

Contentwise Complexity of Inferences in Epistemic Logic

Mamoru Kaneko, University of Tsukuba, kaneko@shako.sk.tsukuba.ac.jp
Nobu-Yuki Suzuki, Shizuoka University, smnsuzu@ipc.shizuoka.ac.jp

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Decision making in a game needs **interpersonal** epistemic introspections as well as **intrapersonal** inferences. Here a game means a normal form game with multiple players in the sense of game theory, and each player makes a decision of choosing a strategy based on his decision-prediction criterion. This decision making needs his logical inferences. In this paper, we focus on the intrapersonal inferences, while we have already developed the framework of epistemic logics GL_{EF} for the interpersonal epistemic introspections in Kaneko-Suzuki [1], [2] and [3] in both proof-theoretic and semantic manners. We will develop the concept of the contentwise complexity measure η of intrapersonal inferences, while applying the concept to game theoretical decision making in various examples.

For the development of our theory, we first adopt epistemic logic IG_{EF} which is obtained from GL_{EF} by substituting intuitionistic logic as the base logic for classical logic. This substitution is crucial for the development of the theory of the contentwise complexity measure, though the definition of the measure is possible independent of a base logic. The subscripts E and F are called **descriptive and inferential epistemic structures**, which are constraints on interpersonal epistemic introspections. The first E constrains the description of formulae or sequents just syntactically, but F constrains how deep a player thinks about other players' minds, in which sense the latter is more important.

Epistemic logic IG_{EF} or GL_{EF} suffices for the consideration of interpersonal introspections required for decision making in a game. By provability or unprovability defined in IG_{EF} or GL_{EF} , we can discuss a player's capability of decision making in a game. However, in this discussion, we inevitably ignore the differences in proofs: some proofs are more or less complex than others. To capture such differences, we will introduce the contentwise complexity for inferences in IG_{EF} .

First, we formulate IG_{EF} and GL_{EF} as sequent calculi in the Gentzen-style. Then, the additional restriction that the succedent of any sequent is at most one formula differentiates IG_{EF} from GL_{EF} . The cut-elimination theorem holds for IG_{EF} as well as GL_{EF} . The cut-elimination theorem is crucial for the development of our theory.

Given a proof P in IG_{EF} , we define the **contentwise complexity** $\eta(P)$ of P to be the number of occurrences of initial sequents of P . Let $\Gamma \rightarrow \Theta$ be an admissible sequent in IG_{EF} . Then we define the contentwise complexity $\eta(\Gamma \rightarrow \Theta)$ by

$$\eta(\Gamma \rightarrow \Theta) = \begin{cases} \min \{ \eta(P) : P \text{ is a proof of } \Gamma \rightarrow \Theta \} & \text{if } \Gamma \rightarrow \Theta \text{ is provable} \\ +\infty & \text{otherwise,} \end{cases}$$

where we restrict proofs to cut-free ones. In the following, we regard the contentwise complexity measure η as applied to sequents, and the application of η to proofs is regarded as an intermediate step. Incidentally, if we allow proofs with cuts, $\eta(\Gamma \rightarrow \Theta)$ is denoted by $\eta_{\text{C}}(\Gamma \rightarrow \Theta)$.

When $\Gamma \rightarrow \Theta$ is given, the contentwise complexity $\eta(\Gamma \rightarrow \Theta)$ expresses the number of indispensable contents of $\Gamma \rightarrow \Theta$ to prove this sequent. It measures the contents of $\Gamma \rightarrow \Theta$ from the viewpoint of inferences. It should be emphasized that the contentwise complexity measure η talks about complexity of a single sequent $\Gamma \rightarrow \Theta$, instead of an algorithm applied to a class of sequents.

Although the contentwise complexity measure η for sequents is well defined, it is generally difficult to calculate the value $\eta(\Gamma \rightarrow \Theta)$ for an arbitrarily given sequent $\Gamma \rightarrow \Theta$. We would like to give some method to calculate the value $\eta(\Gamma \rightarrow \Theta)$. In fact, we will specify a class of sequents so that any sequent in the class, the contentwise complexity measure can be calculated in a quite mechanical manner. The main point of this paper is the development of various theorems to enable us to calculate $\eta(\Gamma \rightarrow \Theta)$.

Once the method of calculation is developed, we can apply the contentwise complexity measure η to measure the required inferences for decision making in a game. There are abundant examples, in which the complexity values given by η give insights to our study of such game theoretical decision making. Also, this measure reflects the interactive nature of intrapersonal inferences and interpersonal epistemic introspection. That is, in some examples, interpersonal introspections would decrease contentwise complexity, while in other cases, it would be more complex. The full development of this part remains for the future study.

References

- [1] Kaneko, M. and N.-Y. Suzuki, Bounded interpersonal inferences and decision making, *Economic Theory* 19 (2002), 63-103.
- [2] Kaneko, M. and N.-Y. Suzuki, Epistemic logic of shallow depths and game-theoretical applications, *Advances in Modal Logic* Vol.3 (2002), F. Wolter et al eds. World Scientific Publishing Co. London. 279-298.

- [3] Kaneko, M. and N.-Y. Suzuki, Epistemic models of shallow depths and game theoretical decision making: Horticulture, **Journal of Symbolic Logic** 68 (2003), 163-186.