Completeness via labelled sequent calculi for bimodal logics with irreflexive modality

Ryo Kashima
kashima@is.titech.ac.jp
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology

Sano[1] gives the completeness results for bimodal logics with irreflexive modality. In this note, we show that these results can be also proved by making use of labelled sequent calculi.

Kripke model: $\mathcal{M} = \langle W, R, \mathcal{I} \rangle$

W: non-empty set
$R \subseteq W \times W$
$\mathcal{I} : W \times \{\text{propositional variables}\} \to \{\mathsf{True}, \mathsf{False}\}$

Truth value of a formula X at a world $w \in W$:

$\mathcal{M}(w, p) = \mathcal{I}(w, p)$. ($p$ is a propositional variable)
$\mathcal{M}(w, \neg A) = \mathsf{True} \iff \mathcal{M}(w, A) = \mathsf{False}$.
$\mathcal{M}(w, A \wedge B) = \mathsf{True} \iff \mathcal{M}(w, A) = \mathsf{True}$ and $\mathcal{M}(w, B) = \mathsf{True}$.
$\mathcal{M}(w, \Box A) = \mathsf{True} \iff \mathcal{M}(x, A) = \mathsf{True}$ for any x s.t. wRx.
$\mathcal{M}(w, \blacksquare A) = \mathsf{True} \iff \mathcal{M}(x, A) = \mathsf{True}$ for any x s.t. (wRx and $w \neq x$).

Logic $\mathbf{K}_{\Box\blacksquare}$ (Hilbert style):

$$\text{Tautologies} \;+\; \Box(A{\to}B){\to}(\Box A{\to}\Box B) \;+\; \frac{A{\to}B \quad A}{B} \;+\; \frac{A}{\Box A}$$
$$+\; \blacksquare(A{\to}B){\to}(\blacksquare A{\to}\blacksquare B) \;+\; \Box A{\to}\blacksquare A \;+\; (A \wedge \blacksquare A){\to}\Box A$$

For many logics over $\mathbf{K}_{\Box\blacksquare}$, Sano[1] proves the completeness theorems via canonical models. We can prove these completeness results via labelled sequent calculi. For example:

$\mathbf{K}_{\Box\blacksquare} \vdash X$ iff X is valid in any model.

$\mathbf{K}_{\Box\blacksquare} + \mathbf{4}_{\blacksquare} \vdash X$ iff X is valid in any transitive-antisymmetric model.

$\mathbf{K}_{\Box\blacksquare} + \mathbf{S5}_{\Box} \vdash X$ iff X is valid in any reflexive-transitive-symmetric model.

where $\mathbf{4}_{\blacksquare} = \blacksquare A{\to}\blacksquare\blacksquare A$, and $\mathbf{S5}_{\Box} = \Box A{\to}A \;+\; \Box A{\to}\Box\Box A \;+\; \Diamond A{\to}\Box\Diamond A$.

Definitions
α is a **label** iff $\alpha = \langle a_1, \ldots, a_n \rangle$ (finite sequence of natural numbers). If $\alpha = \langle a_1, \ldots, a_n \rangle$, the label $\langle a_1, \ldots, a_n, b \rangle$ is denoted by $\alpha \cdot b$. α is a **predecessor** of β (β is a **successor** of α) iff $\beta = \alpha \cdot n$ for some n. $\mathcal{L}$ is a **tree** iff $\mathcal{L}$ is a set of labels s.t. $\langle\rangle \in \mathcal{L}$, and ($\alpha \cdot n \in \mathcal{L}$ implies $\alpha \in \mathcal{L}$). $\alpha : A$ is a **labelled formula** iff α is a label and A is a formula. $\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta$ is a **labelled sequent** iff Γ and Δ are finite sets of labelled formulas, $\mathcal{L}$ is a tree, and each label in Γ, Δ is an element of $\mathcal{L}$.

Translation of labelled sequent into formulas

$\Gamma_\alpha \;=\; \{A \mid \alpha{:}A \in \Gamma\}$ (the set of formulas in Γ whose label is α).

$[\![\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta]\!]_\alpha \quad\equiv\quad \bigvee\Big(\neg\Gamma_\alpha,\ \Delta_\alpha,\ \blacksquare[\![\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta]\!]_{\beta_1},\ldots,\blacksquare[\![\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta]\!]_{\beta_k}\Big)$

where $\{\beta_1,\ldots,\beta_k\}$ is the set of successors of α in $\mathcal{L}$.

Labelled sequent calculus $\mathcal{T}\mathbf{K}_{\Box\blacksquare}$

Initial sequent:

$$\alpha : A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : A$$

Inference rules:

$$\frac{\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : A}{\alpha : \neg A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}\ (\neg\text{left}) \qquad \frac{\alpha : A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}{\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : \neg A}\ (\neg\text{right})$$

$$\frac{\alpha : A,\ \alpha : B,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}{\alpha : A\wedge B,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : A \qquad \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : B}{\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : A\wedge B}\ (\wedge\text{right})$$

$$\frac{\alpha\cdot n : A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}{\alpha : \Box A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}\ (\Box\ \text{left})$$

$$\frac{\Gamma \overset{\mathcal{L}\cup\{\alpha\cdot n\}}{\Rightarrow} \Delta,\ \alpha\cdot n : A \qquad \alpha : \Pi,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : A}{\alpha : \Box\Pi,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : \Box A}\ (\Box\ \text{right})^\ddagger$$

$$\frac{\alpha\cdot n : A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}{\alpha : \blacksquare A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}\ (\blacksquare\ \text{left}) \qquad \frac{\Gamma \overset{\mathcal{L}\cup\{\alpha\cdot n\}}{\Rightarrow} \Delta,\ \alpha\cdot n : A}{\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : \blacksquare A}\ (\blacksquare\ \text{right})^\ddagger$$

($\ddagger$) proviso: $\alpha\cdot n$ does not appear in the lower sequent.

Completeness theorem for $\mathcal{T}\mathbf{K}_{\Box\blacksquare}/\mathbf{K}_{\Box\blacksquare}$

(1) If X is True at any world in any model, then $\mathcal{T}\mathbf{K}_{\Box\blacksquare} \vdash \overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle{:}X$.

(2) If $\mathcal{T}\mathbf{K}_{\Box\blacksquare} \vdash \overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle{:}X$, then $\mathbf{K}_{\Box\blacksquare} \vdash X$.

Proof of (2)

Prove "if $\mathcal{T}\mathbf{K}_{\Box\blacksquare} \vdash \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta$, then $\mathbf{K}_{\Box\blacksquare} \vdash [\![\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta]\!]_{\langle\rangle}$" by induction.

Definition: $\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta$ is **$\mathcal{T}\mathbf{K}_{\Box\blacksquare}$-saturated** iff

- $\mathcal{T}\mathbf{K}_{\Box\blacksquare} \nvdash \Gamma' \overset{\mathcal{L}'}{\Rightarrow} \Delta'$ for any finite subsets $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, and $\mathcal{L}' \subseteq \mathcal{L}$;
- $\alpha{:}\neg A \in \Gamma$ implies $\alpha{:}A \in \Delta$;
- $\alpha{:}\neg A \in \Delta$ implies $\alpha{:}A \in \Gamma$;
- $\alpha{:}A\wedge B \in \Gamma$ implies $\alpha{:}A \in \Gamma$ and $\alpha{:}B \in \Gamma$;
- $\alpha{:}A\wedge B \in \Delta$ implies $\alpha{:}A \in \Delta$ or $\alpha{:}B \in \Delta$;
- $\alpha{:}\Box A \in \Gamma$ implies $\beta{:}A \in \Gamma$ for any successor β of α in $\mathcal{L}$;
- $\alpha{:}\Box A \in \Delta$ implies
  - $\beta{:}A \in \Delta$ for some successor β of α in $\mathcal{L}$; **or**
  - α is reflexive in $\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta$ and $\alpha{:}A \in \Delta$;
- $\alpha{:}\blacksquare A \in \Gamma$ implies $\beta{:}A \in \Gamma$ for any successor β of α in $\mathcal{L}$; and
- $\alpha{:}\blacksquare A \in \Delta$ implies $\beta{:}A \in \Delta$ for some successor β of α in $\mathcal{L}$.

Proof of (1)

Suppose $\mathcal{TK}_{\Box\blacksquare} \nvdash \overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle{:}X$. We expand this sequent into a $\mathcal{TK}_{\Box\blacksquare}$-saturated sequent $\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta$; then we construct a model $\mathcal{M} = \langle W, R, \mathcal{I}\rangle$:

$W = \mathcal{L}$.

$\alpha R \beta$ iff $((\alpha$ is a predecessor of $\beta)$ or $(\alpha$ is **reflexive** in $\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta$ and $\alpha = \beta))$, where α is reflexive in $\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta$ iff $\forall A\Big[\alpha{:}\Box A \in \Gamma$ implies $\alpha{:}A \in \Gamma\Big]$.

$\mathcal{I}(\alpha, p) = \mathsf{True}$ iff $\alpha{:}p \in \Gamma$.

In this model,

$\alpha{:}A \in \Gamma$ implies $\mathcal{M}(\alpha, A) = \mathsf{True}$; and
$\alpha{:}A \in \Delta$ implies $\mathcal{M}(\alpha, A) = \mathsf{False}$.

Therefore, $\mathcal{M}(\langle\rangle, X) = \mathsf{False}$.

Labelled sequent calculus $\mathcal{TK}_{\Box\blacksquare}\mathbf{4}_{\blacksquare}$

$\mathcal{TK}_{\Box\blacksquare}$ +

$$\frac{\alpha\cdot n : \Box A,\ \alpha\cdot n : A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}{\alpha : \Box A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}\ (\Box \text{ left transitive})$$

$$\frac{\alpha\cdot n : \blacksquare A,\ \alpha\cdot n : A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}{\alpha : \blacksquare A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}\ (\blacksquare \text{ left transitive})$$

Completeness theorem for $\mathcal{TK}_{\Box\blacksquare}\mathbf{4}_{\blacksquare}/\mathbf{K}_{\Box\blacksquare}+\mathbf{4}_{\blacksquare}$

(1) If X is True at any world in any transitive-antisymmetric model, then $\mathcal{TK}_{\Box\blacksquare}\mathbf{4}_{\blacksquare} \vdash \overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle{:}X$.

(2) If $\mathcal{TK}_{\Box\blacksquare}\mathbf{4}_{\blacksquare} \vdash \overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle{:}X$, then $\mathbf{K}_{\Box\blacksquare} + \mathbf{4}_{\blacksquare} \vdash X$.

Labelled sequent calculus $\mathcal{TK}_{\Box\blacksquare}\mathbf{S5}_{\Box}$

$\mathcal{TK}_{\Box\blacksquare}$ +

$$\frac{\beta : A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}{\alpha : \Box A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta}\ (\Box \text{ left universal})$$

$$\frac{\Gamma \overset{\mathcal{L}\cup\{\alpha\cdot n\}}{\Rightarrow} \Delta,\ \alpha\cdot n : A \qquad \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : A}{\Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha : \Box A}\ (\Box \text{ right reflexive})^{\ddagger}$$

(‡) proviso: $\alpha\cdot n$ does not appear in the lower sequent.

$$\frac{\beta : A,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta \qquad \alpha{:}\Theta,\ \beta{:}\Sigma,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha{:}\Lambda,\ \beta{:}\Pi}{\alpha : \blacksquare A,\ \alpha{:}\Sigma,\ \beta{:}\Theta,\ \Gamma \overset{\mathcal{L}}{\Rightarrow} \Delta,\ \alpha{:}\Pi,\ \beta{:}\Lambda}\ (\blacksquare \text{ left universal})$$

Completeness theorem for $\mathcal{TK}_{\Box\blacksquare}\mathbf{S5}_{\Box}/\mathbf{K}_{\Box\blacksquare}+\mathbf{S5}_{\Box}$

(1) If X is True at any world in any S5 model, then $\mathcal{TK}_{\Box\blacksquare}\mathbf{S5}_{\Box} \vdash \overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle{:}X$.

(2) If $\mathcal{TK}_{\Box\blacksquare}\mathbf{S5}_{\Box} \vdash \overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle{:}X$, then $\mathbf{K}_{\Box\blacksquare} + \mathbf{S5}_{\Box} \vdash X$.

Concluding remarks

Compared with the canonical model method, labelled sequent calculus method is simpler in some cases, e.g., $\mathbf{K}_{\Box\blacksquare} + \mathbf{4}_{\blacksquare}$ and predicate extensions.

References

[1] Katsuhiko Sano, Bimodal logics with irreflexive modality, in this proceedings.