Bimodal Logics with Irreflexive Modality

Katsuhiko SANO
ksano@sings.jp
Department of Philosophy and History of Science,
Graduate School of Letters, Kyoto University, Sakyo, Kyoto, 606-8501, Japan.

As is well known, standard modal propositional logic cannot define all the natural assumptions related to ordered set: irreflexivity, antisymmetry and etc. In order to overcome this lack of expressive power, various additional tools have been proposed, e.g., difference operator D [6, 7], nominals and satisfaction operator $@_i$ [1, ch. 7.3]. In this paper, we propose a new extension of standard modal logic. Our extension consists in adding an operator $\blacksquare$ whose semantics is based on the intersection of inequality $\neq$ and accessibility relation R. In our language, $\blacksquare p \supset \blacksquare\blacksquare p$ corresponds to transitivity and antisymmetry of R and $\blacksquare p \supset \Box\Box p$ to strict partial order (SPO). As space is limited, however, we are not concerned with frame definability in this paper. Here we will concentrate on proving our two results in our new extension. First, we propose a formal system $\mathrm{K}_{\Box\blacksquare}$ and prove Kripke completeness for $\mathrm{K}_{\Box\blacksquare}$ plus Lemmon-Scott Axioms: $\Diamond^m\Box^n A \supset \Box^j\Diamond^k A$. Second, we show that some logics of $\mathrm{K}_{\Box\blacksquare} + \Diamond^m\Box^n A \supset \Box^j\Diamond^k A$ enjoy the finite model property.

1 Preliminaries

The language $\mathcal{L}(\Box,\blacksquare)$ is defined using (i) the set of propositional variables: $\mathsf{Prop} = \{p_i \mid i \in \omega\}$, (ii) the propositional connectives: $\sim, \supset$ and (iii) the unary modal operators: $\Box, \blacksquare$. The well-formed formulas of $\mathcal{L}(\Box,\blacksquare)$ are defined as usual.

A *bimodal frame* is a triple $\mathfrak{F} = \langle W, R, S\rangle$, where W is a non-empty set and R, S are binary relations on W. A *bimodal model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$, where $\mathfrak{F}$ is a bimodal frame and V a function $V : \mathsf{Prop} \to \mathcal{P}(W)$. For any bimodal model $\mathfrak{M} = \langle W, R, S, V\rangle$, any $w \in W$ and any formula A of $\mathcal{L}(\Box,\blacksquare)$, the relation $\Vdash$ is defined indectively:

$$\begin{array}{lll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p).\\
\mathfrak{M}, w \Vdash \sim A & \text{iff} & \mathfrak{M}, w \nVdash A.\\
\mathfrak{M}, w \Vdash A \supset B & \text{iff} & \mathfrak{M}, w \nVdash A \text{ or } \mathfrak{M}, w \Vdash B.\\
\mathfrak{M}, w \Vdash \Box A & \text{iff} & (\forall w' \in W)\,[wRw' \text{ implies } \mathfrak{M}, w' \Vdash A].\\
\mathfrak{M}, w \Vdash \blacksquare A & \text{iff} & (\forall w' \in W)\,[wSw' \text{ implies } \mathfrak{M}, w' \Vdash A].
\end{array}$$

A bimodal frame satisfying $S = (R \cap \neq)$ is called $\mathcal{L}(\Box,\blacksquare)$*-frame*, where $w(R\cap \neq)w'$ means that wRw' and $w \neq w'$. $\mathcal{L}(\Box,\blacksquare)$*-model* is defined similarly. In $\mathcal{L}(\Box,\blacksquare)$-model,

$$\mathfrak{M}, w \Vdash \blacksquare A \text{ iff } (\forall w' \in W)\,[w(R\cap \neq)w' \text{ implies } \mathfrak{M}, w' \Vdash A].$$

Remark that $\mathcal{L}(\Box,\blacksquare)$-frame (or model) can be regarded as unimodal frame (or model, respectively).

A formula A is *valid in a model* $\mathfrak{M}$ (notation: $\mathfrak{M} \Vdash A$) if $\mathfrak{M}, w \Vdash A$, for any $w \in W$. A formula A is *satisfiable in a model* $\mathfrak{M}$ if $\mathfrak{M} \nVdash \sim A$. A formula A is *valid in a frame* $\mathfrak{F}$ (notation: $\mathfrak{F} \Vdash A$) if $\langle \mathfrak{F}, V\rangle, w \Vdash A$, for any $w \in \mathfrak{F}$ and any valuation V.

Definition 1 (Bimodal p-morphism). Let $\mathfrak{F} = \langle W, R, S\rangle$, $\mathfrak{F}' = \langle W', R', S'\rangle$ be bimodal frames. A mapping $f : W \to W'$ is a *bimodal p-morphism* if it satisfies the following conditions:

$$\begin{array}{rl}
\text{(i)} & (\forall w_1, w_2 \in W)\,[w_1 R w_2 \text{ implies } f(w_1) R' f(w_2)],\\
\text{(ii)} & (\forall w_1 \in W)\,(\forall v' \in W')\,[f(w_1) R' v' \text{ implies } (\exists w_2 \in W)\,(w_1 R w_2 \text{ and } f(w_2) = v')],
\end{array}$$

and the similar conditions about S, S'.

Fact 2. *Let $\mathfrak{F}$, $\mathfrak{F}'$ be bimodal frame and $f : \mathfrak{F} \to \mathfrak{F}'$ be surjective bimodal p-morphism. For any formula A of $\mathcal{L}(\Box,\blacksquare)$, $\mathfrak{F} \Vdash A$ implies $\mathfrak{F}' \Vdash A$.*

2 Kripke Completeness

Definition 3. Hilbert Calculus $\mathbf{K}_{\Box\blacksquare}$ consists of the following axiom schemata and rules:

$(A1)$	$A \supset (B \supset A)$	$(\blacksquare 1)$	$\blacksquare(A \supset B) \supset (\blacksquare A \supset \blacksquare B)$
$(A2)$	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	$(M1)$	$A \wedge \blacksquare A \supset \Box A$
$(A3)$	$(\sim A \supset \sim B) \supset (B \supset A)$	$(M2)$	$\Box A \supset \blacksquare A$
(MP)	From $A \supset B$ and A, we may infer B		
$(\Box 1)$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$		
($\Box$-rule)	From A, we may infer $\Box A$		

Hilbert Calculus $\mathbf{K}_{\Box\blacksquare} + \mathrm{G}'_{(m,n,j,k)}$ consists of the above all schemata, rules and the Lemmon-Scott Axioms $\mathrm{G}'_{(m,n,j,k)}$: $\Diamond^m \Box^n A \supset \Box^j \Diamond^k A$. $\vdash$ is defined as usual.

Theorem 4. *Let F be the class of all $\mathcal{L}(\Box, \blacksquare)$-frames. For any formula A, $\mathfrak{F} \Vdash A$ for any $\mathfrak{F} \in \mathsf{F}$, implies $\vdash_{\mathbf{K}_{\Box\blacksquare}} A$.*

Proof. Suppose that $\sim A$ is consistent. It follows from Lindenbaum's Lemma [1, p.197] that there is a maximal consistent set Δ such that $\sim A \in \Delta$. Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ be the canonical model of $\mathrm{K}_{\Box\blacksquare}$. Note that this canonical model is a bimodal model. From Truth Lemma [1, p.199], we may infer that $\mathfrak{M}, \Delta \Vdash \sim A$. Thus, $\mathfrak{F} \nVdash A$. It is to be noted that $\mathfrak{M}$ satisfies $(R \cap \neq) \subset S$ due to (M1) and $S \subset R$ to (M2).

We have to eliminate each S-reflexive point from $\mathfrak{M}$. Remark that it follows from $S \subset R$ that each S-reflexive point is also R-reflexive point. Thus we replace each subframe $\langle \{c\}, \{\langle c, c\rangle\}, \{\langle c, c\rangle\}\rangle$ of $\mathfrak{F}$ with the new subframe consisting of two points c_1, c_2 where S is symmetric and R is symmetric and reflexive. By construction as this, we can obtain 'bulldozed model' $\mathfrak{M}' = \langle \mathfrak{F}', V' \rangle$. Then we can prove that $S' = (R' \cap \neq)$, whence $\mathfrak{M}'$ is $\mathcal{L}(\Box, \blacksquare)$-model. Finally, we can claim that f is surjective bimodal p- morphism. From Fact 2 this means that $\mathfrak{F}' \Vdash A$ implies $\mathfrak{F} \Vdash A$. Thus, it follows from $\mathfrak{F} \nVdash A$ that $\mathfrak{F}' \nVdash A$. We have thus proved the theorem. QED

Definition 5 (Lemmon-Scott Properties). R^n is defined inductively as follows: wR^0w' iff $w = w'$, $R^{n+1} = R^n \circ R$, where $\circ$ is composition. The Lemmon-Scott Axioms $\mathrm{G}'_{(m,n,j,k)}$: $\Diamond^m \Box^n A \supset \Box^j \Diamond^k A$ corresponds to the following properties:

$$(\mathrm{C}_{m,n,j,k}) \qquad (\forall x, y, z)\,[[xR^m y \text{ and } xR^j z] \text{ imply } (\exists w)\,[yR^n w \text{ and } zR^k w]],$$

where $m, j, n, k \in \omega$. Note that $(\mathrm{C}_{m,n,j,k})$ contains equality in the case $n = k = 0$.

In general, a bimodal model $\mathfrak{M} = \langle W, R, S, V \rangle$ can be regarded as the first-order structure with signature $\langle \boldsymbol{R}, \boldsymbol{S}, \{\boldsymbol{P}_i \mid i \in \omega\} \rangle$ by the following identification: $|\mathfrak{M}| = W$, $\boldsymbol{R}^{\mathfrak{M}} = R$, $\boldsymbol{S}^{\mathfrak{M}} = S$ and $\boldsymbol{P}_i^{\mathfrak{M}} = V(p_i)$, for any $i \in \omega$.

In the Proof of Theorem 4, consider reduct structures $\langle W, R \rangle$ of the canonical model $\mathfrak{M}$ and $\langle W', R' \rangle$ of the transformed model $\mathfrak{M}'$. By construction, we can show that $wR'v$ iff $f(w)Rf(v)$. Thus, f is surjective homomorphism [2, p.94] of $\langle W', R' \rangle$ onto $\langle W, R \rangle$.

Lemma 6. *For any sentence σ not containing the equality symbol in the language with signature $\langle \boldsymbol{R} \rangle$, $\langle W', R' \rangle \models \sigma$ iff $\langle W, R \rangle \models \sigma$, where $\models$ is the first-order satisfaction relation.*

Proof. For any formula in the language with signature $\langle \boldsymbol{R} \rangle$, See [2, p.96, Homomorphism Theorem]. QED

From Theorem 4 and Lemma 6, we can deduce that:

Theorem 7. *Except the case $n = k = 0$, $\mathrm{K}_{\Box\blacksquare} + \mathrm{G}'_{(m,n,j,k)}$ are sound and complete with the class of $\mathcal{L}(\Box, \blacksquare)$-frames satisfying $(\mathrm{C}_{m,n,j,k})$.*

3 Finite Model Property

Definition 8. $\mathcal{L}(\Box, \blacksquare)$ has *the finite model property* (with respect to the class of any models) if the following holds: for any formula A, A is satisfiable in some model, then it is satisfiable in a finite model.

Definition 9. A set Σ of $\mathcal{L}(\Box, \blacksquare)$-formulas is *subformula closed* if for all formulas A, B: if $\sim A \in \Sigma$ then so is A; if $A \supset B \in \Sigma$ then so are A, B; if $\Box A \in \Sigma$ then so is A; if $\blacksquare A \in \Sigma$ then so is A.

Definition 10 (Filtrations). Let $\mathfrak{M} = \langle W, R, V \rangle$ be $\mathcal{L}(\Box, \blacksquare)$-model and Σ be subformula closed set of formulas. For any $w \in W$, let $\Sigma_w = \{ A \in \Sigma \mid \mathfrak{M}, w \Vdash A \}$. Define the equivalence relation $w \sim_\Sigma w'$ by $\Sigma_w = \Sigma_{w'}$. We denote the equivalence class of a state w of $\mathfrak{M}$ with respect to $\sim_\Sigma$ by $[w]$.

Let $W_\Sigma = \{ [w] \mid w \in W \}$. Suppose $\mathfrak{M}^f_\Sigma$ is any model $\langle W^f, R^f, V^f \rangle$ such that:

(i) $W^f = W_\Sigma$.

(ii) If wRw' then $[w]R^f[w']$.

(iii) If $[w]R^f[w']$ then $[\mathfrak{M}, w \Vdash \Box B$ implies $\mathfrak{M}, w' \Vdash B]$ for any $\Box B \in \Sigma$.

(iv) $V^f(p) = \{ [w] \mid \mathfrak{M}, w \Vdash p \}$ for any $p \in \Sigma$.

Then $\mathfrak{M}^f_\Sigma$ is called a *filtration of* $\mathfrak{M}$ *through* Σ.

The *Finest filtration* R^s is defined as follows: $[w]R^s[w']$ iff $(\exists x \in [w])(\exists y \in [w'])\, xRy$.

Theorem 11. $\mathcal{L}(\Box, \blacksquare)$ *has the finite model property.*

Proof. Suppose that $\mathfrak{M}, w \Vdash A$, where $\mathfrak{M} = \langle W, R, V \rangle$. Let Σ be the set of all subformulas of A. An element of $W_\Sigma = \{ [w] \mid w \in W \}$ satisfies either (a) $(\exists u \in [w])(\exists v \in [w])\, u(R\cap \neq)v$ or (b) $(\forall u \in [w])(\forall v \in [w])\,[uRv$ implies $u = v]$. In the case (a), we choose the state $v_{[w]} \in [w]$ such that $u(R\cap \neq)v_{[w]}$. In the case (b), we choose an arbitrary state $v_{[w]} \in [w]$. Let $D = \{ v_{[w]} \mid [w] \in W_\Sigma \}$. Fix one propositional variable $d \notin \Sigma$ and consider d-variant valuation V' such that $V'(d) = D$ and $V'(p) = V(p)$ for any $p \neq d$. Write $\mathfrak{M}' = \langle W, R, V' \rangle$. Obviously, for any formula $B \in \Sigma$ and any $w \in W$, $\mathfrak{M}, w \Vdash B$ iff $\mathfrak{M}', w \Vdash B$. From $\mathfrak{M}, w \Vdash A$ we may infer that $\mathfrak{M}', w \Vdash A$.

Take the finest filtration $\mathfrak{M}'^f_{\Sigma\cup\{d\}}$. Then, by induction, we can show that for any $B \in \Sigma \cup \{ d \}$ and any $w \in W$, $\mathfrak{M}', w \Vdash B$ iff $\mathfrak{M}'^f_{\Sigma\cup\{d\}}, |w| \Vdash B$, where $|w|$ is the equivalence class of a state w with respect to $\sim_{\Sigma\cup\{d\}}$ (As for formulas of the form $\blacksquare C$, we need the finest filtration). It follows from $\mathfrak{M}', w \Vdash A$ that $\mathfrak{M}'^f_{\Sigma\cup\{d\}}, |w| \Vdash A$. Thus, we can conclude that A is satisfiable in the finite model $\mathfrak{M}'^f_{\Sigma\cup\{d\}}$. QED

Fact 12. *The finest filtration R^s preserves the Lemmon-Scott Properties ($C_{m,n,j,k}$), where $m, j \leq 1$ and $(m, j) \neq (1, 1)$.*

From Theorem 7, 11 and Fact 12, we can conclude that:

Corollary 13. *With the case $m, j \leq 1$, $(m, j) \neq (1, 1)$ and except the case $n = k = 0$, $\mathrm{K}_{\Box\blacksquare} + \mathrm{G}'_{(m,n,,j,k)}$ are sound and complete with the class of finite $\mathcal{L}(\Box, \blacksquare)$-frames satisfying ($C_{m,n,j,k}$).*

Corollary 14. *With the case $m, j \leq 1$, $(m, j) \neq (1, 1)$ and except the case $n = k = 0$, $\mathrm{K}_{\Box\blacksquare} + \mathrm{G}'_{(m,n,,j,k)}$ are decidable.*

References

[1] Blackburn, P., de Rijke, M. and Venema, Y., *Modal Logic*, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2001.

[2] Enderton, B., *A Mathematical Introduction to Logic*, Academic Press, 2001.

[3] Kashima, Ryo., *On the completeness of Bimodal Logics with Irreflexive Modality (in Japanese)*, in the proceedings of Mathematical Society of Japan Annual Colloquium of Foundation of Mathematics, pp.24-5, 2004.

[4] Koymans, R., *Specifying Message Passing and Time-Critical Systems with Temporal Logic*, LNCS 651, Springer-Verlag, Berlin, 1992.

[5] Mouri, M., *Proof theoretical Study of Non-Classical Logics*, Ph D thesis, School of Information Science, JAIST, 2002.

[6] de Rijke, M., The Modal Logic of Inequality, *the Journal of Symbolic Logic*, vol.57, pp.566-84, 1992.

[7] de Rijke, M., *Extending Modal Logic*, Ph D thesis, ILLC, University of Amsterdam, 1993.