftarrow$) Suppose that $\Box C \in w$. Assume that $w\boldsymbol{R}x$. By the definition of $\boldsymbol{R}$, (1) $[w = x$ or $\{\, B$ of $\mathcal{L}_w \mid \Box B \in w \,\}]$ or (2) wSx. (Case (1)) From $\Box C \in w$ and (1), we have $C \in w$. (Case (2)) It follows from $\Box C \in w$ and $\mathbf{Ks} \vdash \Box C \supset \blacksquare C$ that $\blacksquare C \in w$. Since wSx, we have $C \in w$. Thus we deduce that $C \in w$. Note that $\overrightarrow{a_i} \in D(w) \subset D(x)$ because of the inclusion requirement of S. Therefore, by induction hypothesis we have $\mathfrak{M}, x \Vdash C[\underline{\overrightarrow{a_i}}/\overrightarrow{a_i}]$ hence $\mathfrak{M}, w \Vdash \Box C[\underline{\overrightarrow{a_i}}/\overrightarrow{a_i}]$.

($\Longrightarrow$) We prove the contraposition. Suppose that $\Box C \notin w$. Then, we deduce that (1) there is a MCS $w' \in W$ with the $\forall$-property in a language $\mathcal{L}_{w'} \sqsupset \mathcal{L}_w$ such that $\{\, \sim C \,\} \cup \{\, B$ of $\mathcal{L}_w \mid \blacksquare B \in w \,\} \subset w'$, or (2) $\{\, \sim C \,\} \cup \{\, B$ of $\mathcal{L}_w \mid \Box B \in w \,\} \subset w$. (Case (1)) Observe that C is a formula of $\mathcal{L}_{w'}$. Since $\sim C \in w'$, we have $C \notin w'$. From $\overrightarrow{a_i} \in D(w) \subset D(w')$, we deduce that $\mathfrak{M}, w' \nVdash C[\underline{\overrightarrow{a_i}}/\overrightarrow{a_i}]$ by induction hypothesis. It follows from wSw' and the definition of $\boldsymbol{R}$ that $w\boldsymbol{R}w'$. Thus, we have $\mathfrak{M}, w \nVdash \Box C[\underline{\overrightarrow{a_i}}/\overrightarrow{a_i}]$. (Case (2)) Clearly, $\sim C \in w$. Note that $\overrightarrow{a_i} \in D(w)$. By induction hypothesis, we have $\mathfrak{M}, w \nVdash C[\underline{\overrightarrow{a_i}}/\overrightarrow{a_i}]$. It follows from $\{\, B$ of $\mathcal{L}_w \mid \Box B \in w \,\} \subseteq w$ that $w\boldsymbol{R}w$. Therefore, from I.H., we have $\mathfrak{M}, w \nVdash \Box C[\underline{\overrightarrow{a_i}}/\overrightarrow{a_i}]$. QED

Theorem 14. *For any formula* A *of* $\mathcal{L}$*,* $\mathsf{K}_1 \Vdash A$ *implies* $\mathbf{Ks} \vdash A$*.*

Proof. Suppose that $\mathbf{Ks} \nvdash A$. There exists a MCS w of $\mathcal{L}_w \sqsupset \mathcal{L}$ such that w has $\forall$-property and $\sim A \in w$. Construct the modified canonical frame $\mathfrak{F}$ of **Ks**. $S \subset \boldsymbol{R}$ and $(\boldsymbol{R} \cap \neq) \subset S$ hold trivially. Transform $\mathfrak{F}$ into $\mathfrak{G} \in \mathsf{K}_1$ by Proposition 6. we have $\mathfrak{G} \twoheadrightarrow \mathfrak{F}$ and $\mathfrak{F} \nVdash A$ (by Truth Lemma and $\sim A \in w$). Thus, we conclude that $\mathfrak{G} \nVdash A$ by Proposition 4. QED

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