A natural extension of Shannon type normal form expansion base for multi-modal logics

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In this paper, we extend naturally Shannon type normal form expansion theorem of the propositional logic to multi-modal logics, and treat the validity checking for these logics.

[I] A ramification of formulas on n-modal operators and the expansion structures

In the following, we fix the number $n(\geqq 1)$ of modal operators $K_1,\ldots,K_n$ $(\Box_1,\ldots,\Box_n)$

[1] For each $m(\geqq 1)$ and $k(\geqq 0)$, let ${}^{(m)}\mathcal{L}^{(k)}$ be the set of formulas having at most m variables $\{p_1,\ldots,p_m\}$ and having degree k, that is, having a least one k nesting of modal operators and not having more large nesting than k of these.

[2] In order to give the normal form expansions of formulas for multi modal logics, for each $m\geqq 1$, an expansion structure $<\underline{{}^{(m)}W^{(k)}\ (k=0,1,\ldots),{}^{(m)}w,\ *}>$ is defined on ${}^{(m)}\mathcal{L}^{(k)}$ $(k=0,1,\ldots)$, in the following Definition 1,4 and 2, where ${}^{(m)}W^{(k)}$ called the expansion base set on ${}^{(m)}\mathcal{L}^{(k)}$, consists of the extended mini-terms of m-rank and k-degree, and ${}^{(m)}w$ and $*$, are mappings relating to expansions of formulas.

(Definition.1) For each $m\geqq 1$, the expansion base ${}^{(m)}W^{(k)}(k=0,1,\ldots)$ is defined k-inductively.

(1) ${}^{(m)}W^{(0)}=\{p_1^{\xi 1}\cdot\cdots\cdot p_m^{\xi m}\mid \xi_1,\ldots,\xi_m\in\{0,1\}\}(={}^{(m)}W)$, (the logical meaning of $\cdot$ is $\wedge$)

(2) ${}^{(m)}W^{(k+1)}=\{\langle \mathbf{y}\cdot[\Pi_{(\underline{1\leqq i\leqq n\&X\subseteq {}^{(m)}W^{(k)}})}\ Ki[X]^{\xi iX}]\rangle \mid \mathbf{y}\in{}^{(m)}W^{(k)}\ \xi_{iX}\in\{0,1\}(1\leqq i\leqq n, X\subseteq{}^{(m)}W^{(k)})\}$,

where $X\subseteq{}^{(m)}W^{(k)}$ means that X is an arbitrary one of $\mathbf{N(m,k)}=2^{|{}^{(m)}W^{(k)}|}$ subsets of ${}^{(m)}W^{(k)}$ and and $[\Pi_{(\underline{1\leqq i\leqq n\&X\subseteq {}^{(m)}W^{(k)}})}\ Ki[X]^{\xi iX}]$ denotes the following $\cdot$ product of $n\times\mathbf{N(m,k)}$ literals of the form $Ki[X]^{\xi iX}$ $(1\leqq i\leqq n, X\subseteq{}^{(m)}W^{(k)})$:

$$\begin{bmatrix} K_1[S_1]^{\xi 1S1}\cdot\ldots\cdot K_1[S_{N(m,k)}]^{\xi 1SN(m,k)} \\ \vdots \qquad\qquad \vdots \\ K_n[S_1]^{\xi nS1}\cdot\ldots\cdot K_n[S_{N(m,k)}]^{\xi nSN(m,k)} \end{bmatrix}$$

, where $2^{{}^{(m)}W^{(k)}}=\{S_1,\ldots,S_{N(m,k)}\}$, $S_1=\varnothing$, $S_{N(m,k)}={}^{(m)}W^{(k)}$ and '$\cdot$' satisfies associative and commutative laws .

(Notation) For any formula $A\in{}^{(m)}\mathcal{L}^{(k)}$, A^{δ} denotes A or $\neg A$, corresponding to $\delta=1$ or 0.

(Definition 2) $\underline{*}$: $\underline{2^{{}^{(m)}W^{(k)}}\rightarrow{}^{(m)}\mathcal{L}^{(k)}\cup\{\bot\}}$ $(m\geqq 1,k\geqq 0)$ is the following mapping to assign any set of mini-terms in ${}^{(m)}W^{(k)}$, a formula in ${}^{(m)}\mathcal{L}^{(k)}\cup\{\bot\}$.

(1) For each $S=\{f_1,\cdots,f_r\}\subseteq{}^{(m)}W^{(k)}$, $*S={}^*f_1\vee\cdots\vee{}^*f_r$ $(r\geq 1)$ (2) For $S=\varnothing$, $*\varnothing=\bot$

(3) For each $f=p_1^{\delta 1}\cdot\cdots\cdot p_m^{\delta m}\in{}^{(m)}W^{(0)}$, $*f=p_1^{\delta 1}\wedge\cdots\wedge p_m^{\delta m}\in{}^{(m)}\mathcal{L}^{(0)}$.

(4) For each $f=<g\cdot[\Pi_{1\leqq i\leqq n\ \underline{\&}\ Y\subseteq{}^{(m)}W^{(k)}}\ Ki[Y]^{\delta iY}]>\in{}^{(m)}W^{(k+1)}$,

$$*f = *g\wedge\bigwedge\nolimits_{1\leqq i\leqq n\ \underline{\&}\ Y\subseteq{}^{(m)}W^{(k)}}\ Ki\,(*(Y)_{(m,k)})^{\delta iY}\ \ (\cdots\in{}^{(m)}\mathcal{L}^{(k+1)})$$

(5) For each $S\subseteq{}^{(m)}W^{(k)}$ such that $S\neq\varnothing$, $\boxed{*(S)_{(m,k)}=*S\qquad(\in{}^{(m)}\mathcal{L}^{(k)})}$;

and, $\boxed{*(\varnothing)_{(m,k)}=\bot\wedge *{}^{(m)}W^{(k)}(\in{}^{(m)}\mathcal{L}^{(k)})}$.

[Proposition 0] (1) $S\subseteq{}^{(m)}W^{(k)}\Rightarrow\deg(*(S)_{(m,k)})=k$.

(2) In the extended propositional logic **PL**, $S\subseteq{}^{(m)}W^{(k)}\Rightarrow\mathbf{PL}\vdash *(S)_{(m,k)}\equiv *S$,

where **PL** is the propositional logic extended on $\mathcal{L}$ $(=\cup_{m\geqq 0}\cup_{k\geqq 0}{}^{(m)}\mathcal{L}^{(k)})$.

(Definition3) $(\,)':2^{{}^{(m)}W^{(k)}} \to 2^{{}^{(m)}W^{(k+1)}}$ and $'(\,):2^{{}^{(m)}W^{(k+1)}} \to 2^{{}^{(m)}W^{(k)}}$ $(m\geqq 1, k\geqq 0)$ are as follows:

(1) $U' =\{\ \langle y \cdot \ [\Pi_{\underline{(1\leqq i\leqq n\ \&\ X\subseteq {}^{(m)}W^{(k)})}}\ Ki[X]^{\xi iX}]\rangle \ |y\in U, \xi i_X \in\{0,1\}\,(1\leqq i\leqq n, X\subseteq {}^{(m)}W^{(k)}\ )\}$,

(2) For each $V\subseteq {}^{(m)}W^{(d)}(d\leqq k)$, $\underline{V^{\langle k\rangle} = V'^{\overset{k-d}{\cdots}}{}'}$.

(3) For each $V\subseteq {}^{(m)}W^{(k+1)}$,

$'V=\{\ y^{(m)}W^{(k)} \mid \langle y\cdot [\Pi_{(1\leqq i\leqq n\ \&\ X\subseteq {}^{(m)}W^{(k)})}\ Ki[X]^{\xi iX}]\rangle \in V,\ \ \xi i_X \in\{0,1\}\,(1\leqq i\leqq n, X\subseteq {}^{(m)}W^{(k)}\ )\}$

(Definition4) For each $m\geqq 1$, $\underline{{}^{(m)}w}:\underline{{}^{(m)}\mathcal{L}^{(k)} \to 2^{{}^{(m)}W^{(k)}}}$ $(k\geqq 0)$ is a mapping defined by induction on the construction steps of $A(\in \cup_{k\geqq 0}{}^{(m)}\mathcal{L}^{(k)} = {}^{(m)}\mathcal{L})$].

(1)(i) ${}^{(m)}w[\perp]=\varnothing$. (ii) ${}^{(m)}w[p_i]=\{p_1{}^{\xi 1}\cdot\cdots\cdot p_{i-1}{}^{\xi i-1}\cdot p_i{}^{1}\cdot p_{i+1}{}^{\xi i+1}\cdot\cdots\cdot p_m{}^{\xi m} \mid \xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_m\in\{0,1\}\}$

(2) For $A=B\vee C \in {}^{(m)}\mathcal{L}^{(k)}$, ${}^{(m)}w[B\vee C]= {}^{(m)}w[B]^{\langle k\rangle} \cup {}^{(m)}w[C]^{\langle k\rangle}$

(3) For $A=B\wedge C \in {}^{(m)}\mathcal{L}^{(k)}$, ${}^{(m)}w[B\wedge C]= {}^{(m)}w[B]^{\langle k\rangle} \cap {}^{(m)}w[C]^{\langle k\rangle}$

(4) For $A=\neg B\in {}^{(m)}\mathcal{L}^{(k)}$, ${}^{(m)}w[\neg B]= {}^{(m)}W^{(k)}- {}^{(m)}w[B]$

(5) For $A= B\supset C \in {}^{(m)}\mathcal{L}^{(k)}$, ${}^{(m)}w[B\supset C]=({}^{(m)}W^{(k)}-{}^{(m)}w[B]^{\langle k\rangle})\cup {}^{(m)}w[C]^{\langle k\rangle}$

(6) For $A=K_i(B) \in {}^{(m)}\mathcal{L}^{(k)}$, where $k\geq 1$,

${}^{(m)}w[K_i(B)] = \{\ \langle y\cdot\ [\Pi_{\underline{(j,\ X)\in([1,n]\times(2^{{}^{(m)}W(k-1)})-\{(i,\ {}^{(m)}w[B])\})}}\ K_j\ [X]^{\xi jX}]\ \cdot Ki[{}^{(m)}w[B]]^{1}\rangle \ \mid$
$y\in{}^{(m)}W^{(k)} \wedge (\bigwedge_{(\underline{j,\ X)\in(\ [1,n]\times(2^{{}^{(m)}W(k-1)})-\{(i,\ {}^{(m)}w[B])\}})}\ \xi\ j_X \in\{0,1\}\,)\}$

[Theorem 1] For an arbitrary $U\subseteq {}^{(m)}W^{(k)}$, ${}^{(m)}w[{}^*U] = U$ $(m\geqq 1,k\geqq 0)$. [That is, ${}^{(m)}w$ is the inverse mapping of $*$.] **(Proof)** By Induction on the degree k of $U\subseteq {}^{(m)}W^{(k)}$, for each $m\geqq 1$.

[Proposition 2] On the extended propositional logic **PL**, the following are shown clearly.

(1) $f, g \in {}^{(m)}W^{(k)}, f\neq g \ \Rightarrow$ **PL** $\vdash {}^*f\wedge {}^*g\equiv\perp$

(2) $U,V\in{}^{(m)}W^{(k)}, U\cap V=\varnothing \ \Rightarrow$ **PL** $\vdash {}^*U\wedge {}^*V\equiv\perp$

(3) $U,V\in{}^{(m)}W^{(k)} \ \Rightarrow$ **PL** $\vdash {}^*(U\cup V)\equiv {}^*U\vee {}^*V$

(4) $U,V\in{}^{(m)}W^{(k)} \ \Rightarrow$ **PL** $\vdash {}^*(U\cap V)\equiv {}^*U\wedge {}^*V$

(5) $U,V\in{}^{(m)}W^{(k)},\ U\subseteq V \Rightarrow$ **PL** $\vdash {}^*U\supset {}^*V$.

[Theorem 3] (# k) **PL** $\vdash {}^{*(m)}W^{(k)}$ $(m\geqq 1,k\geqq 0)$

(Proof) For an arbitrary $m\geqq 1$, we prove (# k) by Induction on k.

(The basic step) The result is clear, as ${}^{*(m)}W^{(0)}$ is the disjunction of all m variable mini-terms in $\mathbf{PL_0}$ [the propositional logic in the narrow sense].

(The inductive step) The following holds in **PL** by the distributives of $\wedge$ for $\vee$.

${}^{*(m)}W^{(k+1)}= \bigvee_{y\in{}^{(m)}W^{(k)}} \bigvee_{\xi_{1S1}\in\{0,1\}} \vee\cdots\vee_{\xi_{iSj}\in\{0,1\}}\cdots \bigvee_{\xi_{nSN(m,k)}\in\{0,1\}}$

$({}^*y\wedge K_1({}^*(S_1)^{(m,k)}))^{\xi 1S1}\wedge\cdots\wedge\ (K_i({}^*(S_j)^{(m,k)}))^{\xi iSj}\wedge\cdots\cdots\wedge(K_n({}^*(S_{N(m,k)})^{(m,k)}))^{\xi n\,SN(m,k)})$

$\equiv(\bigvee_{y\in{}^{(m)}W^{(k)}} {}^*y)\wedge(\bigvee_{\xi 1S1\in\{0,1\}}\ (K_1({}^*(S_1)^{(m,k)}))^{\xi 1S1})\wedge\cdots\wedge(\bigvee_{\xi iSj\in\{0,1\}}\ (K_i({}^*(S_j)^{(m,k)}))^{\xi iSj})$

$\wedge\cdots\cdots\wedge(\bigvee_{\xi nSN(m,k)\in\{0,1\}}\ (K_n({}^*(S_{N(m,k)})^{(m,k)}))^{\xi n\,S\,N(m,k)})$, where $2^{{}^{(m)}W(k)}=\{S_1,\ldots,S_{N(m,k)}\}$,

All the formulas except of the first formula in $\wedge$connections are tautologies, because these are of the form $B\vee\neg B$. Then **PL** $\vdash {}^{*(m)}W^{(k+1)} \equiv(\bigvee_{y\in{}^{(m)}W^{(k)}} {}^*y)\equiv {}^{*(m)}W^{(k)}$. Thus , by the induction hypothesis, **PL** $\vdash {}^{*(m)}W^{(k+1)}$ holds.

[Proposition 4] For an arbitrary $U\subseteq {}^{(m)}W^{(k)}$, **PL** $\vdash {}^*(U') \equiv {}^*U$. $(m\geqq 1,k\geqq 0)$

(Proof) The result is given by replacing ${}^{(m)}W^{(k+1)}$ and ${}^{(m)}W^{(k)}$ in the preceding proof, with U' and U.

[II] Expansion theorems in the extended base sequence and mappings system

(Definition 5) The following multi-modal logics (or deduction systems) are defined by adding modal rules and modal axioms to the extended propositional logic **PL** on $\mathcal{L}=\cup_{m\geqq 1}\cup_{k\geqq 0}{}^{(m)}\mathcal{L}^{(k)}$. .

K_C=PL+modus ponens+$\{\dfrac{\varphi\equiv\psi}{Ki(\varphi)\equiv Ki(\psi)} \mid \varphi,\psi\in\mathcal{L},\ deg(\varphi)=deg(\psi)\ (1\leq i\leq n)\}$

K_1=PL+modus ponens+$\{\dfrac{\varphi}{Ki(\varphi)} \mid \varphi\in\mathcal{L}\ (1\leq i\leq n)\}+\{\dfrac{\varphi\supset\psi}{Ki(\varphi)\supset Ki(\psi)} \mid \varphi,\psi\in\mathcal{L},\ deg(\varphi) = =deg(\psi)\ (1\leq i\leq n)\}$

$K^{\sim}$=PL+modus ponens + $\{Ki(\varphi\supset\psi)\supset(Ki(\varphi)\supset Ki(\psi)) \mid \varphi,\psi\in\mathcal{L},\ deg(\varphi)=deg(\psi)\ (1\leq i\leq n)\}$.

K_C , K_1 and $K^{\sim}$ are called respectively the **congruent logic restricted to equi- degree**, the **quasi-normal logic restricted to equi- degree** and the **normal logic restricted to equi- degree**.

$K =K_0+\{Ki(\varphi\supset\psi)\supset(Ki(\varphi)\supset Ki(\psi)) \mid \varphi,\psi\in\mathcal{L},(1\leq i\leq n)\}$.

K is the logic given by extending the **smallest normal logic** to n modality

We can verify that the following strength order relations hold:, $PL < K_C < K_1 < K^{\sim} < K$.

[Theorem 5] For any logic $L \geqslant K_C$, the following extended expansion theorem holds

For each $m\geqq 1$, (#) $L \vdash *({}^{(m)}w[A]) \equiv A$, for an arbitrary $A\in{}^{(m)}\mathcal{L}\ (=\cup_{k\geqq 0}{}^{(m)}\mathcal{L}^{(k)})$.

(Proof) When L=K_C, for each $m\geqq 1$, (#) is proved by induction on the construction of A.

(1) The basic cases (i) when $A=\bot$, $*{}^{(m)}w[\bot]=*(\varnothing)=\bot$; (ii) when $A=p_i$, the treatment is the same as the propositional logic PL_0 in the narrow sence.

(2) The inductive cases : when $A= B\vee C, B\wedge C, \neg B$ or $B\supset C\in{}^{(m)}\mathcal{L}^{(k)}$, by using the definition of mappings ${}^{(m)}w$ and $*$, Proposition 4 and Theorem 3, the following are obtained in **PL**.

(iii) $*{}^{(m)}w[B\vee C]$	(iv) $*{}^{(m)}w[B\wedge C]$	(v) $*{}^{(m)}w[\neg B]$	(vi) $*{}^{(m)}w[B\supset C]$
$\equiv *{}^{(m)}w[B]\vee *{}^{(m)}w[C]$	$\equiv *{}^{(m)}w[B]\wedge *{}^{(m)}w[C]$	$\equiv\neg *{}^{(m)}w[B]$	$\equiv *{}^{(m)}w[B]\supset *{}^{(m)}w[C]$

Therefore, for each case, the following is shown in K_C, by the induction hypothesis.

$*{}^{(m)}w[B\vee C]\equiv B\vee C$,	$*{}^{(m)}w[B\wedge C]\equiv B\wedge C$,	$*{}^{(m)}w[\neg B]\equiv\neg B$,	$*{}^{(m)}w[B\supset C]\equiv B\supset C$.

(vii) When $A=Ki(B)\in{}^{(m)}\mathcal{L}^{(k)}$, $k\geq 1$, ${}^{(m)}w[B]=Sd(\subseteq{}^{(m)}W^{(k-1)})$, for some $1\leq d\leq N(m,k-1)$. Then, applying a similar way to the proof of Theorem 3, for $*{}^{(m)}w[Ki(B)]$, all literals in the matrix part, except of $Ki[S\,d]^1$ $(=Ki[{}^{(m)}w[B]]^1)$, are erased by $*$. Thus $PL \vdash *{}^{(m)}w[Ki(B)]\equiv Ki(*({}^{(m)}w[B])^{(m,k-1)})$ …①. In other hand, by Proposition0(2), $PL \vdash *({}^{(m)}w[B])^{(m,k-1)}\equiv *({}^{(m)}w[B])$.

So, by the induction hypothesis, $K_C \vdash *({}^{(m)}(w[B])^{(m,k-1)})\equiv B$.

In the other hand, by Proposition 0(1), $deg(*({}^{(m)}(w[B])^{(m,k-1)}))=k-1=deg(B)$.

Thus, by the typical inference of K_C with the condition ($deg(\varphi)=deg(\psi)$): $\dfrac{\varphi\equiv\psi}{Ki(\varphi)\equiv Ki(\psi)}$, $K_C \vdash Ki(*({}^{(m)}w[B])^{(m,k-1)})\equiv Ki(B)$.…②. So, $K_C \vdash *{}^{(m)}w[K_i(B)]\equiv Ki(B)$, by ① and ②.

[III] Characterization sequences of multi-modal logics and decision problems

(Definition 6) For an arbitrary logic **L**, the following double sequence is called the characterization sequence of **L**: ${}^{(m)}W^{(k)}{}_L=\{y\in{}^{(m)}W^{(k)} \mid \text{not } L\vdash \neg *y\}\ (\subseteq{}^{(m)}W^{(k)})$ $(m=1,2,\ldots ; k=0,1,\ldots)$†

[Theorem 6] When $\mathbf{L} \geqslant \mathbf{K_c}$, for each $A \in {}^{(m)}\mathcal{L}^{(k)}$,

$$L \vdash A \quad \Leftrightarrow \quad {}^{(m)}W^{(k)}{}_{L} \subseteq {}^{(m)}w[A] \qquad (m \geqq 1, k \geqq 0)$$

(Proof) For any $A \in {}^{(m)}\mathcal{L}^{(k)}$, $L \vdash {}^{*}({}^{(m)}w[A]) \equiv A$, by **Theorem5**. Thus, this theorem holds, since the following **Lemma6-1** can be easily shown by using Theorem3 and Proposition 2.

Lemma 6-1 For any $\mathbf{L} \geqslant \mathbf{PL}$, for an arbitrary $U \subseteq {}^{(m)}W^{(k)}$,

$$L \vdash {}^{*}(U) \quad \Leftrightarrow \quad {}^{(m)}W^{(k)}{}_{L} \subseteq U \quad \cdots\cdots(\#) \quad (m \geqq 1, k \geqq 0)$$

(Definition 7) A finite set X is said to be **element wise definable,** if all elements of X can be listed.

[Theorem 7] For any logic $\mathbf{L} \geqslant \mathbf{K_c}$, the decision problem of **L**, is affirmatively solvable, if each characterization set ${}^{(m)}W^{(k)}{}_{L}(m \geqq 1; k \geqq 0)$ is element wise definable.

(Proof) By Theorem 6, the decision problem for L is affirmatively solvable, when, for each $m \geqq 1, k \geqq 0$, both finite sets ${}^{(m)}W^{(k)}{}_{L}$ and ${}^{(m)}w[A]$ are element wise definable. In the other hand, the element wise definability of ${}^{(m)}w[A]$ can be shown by the induction on the construction of A.

[Theorem 8] For the congruent logic restricted to equi-degree $\mathbf{K_C}$ and the normal logic restricted to equi-degree $\mathbf{K^{\sim}}$, their decision problems are affirmatively solvable.

(Proof) We can verify by using Theorem 1, 6 and Propositions that their characterization sequences ${}^{(m)}W^{(k)}{}_{K_C}\,(m \geqq 1, k \geqq 0)$ and ${}^{(m)}W^{(k)}{}_{K^{\sim}}(m \geqq 1, k \geqq 0)$ are respectively the sequence ${}^{(m)}\mathbf{W}^{(k)}(m \geqq 1, k \geqq 0)$ and the following sequence ${}^{(m)}\mathbf{R}^{(k)}(m \geqq 1, k \geqq 0)$.

$${}^{(m)}\mathbf{R}^{(0)} = {}^{(m)}\mathbf{W}^{(0)} \; ;$$

$${}^{(m)}\mathbf{R}^{(k+1)} = \{ < x, \left[\Pi_{1 \leqq i \leqq n} \left(\left(\Pi_{(\underline{X_i \subseteq Y \subseteq {}^{(m)}W^{(k)}})} K_i[\underline{Y}]^{1} \cdot \Pi_{(\underline{X_i \not\subseteq Y \subseteq {}^{(m)}W^{(k)}})} K_i[\underline{Y}]^{0} \right) \right) \right] > \mid x \in {}^{(m)}\mathbf{R}^{(k)} \text{ and } X_i \subseteq {}^{(m)}\mathbf{R}^{(k)} \; (1 \leqq i \leqq n) \}.$$

[IV] On characterization sequences and decision problems for more strong logics than $\mathbf{K^{\sim}}$

For any logic $\mathbf{L} \geqslant \mathbf{K^{\sim}}$, if we use the above ${}^{(m)}\mathbf{R}^{(k)}$ as the common base sets, then treatments for characterization sequences and the decision problems for those logics, become more simple than the treatments corresponding for using ${}^{(m)}\mathbf{W}^{(k)}$ in this paper. Several relating results are in [2] – [3].

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