Gentzen-Style Sequent Calculi for Strict Implication

Ryo Ishigaki
ZAN35044@nifty.com
Department of Mathematical and Computing Sciences, Tokyo Institute of Technology

1 Introduction

Strict implication (denoted by $\rightarrow$) is different from *material implication* of classical logic. Strict implication played an important role in the studies of modal logics. We introduce sequent systems for *weak logics with strict implication*, and prove the completeness theorems.

2 Kripke Semantics

Definition 2.1 The propositional language L^W has the symbols $\mathcal{PV} \cup \{\wedge, \vee, \top, \bot, \rightarrow\}$, where $\mathcal{PV}$ is a countable infinite set of propositional variables. Propositional variables are denoted by $p, q, \ldots$ *Formulae* and their *subformulae* are defined in the usual way, and $A, B, \ldots$ are meta-variables for formulae. $\Gamma, \Delta, \ldots$ and $x, y, \ldots$ are meta-variables for finite sets of formulae. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. While sequents are concerned, we usually write $A_1, \ldots, A_n$ for $\{A_1, \ldots, A_n\}$, and Γ, Δ for $\Gamma \cup \Delta$, and so on. *Strict negation* $\neg A$ is defined as $A \rightarrow \bot$. $\mathrm{Sub}(x)$ is defined as the set of all subformulae which occurs in the set x.

Definition 2.2 A *model* $\mathcal{M}$ is a triple $\langle W, R, V \rangle$, where W is a non-empty set, R is a binary relation on W, and V is a function from $\mathcal{PV}$ to 2^W.

Definition 2.3 Let $\mathcal{M} = \langle W, R, V \rangle$ be any model. $\mathcal{M}$ is *isolated*, if $\forall x, y \in W$ [xRy implies $x = y$]. $\mathcal{M}$ is *truth-preserving*, if $\forall x, y \in W$ [$x \in V(p)$ and xRy imply $y \in V(p)$] for any $p \in \mathcal{PV}$. $\mathcal{M}$ is *falsity-preserving*, if $\forall x, y \in W$ [$x \notin V(p)$ and xRy imply $y \notin V(p)$] for any $p \in \mathcal{PV}$.

Definition 2.4 Given a model $\mathcal{M} = \langle W, R, V \rangle$, the *truth-valuation* of formulae is a function from $W \times \mathcal{FORM}$ (the set of all formulae) to $\{\mathtt{True}, \mathtt{False}\}$, and is defined inductively as follows:

- $\mathcal{M}(x, p) = \mathtt{True}$ iff $x \in V(p)$ for any $p \in \mathcal{PV}$, $\mathcal{M}(x, \top) = \mathtt{True}$, $\mathcal{M}(x, \bot) = \mathtt{False}$,
- $\mathcal{M}(x, A \wedge B) = \mathtt{True}$ iff $\mathcal{M}(x, A) = \mathtt{True}$ and $\mathcal{M}(x, B) = \mathtt{True}$,
- $\mathcal{M}(x, A \vee B) = \mathtt{True}$ iff $\mathcal{M}(x, A) = \mathtt{True}$ or $\mathcal{M}(x, B) = \mathtt{True}$,
- $\mathcal{M}(x, A \rightarrow B) = \mathtt{True}$ iff $\forall y \in W$ [xRy and $\mathcal{M}(y, A) = \mathtt{True}$ imply $\mathcal{M}(y, B) = \mathtt{True}$].

A formula F is *valid* in $\mathcal{M}$, if $\mathcal{M}(x, F) = \mathtt{True}$ for any $x \in W$.

Definition 2.5 Given a model $\mathcal{M} = \langle W, R, V \rangle$, the truth-valuation of sequents is a function from $W \times \mathcal{SEQ}$ (the set of all sequents) to $\{\mathtt{True}, \mathtt{False}\}$, and is defined as follows:

- $\mathcal{M}(x, \Gamma \Rightarrow \Delta) = \mathtt{True}$ iff $\forall C \in \Gamma$ [$\mathcal{M}(x, C) = \mathtt{True}$] implies $\exists D \in \Delta$ [$\mathcal{M}(x, D) = \mathtt{True}$].

A sequent $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$, if $\mathcal{M}(x, \Gamma \Rightarrow \Delta) = \mathtt{True}$ for any $x \in W$.

Lemma 2.6 For any transitive and truth-preserving model $\mathcal{M} = \langle W, R, V \rangle$, and any formula F of the language L^W, we have $\forall x, y \in W$ [$\mathcal{M}(x, F) = \mathtt{True}$ and xRy imply $\mathcal{M}(y, F) = \mathtt{True}$].

Lemma 2.7 For any euclidean and falsity-preserving model $\mathcal{M} = \langle W, R, V \rangle$, and any formula F of the language L^W, we have $\forall x, y \in W$ [$\mathcal{M}(x, F) = \mathtt{False}$ and xRy imply $\mathcal{M}(y, F) = \mathtt{False}$].

3 Sequent Calculi for Weak Logics with Strict Implication

Definition 3.1 A sequent system $\mathbf{G}\mathsf{K}^W$ (the fragment of $\mathbf{G}\mathsf{K}^I$, introduced in Kashima [3]) is defined from following axioms (initial sequents) and rules:

$$\frac{}{A \Rightarrow A}\text{(Identity)} \quad \frac{}{\Rightarrow \top}\text{(Truth)} \quad \frac{}{\bot \Rightarrow}\text{(Falsity)}$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\text{(Weakening L)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\text{(Weakening R)} \quad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{(cut)}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}(\wedge\text{L}) \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}(\wedge\text{R}) \quad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}(\vee\text{L}) \quad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}(\vee\text{R})$$

$$\frac{\Delta_1, A \Rightarrow B, \Gamma_1 \quad \Delta_2, A \Rightarrow B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, A \Rightarrow B, \Gamma_{2^n}}{C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B}(\to \mathsf{K})$$

where $n \geq 0$, the rule $(\to \mathsf{K})$ has 2^n premises, $\Gamma_i = \{C_j \mid j \in \gamma(i)\}$, $\Delta_i = \{D_j \mid j \in \delta(i)\}$, and the sets of natural numbers $\gamma(i)$ and $\delta(i)$ are defined as follows: We enumerate all of the subsets of $\{1, \cdots, n\}$. $\delta(i)$ is the ith subset, and $\gamma(i)$ is $\{1, \cdots, n\} \backslash \delta(i)$.

Example 3.2 When $n = 2$, the rule $(\to \mathsf{K})$ takes the form as follows:

$$\frac{A \Rightarrow B, C_1, C_2 \quad D_1, A \Rightarrow B, C_2 \quad D_2, A \Rightarrow B, C_1 \quad D_1, D_2, A \Rightarrow B}{C_1 \to D_1, C_2 \to D_2 \Rightarrow A \to B}(\to \mathsf{K})$$

Theorem 3.3 (Completeness 1) Let $\Gamma \Rightarrow \Delta$ be any sequent.

1. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W$ iff it is valid in every finite model.
2. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{D})$ iff it is valid in every finite serial model.

$$\frac{}{\top \to \bot \Rightarrow}(\text{Seriality, } \mathsf{D}) \quad \frac{}{A, A \to B \Rightarrow B}(\text{Reflexivity, } \mathsf{T}) \quad \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}(\to \text{L})$$

3. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{T})$ or $\mathbf{G}\mathsf{K}^W + (\to \text{L})$ iff it is valid in every finite reflexive model.
4. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{4})$ iff it is valid in every finite transitive model.

$$\frac{}{A \to B \Rightarrow C \to (A \to B)}(\text{Transitivity, } \mathsf{4}) \quad \frac{}{A \Rightarrow (A \to B) \to C, B}(\text{Symmetricity, } \mathsf{B})$$

5. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{D}) + (\mathsf{4})$ iff it is valid in every finite serial and transitive model.
6. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{T}) + (\mathsf{4})$ iff it is valid in every finite reflexive and transitive model ($\mathsf{S4}$-model).
7. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{B})$ iff it is valid in every finite symmetric model.
8. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{D}) + (\mathsf{B})$ iff it is valid in every finite serial and symmetric model.
9. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{T}) + (\mathsf{B})$ iff it is valid in every finite reflexive and symmetric model.
10. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{5})$ iff it is valid in every finite euclidean model.

$$\frac{}{\Rightarrow (A \to B) \to C, A \to B}(\text{Euclidity, } \mathsf{5})$$

11. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{D}) + (\mathsf{5})$ iff it is valid in every finite serial and euclidean model.
12. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{4}) + (\mathsf{5})$ iff it is valid in every finite transitive and euclidean model.
13. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{D}) + (\mathsf{4}) + (\mathsf{5})$ iff it is valid in every finite selial, transitive and euclidean model.
14. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{B}) + (\mathsf{4})$ iff it is valid in every finite symmetric, transitive and euclidean model.
15. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{T}) + (\mathsf{5})$ iff it is valid in every finite reflexive, euclidean and transitive model (equivalence model, $\mathsf{S5}$-model).

Theorem 3.4 (Completeness 2) Let $\Gamma \Rightarrow \Delta$ be any sequent.

1. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W$ + (Heredity) + (EM) or $\mathbf{G}\mathsf{K}^W + (\rightarrow \mathrm{R})$ iff it is valid in every finite isolated model.

$$\frac{}{A \Rightarrow B \rightarrow A}\ (\text{Heredity}) \qquad \frac{}{\Rightarrow A \rightarrow \bot, A}\ (\text{EM, Excluded Middle}) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ (\rightarrow \mathrm{R})$$

2. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W$ + (Heredity) iff it is valid in every finite transitive and truth-preserving model (Basic Propositional Logic BPL-model).
3. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{T})$ + (Heredity) iff it is valid in every finite reflexive, transitive and truth-preserving model (Intuitionistic Logic Int-model).
4. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W$ + (EM) iff it is valid in every finite euclidean and falsity-preserving model.
5. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\mathsf{T}) + (\mathsf{5})$ + (Heredity) or $\mathbf{G}\mathsf{K}^W + (\mathsf{T})$ + (EM) iff it is valid in every finite reflexive and isolated model (Classical Logic Cl-model).
6. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W + (\rightarrow \mathsf{GL})$ iff it is valid in every finite transitive model without infinite ascending chains (Gödel-Löb's Logic GL-model).

$$\frac{\Delta_1, \Psi, A \rightarrow B, A \Rightarrow B, \Gamma_1 \quad \cdots \quad \Delta_{2^n}, \Psi, A \rightarrow B, A \Rightarrow B, \Gamma_{2^n}}{C_1 \rightarrow D_1, \ldots, C_n \rightarrow D_n \Rightarrow A \rightarrow B}\ (\rightarrow \mathsf{GL})$$

where $n \geq 0$, $\Psi \equiv \{C_1 \rightarrow D_1, \ldots, C_n \rightarrow D_n\}$, and Γ_i and Δ_i are as in $(\rightarrow \mathsf{K})$.

7. $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{G}\mathsf{K}^W$ + (Heredity) + $(\rightarrow \mathsf{GL})$ iff it is valid in every finite transitive and truth-preserving model without infinite ascending chains (Formal Propositional Logic FPL-model).

We can also construct cut-free sequent systems for $\mathbf{G}\mathsf{K}^W$, $\mathbf{G}\mathsf{K}^W + (\mathsf{D})$, $\mathbf{G}\mathsf{K}^W + (\mathsf{T})$, $\mathbf{G}\mathsf{K}^W + (\mathsf{4})$, $\mathbf{G}\mathsf{K}^W + (\mathsf{D}) + (\mathsf{4})$, $\mathbf{G}\mathsf{K}^W + (\mathsf{T}) + (\mathsf{4})$, and so on.

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