Generalizations of Stone's Representation - A Survey

SATO Kentaro *

(Department of Mathematics, Univ. of Michigan, USA)

1 Introduction

The algebraic completeness is quite cheap (in many cases), because we can obtain it by Lindenbaum construction, which seems nothing more than syntactical operation. On the other hand, to obtain the 2-valued completeness (for classical logic), something more than syntactical methods is needed, non-deterministic principle, e.g., Weak König's Lemma, Principle of ultrafilter, Axiom of Choice. Similarly, for intuitionistic logic, while completeness with Heyting algebras is cheap, completeness with prime filters or, equivalently, with topological spaces, is properly semantical, from this point of view.

Stone's representation shows how the logic (or syntax, cheap semantics) is embedded into the family (precisely, product) of proper semantics. The completeness of 2-valued semantics essentially says that any Boolean algebra $\mathbb{B}$ is embedded into the product $\prod_{\mathfrak{u}\in\mathbf{Stone}(\mathbb{B})}\mathbf{2}$ of the simplest non-trivial Boolean algebra $\mathbf{2}$. Stone's representation theorem claims more: the image of this embedding is the set of continuous functions $\mathbf{Stone}(\mathbb{B})\to\mathbf{2}$, where $\mathbf{2}$ is equipped with the discrete topology. In other words, the "cheap semantics" is the "continuous patching" of "proper semantics". In terms of sheaf theory, $\mathbb{B}$ is represented as the global sections of the sheaf consisting of continuous functions from $\mathbf{Stone}(\mathbb{B})$ to $\mathbf{2}$ and this sheaf has value $\mathbb{B}/(a\!\uparrow)$ on a basic open set $\langle a\rangle=\{\mathfrak{u}\in\mathbf{Stone}(\mathbb{B})\,|\,a\in\mathfrak{u}\}$ and has the stalk $\mathbb{B}/\mathfrak{u}(\cong\mathbf{2})$ at a point $\mathfrak{u}$, where $a\!\uparrow=\{b\in\mathbb{B}\,|\,b\geq a\}$.

Stone duality has a more content, duality: any Boolean algebra is the set of all global sections of the sheaf of continuous functions from some 0-dimensional compact Hausdorff space to $\mathbf{2}$; any 0-dimensional compact Hausdorff space is the Stone space of some Boolean algebra, and this correspondence is natural (in the category theoretical sense). In other words, Stone(-) and Cont(-, $\mathbf{2}$) provide the equivalence **Bool-alg** $\simeq$ $(\mathbf{0}\text{-}\mathbf{CH})^{\mathrm{op}}$. However beautiful mathematically this duality is, the duality might conceal the essence of the schema "logic is the continuous patching of proper semantics". It is best that this schema provides also the duality as in the classical logic, but this is not the case for general substructural logics as shown below.

This survey overviews several generalizations of Stone's representation from the viewpoint that it has two contents, "sheaf representation" (or "continuous patching") and "duality", and overviews how we can retrieve the unity of these two contents in the cases of the 4-valued logic **Str4Val** and variants of it.

Acknowledgements. The author is very grateful to the organizer of 39th MLG meeting for giving the author an opportunity to have stimulating discussions on the subject with N. Galatos and K. Terui, whom the author would like to thank for invaluable comments. His thanks are also due to A. Blass.

2 Generalizations to Predicate Logics

2.1 Makkai's Works on the Duality for Classical First Order Logic

In [9] Makkai gave a kind of duality, for classical first order predicate logic. Because his definitions and theorems are too complicated to express here, we only see the correspondence between the propositional and predicate cases. First we note that Stone($\mathbb{B}$) = B-Hom($\mathbb{B}, \mathbf{2}$), the set of all Boolean homomorphisms from $\mathbb{B}$ to $\mathbf{2}$, equipped with the topology, and that B-Hom(-, $\mathbf{2}$) : **Bool-alg** $\to$ $(\mathbf{0}\text{-}\mathbf{CH})^{\mathrm{op}}$ and Cont(-, $\mathbf{2}$) : $(\mathbf{0}\text{-}\mathbf{CH})^{\mathrm{op}}$ $\to$ **Bool-alg** provide the equivalence, where $\mathbf{2}$ has the two role: a topological space and a Boolean algebra. He replaced $\mathbf{2}$, **Bool-alg** and **0-CH** in this schema by **Sets** and the categories of pretopoi and of ultracategories respectively, as he mentioned in [8], while it is known that pretopos provides the semantics for first order predicate logics. The notion of ultracategory is what he introduced and it is related to ultraproducts, whereas ultraproducts can be seen as stalks of suitable sheaves [4].

*e-mail: sato@kurt.scitec.kobe-u.ac.jp Partially supported by Japan Association for Mathematical Sciences

2.2 Sheaf and Intuitionistic Predicate Logic

As is well known (e.g. [7]), intuitionistic predicate logics are complete with sheaf semantics and this result can be seen as a generalization of topological completeness of intuitionistic propositional logics. The space (called spectrum) consists of prime theories (theories with disjunction and existence property) equipped with the topology whose basic open sets are $\langle\varphi\rangle = \{T \mid \varphi \in T\}$ and the sheaf consists of constants with relations for atomic formulae. The relations (which themselves are sheaves) for atomic formulae determine interpretations (which are also sheaves) of all formulae, in particular, of sentences, and each of these interpretations of sentences forms a sheaf with value in Heyting algebra. Note that this sheaf-representation works well (even better) for higher order (typed) logics [7], not only for first order logics.

3 Generalizing "Duality" to Substructural Logics

Urquhart showed the duality between relevance algebras and relevance spaces [12], where a relevance space is a Priestley space with a ternary relation. A Priestley space is a ordered topological space $(X, \mathcal{O}, \leq)$ which has the largest and smallest elements w.r.t. $\leq$ and which is compact and order-disconnected, i.e., if $x \not\leq y$ then there is a downward-closed clopen set X with $x \notin X$ and $y \in X$.

This result is based on Priestley's duality (originally from [10]): for a distributive lattice L, $\mathbf{S}(L) = (\mathrm{P}(L), \mathcal{O}_{\mathrm{P}(L)}, \subset)$ is Priestley a space where $\mathrm{P}(L)$ is the set of all prime filters of L and where $\mathcal{O}_{\mathrm{P}(L)}$ is the topology generated by $\langle a\rangle = \{\mathfrak{p} \in \mathrm{P}(L) \mid a \in \mathfrak{p}\}$'s and their compliments; for a Priestley space $S = (X, \mathcal{O}, \leq)$, $\mathbf{L}(S) = (\mathrm{DCO}(S), \cap, \cup)$ is a distributive lattice, where $\mathrm{DCO}(S)$ is the set of all downward-closed clopen subsets of X; and $\mathbf{S} \circ \mathbf{L}$ and $\mathbf{L} \circ \mathbf{S}$ are naturally equivalent to the identity.

Although this is of course a "duality" content, it has also a "sheaf-representation", $L \cong \mathrm{ContMon}(\mathbf{S}(L), \mathbf{2})$, where $\mathbf{2}$ is the lattice $\{0 < 1\}$ and $\mathbf{ContMon}(S_1, S_2)$ is the set of all continuous monotone maps ordered by the point-wise order. Notice that $\mathrm{ContMon}(\text{-}, \mathbf{2})$ forms a sheaf on $\mathbf{S}(L)$ and that $\mathrm{ContMon}(\langle a\rangle, \mathbf{2}) = L/(a\uparrow)$.

However, this "sheaf-representation" works only for (distributive) lattices, not for algebras for substructural logics. The reason is that the relation by which the quotient $L/(a\uparrow)$ is defined is not congruent w.r.t. operations associated with the connectives. (Note that $a\uparrow = \bigcap_{\mathfrak{p} \in \langle a\rangle} \mathfrak{p}$.) This means that the ternary relation, which is introduced in order to deal with such operations, cannot be expressed in terms of sheaves.

In spite of that, this presentation of algebra (by the ternary relation) reduces the meanings of connectives to quite elementary definitions, as Kripke frame reduces the meaning of modality. As one might notice, the duality provides also the completeness of Routley-Meyer semantics for substructural logics with the distributive law, and so it might be possible to say that proper semantics for such substructural logics is Routley-Meyer semantics (note that for the proof of this duality we need some non-deterministic principle), although it lacks "continuous patching" content.

There have been several attempts to obtain the duality and complete Kripke-like elementary semantics for the logics without the distributive law (or non-distributive lattices), based on Urquhart's representation from [11] or on Dunn's gaggle theory from [3]. As done by Hartonas [6], however, such representations no longer require non-deterministic principles, e.g., the axiom of choice. Although the duality and complete Kripke-like elementary semantics are obtained there, they are cheap from the viewpoint mentioned in Introduction. Moreover, it lacks "continuous patching" content for the same reason as the distributive substructural logics.

4 Generalizing "Sheaf Representation" to Substructural Logics

As we have seen in the last section, we need filters with congruency w.r.t. operators associated with connectives. The notion of filter with such congruency has been investigated in [5], called *deductive filter* there. The author of the present survey has defined primeness [SS] for such filters, called FL-*filters*, and obtained several results corresponding to those for classical logic and for intuitionistic logic which are well known: sheaf-representation (on the space, called *spectrum*, of prime FL-filters) of the algebras for substructural logics, the stalk of the sheaf at a prime FL-filter is the quotient by the prime FL-filter, the compactness of the spectrum, the completeness of substructural logics w.r.t. prime FL-filters, etc.

Thus, by deductive filters (or FL-filters), we can convey the difficulty for "continuous patching". However, we cannot obtain a Kripke-like elementary semantics in this line. Indeed, it is impossible to characterize the truth of $\varphi \to \psi$ in a stalk in terms of the truth of φ and ψ there, because the truth of φ is equivalent to that of $\varphi \wedge 1$ and $\varphi * \varphi$, which means that such a characterization for $\to$ yields weakening and contraction. Note that $A/\mathfrak{p} \models \varphi$ (under the canonical valuation) iff $\varphi \in \mathfrak{p}$ i.e., φ is true in the stalk at $\mathfrak{p}$ iff $\varphi \in \mathfrak{p}$.

Besides $\to$, we cannot, in general, obtain characterizations for other connectives, $*, \vee, (\text{-})^{\perp}$ (see [SS]). This means we cannot expect "duality" content along this line. Thus, we have "duality" with the spaces of prime filters and "continuous patching" with those of prime deductive (or FL-) filters, which are separated.

Game semantics from [1],[2] belongs to a family of semantics by prime FL-filters, while Routley-Meyer semantics is by prime filters. In other words, true formulae under a game-assignment form a prime FL-filter. Thus, investigations on prime FL-filters give us new insights into these semantics as shown in [SS].

5 Duality for the 4-valued Logic and Variants

The author has axiomatized the 4-valued logic **Str4Val**, an extension of Belnap's 4-valued logic to the $\neg, \wedge, \vee, \to$-fragment (i.e. additive and multiplicative fragment), in terms of substructural logic in [SF] so that, with weakening, it becomes Łukasiewicz' 3-valued logic (which means that Kleene's 3-valued logic is also embedded into the additive fragment), and that, with contraction, it becomes Priest-style paraconsistent logic (with $\to$). It is also proved there that 4-valued assignments are exactly prime FL-filters in these logics. This means that we have Kripke-like elementary semantics for these logics and so we could expect "duality" theorem in terms of prime FL-filters (for which we have "continuous patching"), not in terms of prime filters.

The trick used here must be mentioned. As shown in the last section, it is true that the truth of $\varphi \to \psi$ at a prime FL-filter $\mathfrak{p}$ cannot be characterized by the truth of φ and ψ at $\mathfrak{p}$. Nevertheless, in the cases of 4-valued logic and variants, the truth of $\varphi \to \psi$ at $\mathfrak{p}$ is characterized by the truth of φ, ψ and $\varphi^{\perp}, \psi^{\perp}$ at $\mathfrak{p}$.

Indeed, "duality" and "continuous patching" hold at the same time for prime FL-filters, in a quite similar way to classical logic [SF]: for any **Str4Val**-algebra A, $A \cong \mathrm{Cont}(\mathrm{Spec}_4(A), (\mathbf{4}\backslash\{H\}, \mathbf{4})) \times \mathrm{Cont}(\mathrm{Spec}_3(A), (2, \mathbf{3}))$, where Spec_4 and Spec_3 are pairs of 0-dimensional compact Hausdorff spaces of all prime FL-filters including $\top \multimap 1$ and avoiding it, respectively, and closed subspaces $\bigcap_{a \in A} \langle a \multimap a \otimes a \rangle$ of them, and where $\mathrm{Cont}((X_1, Y_2), (X_2, Y_2))$ are continuous maps $f : Y_1 \to Y_2$ with f"$X_1 \subset X_2$; obviously $\mathrm{Cont}(\text{-}, (\mathbf{4} \setminus \{H\}, \mathbf{4})) \times \mathrm{Cont}(\text{-}, (2, \mathbf{3}))$ forms a sheaf; $\langle \mathrm{Spec}_4, \mathrm{Spec}_3 \rangle : \textbf{Str4Val-alg} \to ((\textbf{0-CH}^{\hookrightarrow})^2)^{\mathrm{op}}$ provides an equivalence.

If A satisfies weakening, $\mathrm{Spec}_4(A)$ disappears; and if A satisfies contraction, the specified closed subsets coincide with the whole spaces. Thus we can see that Spec_4 is the properly contradictory part; the specified closed subset of Spec_4 is the part of Priest-style paraconsistent logic; Spec_3 is the 3-valued part; the specified closed subset of Spec_3 is the part of classical logic; and the complement of it is the properly 3-valued part. We thus also have duality results: $\textbf{Str4Val}+(\textbf{Weak})\textbf{-alg} \simeq (\textbf{0-CH}^{\hookrightarrow})^{\mathrm{op}}$, $\textbf{Str4Val}+(\textbf{Contr})\textbf{-alg} \simeq (\textbf{0-CH}^2)^{\mathrm{op}}$ and $\textbf{Bool} = \textbf{Str4Val}+(\textbf{Weak})+(\textbf{Contr})\textbf{-alg} \simeq \textbf{0-CH}^{\mathrm{op}}$.

Note that, even in the case of the 4-valued logic, the notions of (prime) filter and (prime) FL-filter differ. (This means that, in the cases of these logics, Routley-Meyer semantics differs from 4-valued semantics.)

References

[1] A. Blass, A game semantics for linear logic, APAL 56 (1992) 183-220
[2] A. Blass, Some semantical aspects of linear logic, J. of Interest Group in Pure and Applied Logic 5 (1998) 115-126
[3] M. Dunn, Gaggle theory: an abstract Galois connections and residuation with application to negation and various logical operations, in: Logics in AI, Proc. European Workshop JELIA 1990 (Springer-Verlag, Berlin, 1991)
[4] D. Ellerman, Sheaves of structures and generalized ultraproducts, Ann. of Math. Log. 7 (1974) 163-195
[5] N. Galatos and H. Ono, Algebraization, parameterized local deduction theorem and interpolation for substructural logics over FL, To appear in Studia Logica 83 (2006)
[6] C. Hartonas, Duality for lattice-ordered algebras and for normal algebraizable logics, Studia Logica 58 (1997) 403-450
[7] J. Lambek and P. J. Scott, Introduction to higher order categorical logic, Cambridge University Press (1986)
[8] M. Makkai, A Stone-type representation theory for first order logic, in: J. Gray, ed, Mathematical Applications of Category Theory (Denver, Col., 1984)
[9] M. Makkai, Stone duality for first order logic, Advances in Mathematics 65 (1987) 97-170
[10] H. Priestley, Representation of distributive lattices by means of ordered stone spaces, Bull. Lond. Math. Soc. 2 (1970) pp.186-190
[11] A. Urquhart, A topological representation theorem for lattices, Algebra Universalis 8 (1978) 45-58
[12] A. Urquhart, Duality for algebras of relevance logics, Studia Logica 56 (1996) 263-276
[SS] K. Sato, Prime filters in substructural logics from stalker theoretic point of view, submitted
[SF] K. Sato, The 4-valued logic within linear logic from game and stalker theoretic points of view, submitted