Complexity of tense logics of linear time flows

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Abstract

We prove that all finitely axiomatizable tense logics with temporal operators for 'always in the future' and 'always in the past' and determined by linear flows time are coNP-complete. It follows, for example, that all tense logics containing a density axiom of the form $\Box_F^{n+1}p \to \Box_F^n p$, for some $n \geq 0$, are coNP-complete. Additionally, we prove coNP-completeness of all $\bigcap$-irreducible tense logics. As these classes of tense logics contain many Kripke incomplete bimodal logics, we obtain many natural examples of Kripke incomplete normal bimodal logics which are nevertheless coNP-complete. This distinguishes our result for other general coNP-completeness results obtained before, as they usually were heavily based on polynomial finite model property of the logics involved.

1 Preliminaries

This work is an extended abstract of [1]. Consult [3] or [2] for more information, details of notation and discussion of notions introduced below. Formulas of propositional tense logic are built from propositional variables $p_1, p_2, \ldots$ using the boolean connectives $\wedge$, $\vee$, $\to$, and $\neg$ and the temporal operators $\Box_F$ ('always in the future'), $\Diamond_F$ ('eventually'), $\Box_P$ ('always in the past'), and $\Diamond_P$ ('at some moment in the past'). We interpret this language in *general tense frames* $\mathfrak{F} = \langle W, R, P\rangle$, where $\langle W, R\rangle$ is a linear ordering (i.e., R is transitive and connected

$$\forall x \forall y (xRy \vee yRx \vee x = y))$$

and P is a set of subsets of W closed under intersection, complement, and the operators $\Box_F^{\mathfrak{F}}$ and $\Box_P^{\mathfrak{F}}$ defined by

$$\Box_F^{\mathfrak{F}} X = \{x \in W \mid \forall y \in W(xRy \to y \in X\}$$

and

$$\Box_P^{\mathfrak{F}} X = \{x \in W \mid \forall y \in W(xR^{-1}y \to y \in X\}.$$

A valuation $\mathfrak{V}$ in $\mathfrak{F}$ maps propositional variables to elements of P. The satisfaction of a formula φ in a point $w \in W$ in the model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, in symbols $(\mathfrak{M}, w) \models \varphi$, is defined as usual. φ is called *valid* in $\mathfrak{F}$ if it is true in every point of $\mathfrak{F}$ under every valuation. If $P = 2^W$, then we say that $\mathfrak{F}$ is *a tense frame* and set $\mathfrak{F} = \langle W, R\rangle$.

By **Lin** we denote the set of all formulas which are valid in all tense frames. A *tense logic* is a set of formulas containing **Lin** which is closed under modus ponens, substitution, and necessitation (from φ derive $\Box_P\varphi$ and $\Box_F\varphi$). A tense logic L is called *finitely axiomatizable* if there exists a formula φ such that L is the smallest tense logic containing φ; in this case we set $L = \mathbf{Lin} \oplus \varphi$.

All tense logics are determined by some class of general tense frames and, conversely, every class of general tense frames determines a tense logic. Well-known examples of tense logics are the logics determined by $\langle \mathbb{N}, <\rangle$, $\langle \mathbb{Q}, <\rangle$ and $\langle \mathbb{R}, <\rangle$. Call a tense logic L $\bigcap$-irreducible if, for any sequence $L_0, L_1 \ldots$ of tense logics it follows from $L = \bigcap_{i<\omega} L_i$ that $L = L_i$ for some $i < \omega$. Now we can formulate our main result:

Theorem 1 *(i) All finitely axiomatizable tense logics are coNP-complete.*
(ii) All $\bigcap$-irreducible tense logics are coNP-complete.

Notice that there exist *non-finitely axiomatizable* tense logics of very high complexity: just consider, for any $M \subseteq \mathbb{N}$, the logic L_M determined by the class of frames

$$\{\langle\{1, \dots, m\}, <\rangle \mid m \in M\}.$$

Set $\bot = p \wedge \neg p$ and $\top = p \vee \neg p$. Then the formula

$$\varphi_m = \Box_P \bot \wedge \Box_F^{m+1} \bot \wedge \Diamond_F^m \top$$

is satisfiable in a frame validating L_M iff $m + 1 \in M$.

2 The proof in a nutshell

We begin by defining *quasi-models* for **Lin** — *finite frames with types.* Fix two symbols m and j. A *cluster assignment* is a mapping $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ from the set of clusters of a finite tense frame into $\{\mathsf{m}, \mathsf{j}\} \times \{\mathsf{m}, \mathsf{j}\}$. Irreflexive points are mapped onto (m, m). *A frame with types* is just a frame with cluster assignments. The idea: if a cluster is assigned j in the first coordinate, it cannot contain a maximal point of any subformula of a formula we are trying to refute (or satisfy) under a given valuation. If j is in the second coordinate, the cluster cannot contain a minimal point. Thus, a valuation in a frame with types $(\mathfrak{F}, \mathbf{t})$ will be called *good for* φ if this condition is satisfied.

Fact 2 *For every formula φ there exists a finite family of frames with types $M(\varphi)$ s.t. every countermodel for φ can be filtrated (via minimal/maximal point technique) to a $\neg\varphi$-good model based on a frame from $M(\varphi)$. Moreover, the size of all elements of $M(\varphi)$ is polynomially bounded by $l(\varphi)$ — the cardinality of the closure of the set of all subformulas of φ under single negations.*

Thus, we can devise an apparatus of *canonical formulas for frames with types* — $\alpha(\mathfrak{F}, \mathbf{t})$. They can axiomatize every extension of **Lin**.

The second ingredient of our proof is a family of effective general frames needed for a general completeness result. The family $\mathfrak{B}$ defined below provides building blocks of these frames:

- $\bullet$, $\circ$, ⓚ — finite clusters;
- $\mathfrak{C}(0, ⓚ)$, $\mathfrak{C}(ⓚ, 0)$ — the descriptive frames corresponding to natural numbers with strict/reverse strict order where admissible sets are generated by singletons and sets of the form $\{m \mid m \equiv k (mod\, i)\}$, $i < k$;
- $\mathfrak{C}(n, ⓚ)$, $\mathfrak{C}(ⓚ, n)$ — as above, but now every nth point is reflexive;
- $\mathfrak{C}(0, ①, 0)$ — the descriptive frame corresponding to natural numbers with strict order followed by natural numbers with strict reverse order where admissible sets are finite or cofinite.

$\mathfrak{B}_0 = \mathfrak{B} - \{\mathfrak{C}(0, ①, 0)\}$. For a given class $\mathfrak{C}$, $\mathfrak{C}^*$ is the family of all finite sequences of elements from $\mathfrak{C}$. For a finite sequence of frames $\overline{\mathfrak{F}}$, $[\overline{\mathfrak{F}}]$ denotes *the directed union* of $\overline{\mathfrak{F}}$. $[\mathfrak{B}^*]$ is the family of directed unions of elements of $\mathfrak{B}^*$. $[\mathfrak{B}_0^*]$ is the family of directed unions of elements of $\mathfrak{B}_0^*$. Our completeness result is

Fact 3 *Every extension of **Lin** is complete with respect to some family of elements of $[\mathfrak{B}^*]$. In consequence, every $\bigcap$-irreducible extension of **Lin** is determined by some element of $[\mathfrak{B}^*]$.*

For finitely axiomatizable logics, we can even prove more:

Fact 4 *For every non-theorem φ of a finitely axiomatizable extension of **Lin** L, there exists $\overline{\mathfrak{G}} \in \mathfrak{B}^*$ s.t $\neg\varphi$ is satisfied in $[\overline{\mathfrak{G}}]$, $[\overline{\mathfrak{G}}]$ is a frame for L and*

- *the length of the sequence $\overline{\mathfrak{G}}$,*
- *the upper bound on the cardinality of clusters,*

- *and the distance between reflexive points in those $\mathfrak{G}_i$'s which are not finite clusters*

are polynomial in $l(\varphi)$.

Now, we just have bring those two ingredients together. For $\mathfrak{F} \in \mathcal{B}$, define *$m$-reduct of $\mathfrak{F}$* as follows:

- for $\mathfrak{F}$ a cluster, $r_m(\mathfrak{F}) = (\mathfrak{F}, \mathbf{t})$, where $\mathbf{t}$ assigns (m, m) to the cluster,
- $r_m(\mathfrak{C}(0, \text{ⓚ})) = [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1} \lhd (\text{ⓚ}, (\mathsf{m}, \mathsf{j}))$,
- $r_m(\mathfrak{C}(n, \text{ⓚ})) = [(\circ, (\mathsf{m}, \mathsf{m})) \lhd [(\bullet, (\mathsf{m}, \mathsf{m}))]^{n-1}]^{m+1} \lhd (\text{ⓚ}, (\mathsf{m}, \mathsf{j}))$,
- $r_m(\mathfrak{C}(\text{ⓚ}, n)) = (\text{ⓚ}, (\mathsf{j}, \mathsf{m})) \lhd [[(\bullet, (\mathsf{m}, \mathsf{m}))]^{n-1} \lhd (\circ, (\mathsf{m}, \mathsf{m}))]^{m+1}$,
- $r_m(\mathfrak{C}(0, \text{①}, 0)) = [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1} \lhd (\circ, (\mathsf{j}, \mathsf{j})) \lhd [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$.

For $\overline{\mathfrak{F}} = \langle \mathfrak{F}_1, \ldots, \mathfrak{F}_n \rangle \in \mathcal{B}^*$, $r_m(\overline{\mathfrak{F}}) = r_m(\mathfrak{F}_1) \lhd \cdots \lhd r_m(\mathfrak{F}_m)$.

Fact 5 *For every ψ, $\overline{\mathfrak{G}} \in \mathfrak{B}^*$ and m suitably (but polynomially) larger than $l(\varphi)$, ψ is satisfiable in $[\overline{\mathfrak{G}}]$ iff ψ is satisfiable in $r_m(\overline{\mathfrak{G}})$ under a valuation which is good for ψ and the type assignment of $r_m(\overline{\mathfrak{G}})$*

Proof of Theorem 1 (ii). Given a $\overline{\mathfrak{G}}$ determining a prime logic and φ, take $r_m(\overline{\mathfrak{G}})$ for m given by $l(\varphi)$. The size of the reduct will be polynomial in $l(\varphi)$. Then, generate non-deterministically a valuation in $r_m(\overline{\mathfrak{G}})$, check if it is φ-good and whether it satisfies (refutes) φ. By standard arguments, everything may be done in polynomial time. $\dashv$

For finitely axiomatizable logics more work is necessary. Let us just formulate the missing ingredient:

Fact 6 *If a finite axiomatization for L is given in its canonical form (i.e. as conjunction of $\alpha(\mathfrak{F}, \mathbf{t})$'s), then it may be checked in polynomial time whether it holds under all good valuations in a given frame with types.*

Proof of Theorem 1 (i). Given φ, generate a $\overline{\mathfrak{G}} \in \mathfrak{B}_0^*$, whose length, size of clusters and distance between reflexive points are suitably bounded by φ. Take its m-reduct, for m suitable both for φ and the axioms of L. Check whether the axioms hold under all good valuations, then generate non-deterministically a valuation for φ. $\dashv$

The results above shows that for modal logics of linear orderings, adding the past modality does not, in general, increase the computational complexity. In both cases, we have that the ones corresponding to reflexive or dense orders are finitely axiomatizable and finitely axiomatizable ones are always coNP-complete. And among non-finitely axiomatizable ones, in both cases we have logics of arbitrary complexity; the task of adapting for **K4.3** the counterexample given at the beginning is left as an exercise. An interesting open problem is whether one can obtain a general coNP- (PSPACE- ?) complexity result analogous to Theorem 1 for modal/tense logic of transitive frames of finite width.

References

[1] T. Litak, F. Wolter, *All finitely axiomatizable tense logics of linear time flows are coNP-complete*, submitted.

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[3] M. Zakharyaschev, F. Wolter and A. Chagrov, *Advanced modal logic*, [in:] **Handbook of Philosophical Logic, 2nd Edition**, vol. 3, Kluwer, 2001, pp. 83–266.