Glivenko Properties of Substructural Logics

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In this note, we discuss Glivenko properties of substructural logics. We introduce three types of Glivenko's theorems, and then show when each of these Glivenko's theorems holds between given two substructural logics. We will discuss here mainly extensions of $\mathbf{FL_e}$, i.e. intuitionistic linear logic without exponentials, though results hold essentially also for noncommutative case. The present topics will be comprehensively discussed in our forthcoming papers [2] and [3] with N. Galatos. For general information on substructural logics and corresponding varieties of residuated lattices, consult [4].

1 Glivenko properties

In 1929, V. Glivenko obtained the following well-known result, which means that classical logic can be *embedded* into intuitionistic logic by the double negation translation.

Proposition 1 *For any formula α, $\neg\neg\alpha$ is provable in intuitionistic logic* **INT** *iff α is provable in classical logic* **CL**.

After that, there had not been so much progresses in the study of Glivenko-type theorems, until R. Cignoli and A. Torrens showed the following in [1].

Proposition 2 *1. For any formula α, $\neg\neg\alpha$ is provable* **SBL** *iff α is provable in* **CL**,
2. For any formula α, $\neg\neg\alpha$ is provable in **BL** *iff α is provable in Łukasiewicz infinite valued logic* **Ł**.

Here **BL** is Hájek's basic logic, which is an extension of $\mathbf{FL_{ew}}$, and **SBL** is a logic obtained from **BL** by adding the axiom $(\alpha * (\alpha \to \neg\alpha)) \to \beta$. While $\alpha \to \alpha^2$ is provable in **INT**, it is not provable in **SBL**. On the other hand, though $(\alpha \to \beta) \vee (\beta \to \alpha)$ is provable in **SBL** it is not provable in **INT**. Therefore, these two logics are incomparable. So, it will be natural to ask the following questions:

- Is there any extension of $\mathbf{FL_e}$, which is weaker than both of **INT** and **SBL** and for which Glivenko's theorem holds relative to **CL**?
- If so, is there the least one among such a logic?

Inspired by the paper [1], we have developed a study of Glivenko properties of substructural logics in general setting. Let **L** and **K** be substructural logics over $\mathbf{FL_e}$. We say that *Glivenko property holds for* **L** *relative to* **K**, whenever for any formula α, $\neg\neg\alpha$ is provable in **L** iff α is provable in **K**. Our goal is to answer the following questions.

- When does Glivenko property holds for a given logic **L** relative to another logic **K**?
- What conditions should **K** satisfy, if Glivenko property holds relative to **K**?
- Is there the least logic among logics for which Glivenko property holds relative to **K**?

2 Algebraization and local deduction theorem

Let $\mathcal{CRL}$ be the variety of all commutative residuated lattices. For each substructural logic $\mathbf{L}$ (over $\mathbf{FL_e}$), let $V(\mathbf{L})$ be a subvariety of $\mathcal{CRL}$ determined by the set of equations $\{\varphi \wedge 1 \approx 1 | \varphi \in \mathbf{L}\}$. Conversely, for a given subvariety V of $\mathcal{CRL}$, let $\mathbf{L}(V)$ be a set of formulas $\{\varphi | \ V \models \varphi \wedge 1 \approx 1\}$. Then, $\mathbf{L}(V)$ is a substructural logic over $\mathbf{FL_e}$. Moreover, these two maps V and $\mathbf{L}$ are shown to be mutually inverse dual lattice isomorphisms between the lattice of all extensions of $\mathbf{FL_e}$ and the lattice of all subvarieties of $\mathcal{CRL}$.

For each substructural logic $\mathbf{L}$ over $\mathbf{FL_e}$, define the deducibility relation $\vdash_{\mathbf{L}}$ as follows. For any set of formulas Σ and a formula α, $\Sigma \vdash_{\mathbf{L}} \alpha$ iff the sequent $\Rightarrow \alpha$ is provable in the sequent system obtained from $\mathbf{FL_e}$ by adding sequents $\Rightarrow \varphi$ for $\varphi \in \mathbf{L} \cup \Sigma$. It is obvious that $\vdash_{\mathbf{L}}$ is a consequence relation. Then we have the following.

Theorem 3 *For each substructural logic $\mathbf{L}$, the deductive system determined by $\vdash_{\mathbf{L}}$ is algebraizable (in the sense of Blok and Pigozzi) with respect to a subvariety $V(\mathbf{L})$ of the variety $\mathcal{CRL}$.*

This means that there exists two mutually inverse translations between formulas and equations that interpret the syntactic relation $\vdash_{\mathbf{L}}$ by means of the semantic one $\models_{V(\mathbf{L})}$ and vice versa. By the above theorem with a theorem on filter generation in commutative residuated lattices, we can get the following *local deduction theorem.*

Theorem 4 *For each substructural logic $\mathbf{L}$ over $\mathbf{FL_e}$, the following holds for formulas ψ, φ and for any set of formulas Σ:*

$\Sigma, \psi \vdash_{\mathbf{L}} \phi$ *iff* $\Sigma \vdash_{\mathbf{L}} (\psi \wedge 1)^m \to \varphi$ *for some m.*

3 Glivenko equivalence

A logic $\mathbf{L}$ is *Glivenko equivalent* to a logic $\mathbf{K}$, iff for any formula α

$\vdash_{\mathbf{L}} \neg\alpha$ iff $\vdash_{\mathbf{K}} \neg\alpha$.

Using our algebraization theorem, we can show that $\mathbf{L}$ is Glivenko equivalent to $\mathbf{K}$ iff for any set of equations $E \cup \{s \approx t\}$

$E \models_{V(\mathbf{L})} \neg s \approx \neg t$ iff $E \models_{V(\mathbf{K})} \neg s \approx \neg t$.

Obviously, Glivenko equivalence is an equivalence relation on the class of all extensions of $\mathbf{FL_e}$. For a given logic $\mathbf{L}$, let $\mathcal{E}(\mathbf{L})$ be the Glivenko equivalence class to which $\mathbf{L}$ belongs. Then each $\mathcal{E}(\mathbf{L})$ is shown to be convex, i.e. if $\mathbf{L}_1 \subseteq \mathbf{K} \subseteq \mathbf{L}_2$ for $\mathbf{L}_1, \mathbf{L}_2 \in \mathcal{E}(\mathbf{L})$, then $\mathbf{K} \in \mathcal{E}(\mathbf{L})$.

Lemma 5 *Each Glivenko equivalence class $\mathcal{E}(\mathbf{L})$ contains the least logic $\mathbf{G}(\mathbf{L})$ and the greatest $\mathbf{M}(\mathbf{L})$.*

Note that $\mathbf{G}(\mathbf{L})$ depends on where we look for the least, e.g. among logics over $\mathbf{FL}$, or over $\mathbf{FL_e}$. For the simplicity's sake, we restrict our attention only to substructural logics over $\mathbf{FL_e}$. Then, $\mathbf{G}(\mathbf{L})$ and $\mathbf{M}(\mathbf{L})$ are given explicitly as follows. To show that they are actually the least and the greatest, respectively, we need the local deduction theorem.

- $\mathbf{G}(\mathbf{L})$ is obtained from $\mathbf{FL_e}$ by adding $\{\neg\neg\phi \mid \phi \in \mathbf{L}\}$ as axioms,
- $\mathbf{M}(\mathbf{L})$ is obtained from $\mathbf{L}$ by adding $\{\phi \mid \neg\neg\phi^n \in \mathbf{L}$ for each $n \geq 1\}$.

4 Involutiveness and Glivenko properties

Now, we introduce three types of involutiveness.

1. A logic $\mathbf{L}$ is *Glivenko involutive*, if $\vdash_{\mathbf{L}} \neg\neg\alpha$ implies $\vdash_{\mathbf{L}} \alpha$ for every α,
2. $\mathbf{L}$ is *weakly involutive*, if $\neg\neg\alpha \vdash_{\mathbf{L}} \alpha$ for every α,
3. $\mathbf{L}$ is *involutive*, if $\vdash_{\mathbf{L}} \neg\neg\alpha \to \alpha$ for every α.

Clearly, involutiveness implies weak involutiveness, which in turn implies Glivenko involutiveness. While both involutiveness and weak involutiveness are preserved under extensions, Glivenko involutiveness is not preserved always.

Lemma 6 *Suppose that* $\mathbf{K}$ *belongs to* $\mathcal{E}(\mathbf{L})$ *for a logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$. *If* $\mathbf{K}$ *is Glivenko involutive then it is equal to* $\mathbf{M(L)}$. *Moreover, the converse holds whenever* $\mathbf{L}$ *is a logic over* $\mathbf{FL_{ew}}$.

Next, we introduce three types of *Glivenko properties.*

1. *Glivenko property* holds for $\mathbf{L}$ relative to $\mathbf{K}$, when $\vdash_{\mathbf{L}} \neg\neg\alpha$ iff $\vdash_{\mathbf{K}} \alpha$ for every α,
2. *deductive Glivenko property* holds for $\mathbf{L}$ relative to $\mathbf{K}$, when $\Sigma \vdash_{\mathbf{L}} \neg\neg\alpha$ iff $\Sigma \vdash_{\mathbf{K}} \alpha$ for every set of formulas $\Sigma \cup \{\alpha\}$,
3. *equational Glivenko property* holds for $\mathbf{L}$ relative to $\mathbf{K}$, when $s, t \models_{V(\mathbf{L})} \neg\neg s \approx \neg\neg t$ iff $\models_{V(\mathbf{K})} s \approx t$ for all terms s, t.

Theorem 7 *The following statements are equivalent:*

1. *Glivenko property holds for* $\mathbf{L}$ *relative to* $\mathbf{K}$*,*
2. $\mathbf{L}$ *and* $\mathbf{K}$ *are Glivenko equivalent, and moreover* $\mathbf{K}$ *is Glivenko involutive.*

The similar statement holds also between deductive (equational) Glivenko property and weak involutiveness (involutiveness, respectively). The next theorem tells us how to get $\mathbf{G(K)}$ for a given Glivenko involutive $\mathbf{K}$.

Theorem 8 *Suppose that* $\mathbf{K}$ *is a Glivenko involutive extension of* $\mathbf{FL_e}$*, which is axiomatized by a set of formulas* Σ*. Then, a set of axioms of* $\mathbf{G(K)}$ *is given (relative to* $\mathbf{FL_e}$*) by* $\{\neg\neg\alpha \mid \alpha \in \Sigma\} \cup \{\neg\neg(\neg\neg\beta \to \beta), \neg(\gamma \cdot \delta) \to \neg(\neg\neg\gamma \cdot \neg\neg\delta)\}$*. Thus, if* $\mathbf{K}$ *is finitely axiomatizable then so is* $\mathbf{G(K)}$.

For example, $\mathbf{G(CL)}$ is axiomatized over $\mathbf{FL_e}$ by axioms $\neg\neg(\alpha \cdot \beta \to \alpha), \neg\neg(\alpha \to \alpha^2), \neg\neg(\neg\neg\beta \to \beta)$ and $\neg(\gamma \cdot \delta) \to \neg(\neg\neg\gamma \cdot \neg\neg\delta)$. By a slight modification, similar arguments as above work well also for $\mathbf{FL}$. For instance, let $\mathbf{L}^*$ be the least extension of $\mathbf{FL}$ which is Glivenko equivalent to classical logic. Then, we can show that none of structural rules holds in $\mathbf{L}^*$. In other words, classical logic can be embedded even into a quite weak logic.

References

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