Conjunctive Normal Forms in S4 and S4Grz

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1 Introduction

Here we treat modal formulas with only one propositional variable p on modal logics **S4** and **S4Grz**. The quotient sets of the set of such formulas modulo the provability of these two modal logics are Boolean with respect to the derivation of the logics (cf. Chagrov and Zakharyaschev [CZ97]). We give an inductive construction for the representatives of the generators of the Boolean.

2 Preliminaries

We use lower case Latin letters p, q for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\bot$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation). By $\mathbf{S}(p)$, we mean the set of formulas constructed from p by using $\wedge$, $\vee$, $\supset$ and $\Box$. The depth $d(A)$ of a formula $A \in \mathbf{S}(p)$ is defined inductively as follows:

(1) $d(p) = 0$,

(2) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,

(3) $d(\Box B) = d(B) + 1$.

We put $\mathbf{S}^n(p) = \{A \in \mathbf{S}(p) \mid d(A) \leq n\}$.

Let **ENU** be an enumeration of the formulas in $\mathbf{S}(p)$. For a non-empty finite set $\mathbf{S}$ of formulas, the expressions

$$\bigwedge \mathbf{S} \qquad \text{and} \qquad \bigvee \mathbf{S}$$

denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \qquad \text{and} \qquad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

where $\{A_1, \cdots, A_n\} = \mathbf{S}$ and A_i occurs earlier than A_{i+1} in **ENU**. Also the expression

$$\bigwedge \emptyset$$

denote the formula $p \supset p$.

By **S4**, we mean the smallest set of formulas containing all the tautologies and the axioms

$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$,

$T : \Box p \supset p$,

$4 : \Box p \supset \Box\Box p$

and closed under modus ponens, substitution and necessitation. By **S4Grz**, we mean the logic obtained by adding

$Grz : \Box(\Box(p \supset \Box p) \supset p) \supset p$ (Grzegorczyk axiom),

to **S4**.

Let **L** be either **S4** or **S4Grz**. For formulas A and B, we use the expression $A \equiv_{\mathbf{L}} B$ instead of $(A \supset B) \wedge (B \supset A) \in \mathbf{L}$. We note that $\equiv_{\mathbf{L}}$ is an equivalence relation on a set of formulas. We write $[A] \leq_{\mathbf{L}} [B]$ if there exist $A' \in [A]$ and $B' \in [B]$ such that $A' \supset B' \in \mathbf{L}$. Our main purpose is to elucidate the structure $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{L}}, \leq_{\mathbf{L}} \rangle$. However, it is known that the structure is Boolean. Here we construct a concrete representative of each generator. Representatives of the other equivalent classes can be expressed as conjunctions of the representatives of generators.

3 Construction of representatives for S4Grz

In this section, we give a concrete representative of each generator of $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{S4Grz}}, \leq_{\mathbf{S4Grz}}\rangle$.

Definition 3.1. Formulas F_n $(n = 0, 1, 2, \cdots)$ are defined inductively as follows:
$F_0 = p$,
$F_1 = p \supset \Box p$,
$F_{k+2} = F_k \vee \Box F_{k+1}$.

Definition 3.2. Formulas E_n $(n = 2, 3, 4, \cdots)$ are defined as follows:
$E_n = F_{n-2} \vee (\Box F_{n-1} \supset \Box p)$.

We note that $E_n, F_n \in \mathbf{S}^n(p)$.

Definition 3.3. The sets $\mathbf{G}_n$ $(n = 0, 1, 2, \cdots)$ of formulas are defined inductively as follows:
$\mathbf{G}_0 = \{F_0\}$,
$\mathbf{G}_1 = \{F_0, F_1\}$,
$\mathbf{G}_{k+2} = \{F_{k+1}, F_{k+2}, E_2, \cdots, E_{k+2}\}$.

We note that $\mathbf{G}_n$ has just $n + 1$ elements.

Theorem 3.4.
(1) $\mathbf{S}^n(p)/\equiv_{\mathbf{S4Grz}}= \{[\bigwedge_{A\in\mathbf{S}} A] \mid \mathbf{S} \subseteq \mathbf{G}_n\}$.
(2) *For subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbf{G}_n$*,*
(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge_{A\in\mathbf{S}_1} A] \leq_{\mathbf{S4Grz}} [\bigwedge_{A\in\mathbf{S}_2} A]$,
(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge_{A\in\mathbf{S}_1} A] = [\bigwedge_{A\in\mathbf{S}_2} A]$.
(3) $\mathbf{S}^n(p)/\equiv_{\mathbf{S4Grz}}$ *has just* 2^{n+1} *elements.*

4 Construction of representatives for S4

In this section, we give a concrete representative of each generator of $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{S4}}, \leq_{\mathbf{S4}}\rangle$.

To construct such representatives, we use a sequent. We use Greek letters, Γ and Δ, possibly with suffixes, for finite sets of formulas. The expression $\Gamma^\Box$ denote the set $\{\Box A \mid \Box A \in \Gamma\}$. By a sequent, we mean the expression $(\Gamma \to \Delta)$. We often write $\Gamma \to \Delta$ instead of the expression with the parentheses. By **SEQ**, we mean the set of the sequents. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

We put

$$f(\Gamma \to \Delta) = \begin{cases} \bigwedge \Gamma \supset \bigvee \Delta & \text{if } \Gamma \neq \emptyset \\ \bigvee \Delta & \text{if } \Gamma = \emptyset, \end{cases}$$

and for a set $\mathcal{S}$ of sequents,

$$f(\mathcal{S}) = \{f(X) \mid X \in \mathcal{S}\}.$$

Definition 4.1.

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma, \qquad \mathbf{suc}(\Gamma \to \Delta) = \Delta, \qquad \mathbf{R}(\Gamma \to \Delta) = \{\Sigma \to \Lambda \mid \Gamma^\Box \cap \Lambda^\Box \neq \emptyset\}.$$

Definition 4.2. The sets $\mathcal{S}_n, \mathcal{G}_n, \mathcal{G}_n^*$ $(n = 0, 1, 2.\cdots)$ of sequents and the mappings
$\mathcal{S}^+ : \bigcup_{k=1}^{\infty} \mathcal{S}_k \to \mathbf{SEQ}$,
$\mathcal{T} : \bigcup_{k=1}^{\infty} \mathcal{S}_k \to \mathbf{SEQ}$,
$\mathcal{S} : \bigcup_{k=1}^{\infty} \mathcal{S}_k \to \mathbf{SEQ}$,
$^* : \bigcup_{k=2}^{\infty} \bigcup_{X \in \mathcal{S}_k} \mathcal{S}^+(X) \to \mathbf{SEQ}$
are defined inductively as follows:

$\mathcal{S}_0 = \{\to p\}$, $\mathcal{G}_0 = \mathcal{G}_0^* = \emptyset$,

$\mathcal{S}_1 = \{(p \to \Box p), (\to p)\}$, $\mathcal{G}_1 = \mathcal{G}_1^* = \emptyset$,

Suppose $n \geq 1$. For $X \in \mathcal{S}_n$,

$\mathcal{S}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = f(\mathcal{S}_n), \Gamma \cap \Delta = \emptyset, f(X) \in \Delta\}$,

$\mathcal{T}(X) = \{Y \in \mathcal{S}^+(X) \mid f(Y) \in \mathbf{S4}\}$,

$\mathcal{S}(X) = \mathcal{S}^+(X) - \mathcal{T}(X)$,

for $Z \in \mathcal{S}^+(X)$, $Z^* = \begin{cases} (\Box\Gamma, \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Box Y, \Box X & \text{if } Z = (\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box X, \Box Y) \\ Z & \text{otherwise.} \end{cases}$

And

$$\mathcal{S}_{n+1} = \bigcup_{X \in \mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*)} \mathcal{S}(X),$$

$\mathcal{G}_{n+1} = \{X \in \mathcal{S}_{n+1} \mid \mathbf{R}(X) \cap \mathcal{S}_{n+1} = \mathcal{S}_{n+1} - \{X\}\}$,

$\mathcal{G}_{n+1}^* = \{X \in \mathcal{S}_{n+1} \mid X \neq X^*, X^* \in \mathcal{S}_{n+1}, \mathbf{R}(X) \cap \mathcal{S}_{n+1} = \mathcal{S}_{n+1} - \{X, X^*\}\}$.

Definition 4.3.

$$\mathbf{G}_n^* = f(\mathcal{S}_n \cup \bigcup_{k=0}^{n-1} (\mathcal{G}_n \cup \mathcal{G}_n^*))$$

Theorem 4.4.
(1) $\mathbf{S}^n(p)/\equiv_{\mathbf{S4}} = \{[\bigwedge_{A \in \mathbf{S}} A] \mid \mathbf{S} \subseteq \mathbf{G}_n^*\}$.

(2) *For subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbf{G}_n^*$,

(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge_{A \in \mathbf{S}_1} A] \leq_{\mathbf{S4}} [\bigwedge_{A \in \mathbf{S}_2} A]$,

(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge_{A \in \mathbf{S}_1} A] = [\bigwedge_{A \in \mathbf{S}_2} A]$.

References

[CZ97] A. Chagrov and M. Zakharyaschev, *Modal Logic*, Oxford University Press, 1997.