Halldén Completeness of Substructural Logics

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In this paper, we discuss necessary and sufficient conditions for a substructural logic to be Halldén complete. We show that to substructural logics over $\mathbf{FL_{ew}}$ we can extend most of results on Halldén-completeness of intermediate logics. On the other hand, the lack of weakening will cause some difficulties in extending them to logics over $\mathbf{FL_e}$. We will give a partial result on a modified Halldén-completeness for logics over $\mathbf{FL_e}$.

1 Halldén completeness of $\mathbf{FL_{ew}}$

We say that a logic $\mathcal{L}$ is *Halldén complete* if and only if for every formulas ϕ and ψ which have no variables in common, $\phi \vee \psi \in \mathcal{L}$ implies that $\phi \in \mathcal{L}$ or $\psi \in \mathcal{L}$. The following results are well-known.

PROPOSITION 1 *(see e.g. [1,Theorem 15.22]) For every intermediate logic $\mathcal{L}$ the following are equivalent:*

(i) *$\mathcal{L}$ is Halldén complete,*

(ii) *for any logics $\mathcal{L}_1, \mathcal{L}_2 \supseteq \mathcal{L}$,*

$$if\ \mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2\ then\ \mathcal{L}_1 \subseteq \mathcal{L}_2\ or\ \mathcal{L}_2 \subseteq \mathcal{L}_1$$

PROPOSITION 2 *([2]) For every intermediate logic $\mathcal{L}$ the following are equivalent:*

(i) *$\mathcal{L}$ is Halldén complete,*

(iii) *$\mathcal{L} = L(\mathbf{A})$ for some well-connected Heyting algebra $\mathbf{A}$, i.e. for all $x, y \in \mathbf{A}$,*

$$if\ x \vee y = 1\ then\ x = 1\ or\ y = 1,$$

(iv) *$\mathcal{L} = L(\mathbf{A})$ for some subdirectly irreducible Heyting algebra $\mathbf{A}$.*

Here, $L(\mathbf{A})$ denotes the set of all formulas which are valid in a Heyting algebra $\mathbf{A}$.

One can show the following theorem in the same way as above propositions.

THEOREM 3 *For every logic $\mathcal{L}$ over $\mathbf{FL_{ew}}$ the following are equivalent:*

(i) *$\mathcal{L}$ is Halldén complete,*

(ii) *for any logics $\mathcal{L}_1, \mathcal{L}_2 \supseteq \mathcal{L}$,*

$$if\ \mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2\ then\ \mathcal{L}_1 \subseteq \mathcal{L}_2\ or\ \mathcal{L}_2 \subseteq \mathcal{L}_1$$

(iii) *$\mathcal{L} = L(\mathbf{A})$ for some well-connected commutative integral residuated lattice (CIRL) $\mathbf{A}$.*

It will be interesting to see whether
(iv) $\mathcal{L} = L(\mathbf{A})$ for some subdirectly irreducible CIRL $\mathbf{A}$
is equivalent to Halldén completeness or not.

2 Halldén completeness of $\mathbf{FL_e}$

Theorem 3 doesn't hold always, if we replace $\mathbf{FL_{ew}}$ by $\mathbf{FL_e}$, and CIRLs by commutative residuated lattices (CRLs). In other words, we need to modify definitions of Halldén completeness and well-connectedness so as to make Theorem 3 true.

Let $\mathbf{A}$ be an CRL and $\mathcal{F}$ a subset of A. Then $\mathcal{F}$ is called an *filter* of A iff

1. if $1 \leq x$ then $x \in \mathcal{F}$,
2. if $x, x \to y \in \mathcal{F}$ then $y \in \mathcal{F}$,
3. if $x, y \in \mathcal{F}$ then $x \wedge y \in \mathcal{F}$.

LEMMA 4 *Let $\mathcal{G}$ be a proper filter of CRL $\mathbf{A}$ and $a \notin \mathcal{G}$. Then there exists a filter $\mathcal{F}_a$ which is maximal in the set*

$$\Sigma = \{\mathcal{F} : filter | \mathcal{G} \subseteq \mathcal{F}, a \notin \mathcal{F}\}.$$

Moreover, $\mathcal{F}_a$ satisfies the following condition:

$$if\ (x \wedge 1) \vee (y \wedge 1) \in \mathcal{F}_a\ then\ x \in \mathcal{F}_a\ or\ y \in \mathcal{F}_a.$$

(proof) By Zorn's lemma, Σ has a maximal element. So let $\mathcal{F}_a$ be a maximal element of Σ. We will show that $\mathcal{F}_a$ satisfies the above condition.

Assume $x \notin \mathcal{F}_a$ and $y \notin \mathcal{F}_a$. Define $\mathcal{H}_x$ as follows.

$$\mathcal{H}_x = \{z \in A | (x \wedge 1)^k \cdot (u \wedge 1) \leq z,\ \exists k \in N, \exists u \in \mathcal{F}_a\}$$

Then $\mathcal{H}_x$ is the filter generated by $\mathcal{F}_a \cup \{x\}$. Since $\mathcal{F}_a$ is maximal in Σ and $x \notin \mathcal{F}_a$, $a \in \mathcal{H}_x$. So there exists some $l \in N$ and $u \in \mathcal{F}_a$ such that

$$(x \wedge 1)^l \cdot (u \wedge 1) \leq a.$$

Similarly there exists some $m \in N$ and $v \in F_a$ such that

$$(y \wedge 1)^m \cdot (v \wedge 1) \leq a.$$

Let $t = l + m - 1$. Then, by the distributivity of $\cdot$ with $\vee$

$$((x \wedge 1) \vee (y \wedge 1))^t \cdot (u \wedge 1) \cdot (v \wedge 1)$$

$$= \bigvee_{i=0}^{t} (x \wedge 1)^i \cdot (y \wedge 1)^{t-i} \cdot (u \wedge 1) \cdot (v \wedge 1).$$

Since $i \geq l$ or $t - i \geq m$, either of the following holds:

$$\begin{array}{lll} (1) & & (x \wedge 1)^i \cdot (y \wedge 1)^{t-i} \cdot (u \wedge 1) \cdot (v \wedge 1) \\ & \leq & (x \wedge 1)^l \cdot (u \wedge 1) \\ & \leq & a \end{array}$$

$$\begin{array}{lll} (2) & & (x \wedge 1)^i \cdot (y \wedge 1)^{t-i} \cdot (u \wedge 1) \cdot (v \wedge 1) \\ & \leq & (y \wedge 1)^m \cdot (v \wedge 1) \\ & \leq & a. \end{array}$$

So if $(x \wedge 1) \vee (y \wedge 1) \in \mathcal{F}_a$ then $a \in \mathcal{F}_a$. But this is a contradiction. Hence, $(x \wedge 1) \vee (y \wedge 1) \notin \mathcal{F}_a$. □

Note that the above condition is equal to the following condition:

$$if\ (x \wedge 1) \vee (y \wedge 1) \in \mathcal{F}_a\ then\ x \wedge 1 \in \mathcal{F}_a\ or\ y \wedge 1 \in \mathcal{F}_a.$$

Therefore, when **A** is a commutative integral residuated lattice, i.e., 1 is the greatest element of **A**, the above condition is equal to the condition which says that the filter $\mathcal{F}_a$ is prime, i.e., $\mathcal{F}_a$ satisfies the condition

$$if\ x \vee y \in \mathcal{F}_a\ then\ x \in \mathcal{F}_a\ or\ y \in \mathcal{F}_a.$$

As the above lemma shows, it seems to be necessary to modify the notion of Halldén completeness and well-connectedness. The following conditions **(i)** and **(*)** seem to be strictly weaker than Halldén completeness and well-connectedness, respectively.

THEOREM 5 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. Then the following are equivalent:*

(i) *for every formulas ϕ and ψ which have no variables in common*

$$if\ (\phi \wedge 1) \vee (\psi \wedge 1) \in \mathcal{L}\ then\ \phi \in \mathcal{L}\ or\ \psi \in \mathcal{L},$$

(ii) *for any logics $\mathcal{L}_1, \mathcal{L}_2 \supseteq \mathcal{L}$,*

$$if\ \mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2\ then\ \mathcal{L}_1 \subseteq \mathcal{L}_2\ or\ \mathcal{L}_2 \subseteq \mathcal{L}_1$$

(iii) $\mathcal{L} = L(\mathbf{A})$ *for some CRL* **A** *satisfying the following.*

(*) *for any $x, y \in A^- = \{a \in A | a \leq 1\}$,*

$$if\ x \vee y = 1\ then\ x = 1\ or\ y = 1.$$

References

[1] A.Chagrov and M.Zakharyaschev, Modal Logic, Clarendon Press, Oxford, 1997, pp.482.

[2] A.Wroński, Remarks on Hallden-completeness of modal and intermediate logics, Bulletin of the Section of Logic 5, No.4(1976), pp.126-129.