Strict implication logics (e.g., intuitionistic propositional logic) correspond to removing symmetry from bisimilarity

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The purpose of this contribution is to explain several results on the strict implication fragment (together with falsum, true, conjunction and disjunction) of the modal propositional language. Although the strict implication language is a proper fragment of the modal language, the results that we present suggest that the strict implication fragment is very close to the full modal language. Indeed, the strict implication fragment allows us to recuperate most of the results that we have in the modal language (but in a more complex way).

This contribution explains some of the results contained in the author's Ph.D. Dissertation [Bou04, Chapter 3]. Due to simplicity reasons we restrict ourselves to one of the strict-weak languages introduced in the dissertation, the strict implication language.

Fix a set Prop of *(atomic) propositions*. The set $\mathcal{L}^{mod}$ of *modal formulas* is the smallest set X such that i) $\mathsf{Prop} \subseteq X$, ii) X contains the logical constants $\perp$ (*falsum*) and $\top$ (*true*), and iii) X is closed under the Boolean connectives $\wedge, \vee, \sim, \supset$ (*conjunction, disjunction, negation, implication*) as well as under the unary *necessity operator* $\Box$. The semantics on modal formulas that we consider is based on pointed Kripke models, and it is the standard one [CZ97, BdRV01]. Under this semantics it is well-known that a modal formula φ can be seen as a first-order formula $ST_v(\varphi)$ with one free variable v, where ST refers to *standard translation*. For instance, $\Box p$ corresponds to the first-order formula $\forall v_1(Rvv_1 \supset Pv_1)$.

The *strict implication* fragment is the set $\mathcal{L}^s \subseteq \mathcal{L}^{mod}$ defined as the smallest set X such that i) $\mathsf{Prop} \cup \{\perp, \top\} \subseteq X$, ii) X is closed under $\wedge, \vee$, and iii) if $\varphi, \psi \in X$ then $\Box(\varphi \supset \psi) \in X$. We will write $\varphi \to \psi$ for $\Box(\varphi \supset \psi)$. The semantic condition inherited by the binary new connective of *strict implication* $\to$ on Kripke models is the same as the condition for the implication of intuitionistic logic (and superintuitionistic logics) in its standard Kripke semantics [CZ97], and it also corresponds to the semantic condition given in several subintuitionistic logics [Vis81, Cor87, Došen93, Res94, Wan97, CJ01, Rui91, Rui93].

If we add $\sim$ or $\supset$ to $\mathcal{L}^s$ then the language that we obtain has the same expressive power than $\mathcal{L}^{mod}$. On the other hand, we notice that the strict implication language $\mathcal{L}^s$ is a proper fragment of the modal language since neither $\sim$ nor $\supset$ are definable in this fragment. Since it is a proper fragment it does not seem plausible to obtain information on the full modal language $\mathcal{L}^{mod}$ from what happens in this fragment. However, this is what we claim in Corollary 2 and Theorems 6 and 8.

Comparison of the Expressive Power

Theorem 1 (Standard Form) *For every $\varphi \in \mathcal{L}^{mod}$, there exists $k \in \omega$ and $\mathcal{L}^s$-formulas $\nu_0, \ldots, \nu_{k-1}$, $\pi_0, \ldots, \pi_{k-1}$ such that φ and $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ are equivalent (i.e., satisfied in the same pointed Kripke models).*

Corollary 2 *Two pointed Kripke models satisfy the same $\mathcal{L}^{mod}$-formulas iff they satisfy the same $\mathcal{L}^s$-formulas.*

Corollary 3 *For every $\varphi \in \mathcal{L}^{mod}$, $\Box\varphi$ is equivalent to a $\mathcal{L}^s$-formula.*

Model Theory for the Strict Implication Fragment

First of all we remind what happens in the modal case. There, the main notion to understand the modal language is the bisimilarity relation, which is usually introduced using the notion of bisimulation [BdRV01]. A *bisimulation* between two Kripke models $\mathcal{M}$ and $\mathcal{N}$ is a set $Z \subseteq M \times N$ satisfying i) 'atomic proposition invariance' at Z-corresponding states, ii) if $\langle m,n\rangle \in Z$ and $\langle n,n'\rangle \in R^{\mathcal{N}}$ then exists $m' \in M$ such that $\langle m,m'\rangle \in R^{\mathcal{M}}$ and $\langle m',n'\rangle \in Z$, and iii) if $\langle m,n\rangle \in Z$ and $\langle m,m'\rangle \in R^{\mathcal{M}}$ then exists $n' \in N$ such that $\langle n,n'\rangle \in R^{\mathcal{N}}$ and $\langle m',n'\rangle \in Z$. It is said that $\langle \mathcal{M},m\rangle$ and $\langle \mathcal{N},n\rangle$ are *bisimilar* (notation: $\langle \mathcal{M},m\rangle \simeq \langle \mathcal{N},n\rangle$) if there is a bisimulation Z between $\mathcal{M}$ and $\mathcal{N}$ such that $\langle m,n\rangle \in Z$. We notice that $\simeq$ is a relation (indeed it is a proper class) between pointed Kripke models. The interest on the bisimilarity relation comes from the following result: a first order formula $\alpha(v)$ with one free variable v is invariant under the bisimilarity relation iff it is equivalent to $ST_v(\varphi)$ for a certain $\varphi \in \mathcal{L}^{mod}$. Now we develop the model theory for the strict implication fragment. It results that the adequate notion correspond to removing symmetry from the bisimilarity relation, what it is not the same than removing symmetry from the clauses in the definition of bisimulation.

Definition 4 *$\langle \mathcal{M},m\rangle$ is* quasi bisimilar *into $\langle \mathcal{N},n\rangle$ (notation: $\langle \mathcal{M},m\rangle \preceq \langle \mathcal{N},n\rangle$) if the following conditions hold:*

- *if $p \in \mathsf{Prop}$ and $\langle \mathcal{M},m\rangle \Vdash p$, then $\langle \mathcal{N},n\rangle \Vdash p$.*
- *for every n' such that $\langle n,n'\rangle \in R^{\mathcal{N}}$, there is m' such that $\langle m,m'\rangle \in R^{\mathcal{M}}$ and $\langle \mathcal{M},m'\rangle \simeq \langle \mathcal{N},n'\rangle$.*

Theorem 5 *A first order formula $\alpha(v)$ with one free variable v is preserved under the quasi bisimilarity relation iff it is equivalent to $ST_v(\varphi)$ for a certain $\varphi \in \mathcal{L}^s$.*

The previous theorem also holds when we restrict the class of Kripke models to a subclass that is closed under ultraproducts. In particular, this remark applies to the class of intuitionistic Kripke models.

Theorem 6 *The quasi bisimilarity relation $\preceq$ is a quasi order (i.e., reflexive and transitive) that generates the bisimilarity relation $\simeq$. In particular,*

$$\langle \mathcal{M},m\rangle \simeq \langle \mathcal{N},n\rangle \quad \textit{iff} \quad \langle \mathcal{M},m\rangle \preceq \langle \mathcal{N},n\rangle \textit{ and } \langle \mathcal{N},n\rangle \preceq \langle \mathcal{M},m\rangle.$$

In [Bou04], to prove Theorem 5 it is introduced the notion of strongly Hennessy-Milner class, which is the natural generalization of a Hennessy-Milner class [BdRV01] when we restrict ourselves to the strict implication fragment.

Definition 7 *A class K of Kripke models is a* strongly Hennessy-Milner class *if for all $\mathcal{M},\mathcal{N} \in \mathsf{K}$, all $m \in M$, and all $n \in N$, if*

the satisfiability of $\mathcal{L}^s$-formulas is preserved from $\langle \mathcal{M},m\rangle$ into $\langle \mathcal{N},n\rangle$,

then

$\langle \mathcal{M},m\rangle$ is quasi bisimilar into $\langle \mathcal{N},n\rangle$.

It is obvious that strongly Hennessy-Milner classes are interesting from the perspective of the strict implication language. What it is more surprising is that they are also interesting to understand the full modal language, since we can prove the following theorem.

Theorem 8 *Let $\mathcal{M}$ be a Kripke model. The following statements are equivalent:*

- *for every $m \in M$, there is a single modal formula characterizing $\langle \mathcal{M},m\rangle$ up to bisimilarity.*
- *$\mathcal{M} \in \bigcap\{\mathsf{K} : \mathsf{K}$ is a maximal strongly Hennessy-Milner class$\}$.*

Finally, I mention that the quasi bisimilarity relation has a very natural counterpart in the theory of non-well founded sets under the Antifoundation Axiom [Acz88]. It corresponds to the dual of inclusion in the same way than the bisimilarity relation corresponds to equality.

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