Halldén Completeness and Principle of Variable Separation of Commutative Substructural Logics

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In this paper, we discuss algebraic characterizations of Halldén completeness (HC) and the deductive principle of variable separation (DPVS) for logics over $\mathbf{FL_e}$. We will see that some modifications on the definitions of HC and DPVS become necessary in order to extend results for intermediate logics or normal modal logics given by Lemmon (1966), Wroński (1976) and Maksimova (1995) to logics over $\mathbf{FL_e}$. Thus, we will introduce a new algebraic notion called well-connected pair. By using this, we will give alternative characterizations of HC and DPVS in the original forms for logics over $\mathbf{FL_e}$, and clarify their semantical relations with the disjunction property. This is a joint work with H. Ono.

1 Halldén completeness for logics over $\mathbf{FL_e}$

Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$ (intuitionistic linear logic without exponentials) and $\vdash_{\mathcal{L}}$ the deducibility relation determined by the logic $\mathcal{L}$.

We say that a logic $\mathcal{L}$ is *Halldén complete* if for all formulas ϕ and ψ with no variables in common, $\vdash_{\mathcal{L}} \phi \vee \psi$ implies $\vdash_{\mathcal{L}} \phi$ or $\vdash_{\mathcal{L}} \psi$. Obviously, HC follows from the disjunction property (DP).

We say that an algebra $\mathbf{A}$ is *well-connected* if for every $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$. The following result is given by Lemmon (1966) and Wroński (1976).

PROPOSITION 1 *For any intermediate logic $\mathcal{L}$, the following are equivalent;*

1. *$\mathcal{L}$ is Halldén complete,*
2. *$\mathcal{L}$ is meet-irreducible in the lattice of all intermediate logics, i.e., $\mathcal{L}$ cannot be represented as the intersection of two incomparable logics,*
3. *$\mathcal{L} = L(\mathbf{A})$ for some well-connected Heyting algebra $\mathbf{A}$, where $L(\mathbf{A})$ denotes the set of all formulas which are valid in $\mathbf{A}$.*

Though we can extend the above result to logics over $\mathbf{FL_{ew}}$, some modifications of the definitions of HC and well-connectedness become necessary to make similar result hold for logics over $\mathbf{FL_e}$. More precisely, see below.

THEOREM 2 *For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, the following are equivalent;*

1. *for all formulas ϕ and ψ which have no variables in common,*

$$\vdash_{\mathcal{L}} (\phi \wedge 1) \vee (\psi \wedge 1) \text{ implies } \vdash_{\mathcal{L}} \phi \text{ or } \vdash_{\mathcal{L}} \psi,$$

2. *$\mathcal{L}$ is meet-irreducible in the lattice of all logics over $\mathbf{FL_e}$,*
3. *$\mathcal{L} = L(\mathbf{A})$ for some FL_e-algebra $\mathbf{A}$ satisfying the following;*
   for any $x, y \in A^- = \{a \in A | a \leq 1\}$,

$$x \vee y = 1 \text{ implies } x = 1 \text{ or } y = 1.$$

D. Souma (2004) has shown that a characterization of the disjunction property for intermediate logics given by Maksimova (1984) holds also for logics over $\mathbf{FL_e}$.

PROPOSITION 3 *For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, the following are equivalent;*

1. *$\mathcal{L}$ has the disjunction property,*
2. *for any FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$ there exists a well-connected $\mathbf{C}$ in $V(\mathcal{L})$ such that $\mathbf{A} \times \mathbf{B}$ is a quotient algebra of $\mathbf{C}$, where $V(\mathcal{L})$ denotes the corresponding variety of $\mathcal{L}$.*

These semantical characterizations don't explain why DP implies HC or its modified version, though this implication is clear syntactically. In order to clarify a semantical relation between DP and HC, we will introduce a new algebraic notion called well-connected pair.

Subalgebras $\mathbf{B}, \mathbf{C}$ of $\mathbf{A}$ are said to be a *well-connected pair* if for any elements $x \in B$ and $y \in C$, $x \vee_{\mathbf{A}} y \geq 1_{\mathbf{A}}$ implies $x \geq 1_{\mathbf{A}}$ or $y \geq 1_{\mathbf{A}}$. By using this notion, we can give an alternative characterization of HC in the original form.

THEOREM 4 *For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, the following are equivalent;*

1. *$\mathcal{L}$ is Halldén complete,*
2. *for any non-degenerate FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$, there exists a well-connected pair $\mathbf{C}_1, \mathbf{C}_2$ of some algebra $\mathbf{C}$ in $V(\mathcal{L})$ such that $\mathbf{A}$ and $\mathbf{B}$ are quotient algebras of $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively.*

Note that if $\mathbf{C}$ is a well-connected FL_e-algebra then $\mathbf{C}$ and $\mathbf{C}$ themselves are a well-connected pair of it. Moreover, if $\mathbf{A} \times \mathbf{B}$ is a quotient algebra of $\mathbf{C}$ then $\mathbf{A}$ and $\mathbf{B}$ are quotient algebras of $\mathbf{C}$. This shows why DP implies HC semantically.

2 Deductive principle of variable separation for logics over $\mathbf{FL_e}$

We say that a logic $\mathcal{L}$ has the *deductive principle of variable separation* if for all formulas $\phi_1, \phi_2, \psi_1, \psi_2$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ have no variables in common, the condition $\phi_1 \wedge \psi_1 \vdash_{\mathcal{L}} \phi_2 \vee \psi_2$ implies $\phi_1 \vdash_{\mathcal{L}} \phi_2$ or $\psi_1 \vdash_{\mathcal{L}} \psi_2$. It is easy to see that HC is a special case of DPVS.

A class $\mathcal{K}$ of algebras is said to have the *joint embedding property* (JEP) if for all algebras $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ and monomorphisms $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$.

The following result is given by Maksimova (1995).

PROPOSITION 5 *For any intermediate logic $\mathcal{L}$, the following are equivalent;*

1. *DPVS holds in $\mathcal{L}$,*
2. *the class of all subdirectly irreducible Heyting algebras of $V(\mathcal{L})$ has the JEP.*

As well as in the case of Halldén completeness, though we can extend the above result to logics over $\mathbf{FL_{ew}}$, we need to modify the definition of DPVS when we try to extend it to logics over $\mathbf{FL_e}$. More precisely, see below.

THEOREM 6 *For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, the following are equivalent;*

1. *for all formuals $\phi_1, \phi_2, \psi_1, \psi_2$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ have no variables in common,*
$$\phi_1 \wedge \psi_1 \vdash_{\mathcal{L}} (\phi_2 \wedge 1) \vee (\psi_2 \wedge 1) \text{ implies } \phi_1 \vdash_{\mathcal{L}} \phi_2 \text{ or } \psi_1 \vdash_{\mathcal{L}} \psi_2,$$
2. *the class of all subdirectly irreducible FL_e-algebras of $V(\mathcal{L})$ has the JEP.*

In order to give an alternative characterization of DPVS in the original form and clarify semantical relation between HC and DPVS, we will introduce a new algebraic notion called well-connected joint embedding property.

We say that a class $\mathcal{K}$ of algebras has the *well-connected joint embedding property* (WCJEP) if for each $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ there exist $\mathbf{C}$ in $\mathcal{K}$ and monomorphisms $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$ such that for every $a \in A$ and $b \in B$, $\alpha(a) \vee \beta(b) \geq 1_{\mathbf{C}}$ implies $a \geq 1_{\mathbf{A}}$ or $b \geq 1_{\mathbf{B}}$.

THEOREM 7 *For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, the following are equivalent;*

1. *DPVS holds in $\mathcal{L}$,*
2. *the class of all non-degenerate FL_e-algebras of $V(\mathcal{L})$ has the WCJEP.*

Note that the following condition is equivalent to the WCJEP of the class of all non-degenerate algebras of $V(\mathcal{L})$;

($\star$) for every two non-degenerate FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$, there is a well-connected pair $\mathbf{C}_1, \mathbf{C}_2$ of some algebra $\mathbf{C}$ in $V(\mathcal{L})$ such that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic to $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively.

It is easy to see that ($\star$) implies the algebraic characterization of Halldén completeness given in Theorem 4. This shows why DPVS implies HC semantically.

The following figure shows relations among several logical properties for logics over $\mathbf{FL_e}$, where HC* and DPVS* denote the modified versions of HC and DPVS, respectively.

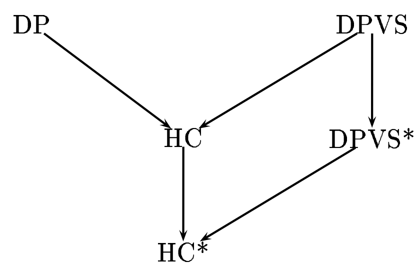

Figure 1: relations among several properties

References

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