Products of 'transitive' modal logics with constant and expanding domains

Extended abstract

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1 Introduction

Products of modal logics—or, more generally, multi-modal languages interpreted in various product-like structures—are very natural and clear formalisms arising in both pure logic and numerous applications. Introduced in the 1970s, products of modal logics have been intensively studied over the last decade; for a comprehensive exposition and references see [1]. The landscape of the obtained results that are relevant to the decision problem for these logics can be briefly outlined as follows. (1) The product of finitely many logics, whose Kripke frames are definable by recursive sets of first-order sentences, is recursively enumerable. (2) Products of two standard logics, where at least one component logic is determined by a class of frames of finite bounded depth (like **S5** or **K**), are usually decidable. (3) Products of two 'linear transitive' logics are undecidable whenever the depth of frames for both component logics cannot be bounded by any fixed $n < \omega$; examples are products of **K4.3**, **S4.3**, **GL.3** or $\mathsf{Log}\,(\omega, <)$. (4) Products of more than two modal logics are usually undecidable. In fact, no logic between $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ is decidable.

Thus, the main gap in our knowledge about the decision problem for product logics is the computational behaviour of products of two 'transitive' logics whose 'depth' is not bounded by any fixed $n < \omega$ and at least one component logic has *branching* frames. Many natural and useful logics, such as $\mathbf{S4} \times \mathbf{S4}$, $\mathbf{S4.3} \times \mathbf{S4}$, and $\mathbf{GL} \times \mathbf{GL}$ belong to this group.

2 Products (with constant domains) and commutators

Given unimodal Kripke frames $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$, their *product* is defined to be the bimodal frame $\mathfrak{F}_1 \times \mathfrak{F}_2 = (W_1 \times W_2, R_h, R_v)$, where $W_1 \times W_2$ is the Cartesian product of W_1 and W_2 and, for all $u, u' \in W_1$, $v, v' \in W_2$,

$$(u,v) R_h (u',v') \quad \text{iff} \quad u R_1 u' \text{ and } v = v',$$
$$(u,v) R_v (u',v') \quad \text{iff} \quad v R_2 v' \text{ and } u = u'.$$

Let L_1 be a normal (uni)modal logic in the language with the box $\boxminus$ and the diamond $\diamondminus$. Let L_2 be a normal (uni)modal logic in the language with the box $\Box$ and the diamond $\Diamond$. Assume also that both L_1 and L_2 are Kripke complete. Then the *product* of the logics L_1 and L_2 is the (Kripke complete) bimodal logic $L_1 \times L_2$ in the language having boxes $\boxminus$, $\boxbar$ and diamonds $\diamondminus$, $\diamondbar$ which is characterised by the class of product frames $\mathfrak{F}_1 \times \mathfrak{F}_2$, where $\mathfrak{F}_i$ is a frame for L_i, $i = 1, 2$. (Here we assume that $\boxminus$ and $\diamondminus$ are interpreted by R_h, while $\boxbar$ and $\diamondbar$ are interpreted by R_v.)

Given Kripke complete unimodal logics L_1 and L_2, their *commutator* $[L_1, L_2]$ is the smallest normal modal logic in the language $\mathcal{ML}_2$ which contains L_1, L_2 and the axioms

$$\diamondbar\diamondminus p \to \diamondminus\diamondbar p, \qquad \diamondminus\diamondbar p \to \diamondbar\diamondminus p, \qquad \diamondminus\boxbar p \to \boxbar\diamondminus p.$$

Clearly, we always have $[L_1, L_2] \subseteq L_1 \times L_2$. For certain pairs of logics, their commutators and products actually coincide (e.g., $[\mathbf{K4}, \mathbf{K4}] = \mathbf{K4} \times \mathbf{K4}$ and $[\mathbf{S4}, \mathbf{S4}] = \mathbf{S4} \times \mathbf{S4}$), but in general this is not the case.

Although product logics $L_1 \times L_2$ are Kripke complete by definition, there can be (and, in general, there are) other, *non-product*, frames for $L_1 \times L_2$. This gives rise to two different types of the finite model property. As usual, a bimodal logic L (in particular, a product logic $L_1 \times L_2$) is said to have the (*abstract*) *finite model property* (*fmp*, for short) if, for every formula $\varphi \notin L$, there is a finite frame $\mathfrak{F}$ for L such that $\mathfrak{F} \not\models \varphi$. And we say that $L_1 \times L_2$ has the *product finite model property* (*product fmp*, for short) if, for every formula $\varphi \notin L_1 \times L_2$, there is a finite *product* frame $\mathfrak{F}$ for $L_1 \times L_2$ such that $\mathfrak{F} \not\models \varphi$. Clearly, the product fmp implies the fmp. Examples of product logics having the product fmp (and so the fmp) are $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{S5}$, and $\mathbf{S5} \times \mathbf{S5}$. On the other hand, there are product logics, such as $\mathbf{K4} \times \mathbf{S5}$ and $\mathbf{S4} \times \mathbf{K}$, that do enjoy the (abstract) fmp, but lack the product fmp. In general, it is well known that many product logics with at least one 'transitive' (but not 'symmetric') component do not have the product fmp.

In this paper we introduce a novel technique for dealing with products of logics with transitive branching frames. Our results say that all products—and quite often even the commutators—of two Kripke complete modal logics with transitive frames of arbitrary finite or infinite depth are undecidable, in many cases these products are not axiomatisable and do not enjoy the (abstract) finite model property, and sometimes they are even Π^1_1-hard. Precise formulations are as follows.

Denote by $\mathsf{Log}\,(\mathcal{C})$ the normal modal logic of a class $\mathcal{C}$ of frames. If $\mathcal{C}$ consists of a single frame $\mathfrak{F}$ then we write $\mathsf{Log}\,\mathfrak{F}$ instead of $\mathsf{Log}\,(\{\mathfrak{F}\})$. Recall that a logic L is *Kripke complete* if $L = \mathsf{Log}\,(\mathcal{C})$ for some class $\mathcal{C}$ of frames.

Theorem 1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of transitive frames both containing frames of arbitrarily large finite or infinite depth. Then $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)$ is undecidable.*

More generally, if L is any Kripke complete bimodal logic containing $[\mathbf{K4}, \mathbf{K4}]$ and having frames of arbitrarily large finite or infinite depth, then L is undecidable.

Theorem 2. *Let L_1 and L_2 be any logics from the list*

$$\mathbf{K4}, \mathbf{K4.1}, \mathbf{K4.2}, \mathbf{K4.3}, \mathbf{S4}, \mathbf{S4.1}, \mathbf{S4.2}, \mathbf{S4.3}, \mathbf{GL}, \mathbf{GL.3}, \mathbf{Grz}, \mathbf{Grz.3}, \mathsf{Log}\,(\omega, <), \mathsf{Log}\,(\omega, \leq).$$

Then both $[L_1, L_2]$ and $L_1 \times L_2$ are undecidable and lack the (abstract) fmp.

In some cases, we can even say a bit more.

Theorem 3. *Let L_1 be like in Theorem 2 and $L_2 \in \{\mathsf{Log}\,(\omega, <), \mathsf{Log}\,(\omega, \leq), \mathbf{GL.3}, \mathbf{Grz.3}\}$. Then any Kripke complete bimodal logic L in the interval*

$$[L_1, L_2] \;\subseteq\; L \;\subseteq\; L_1 \times L_2$$

is Π^1_1-hard. In fact, the product logics $L_1 \times L_2$ are Π^1_1-complete.

We also obtain the following interesting corollary. As the commutator of two recursively axiomatisable logics is recursively axiomatisable by definition, Theorem 3 yields a number of *Kripke incomplete* commutators of Kripke complete and finitely axiomatisable logics:

Corollary 3.1. *Let L_1 and L_2 be like in Theorem 3. Then the commutator $[L_1, L_2]$ is Kripke incomplete.*

3 Products with expanding domains

As the above results show, we have to pay for the strong interaction between the modal operators of the component logics of a product. Now the general research problem we are facing can be formulated as follows: is it possible to reduce the computational complexity of product logics by relaxing the interaction between their components and yet keeping some of the useful and attractive features of the product construction?

One approach to this problem is motivated by structures often used in such areas as temporal and modal first-order logics, temporal data or knowledge bases or logical modelling of dynamic systems. What we mean is models/structures with *expanding domains*: if at a certain time point (or in a world) w we have a 'population' Δ_w of elements (objects), then at every later point (in every accessible world) u the population Δ_u cannot be smaller but can grow—i.e., $\Delta_w \subseteq \Delta_u$.

Here we show that there are natural and useful products with expanding domains which are indeed simpler than their constant domain counterparts. The decidability proof is heavily based on Kruskal's tree theorem and does not establish any elementary upper bound for the time complexity of the decision algorithm. We show that indeed no such upper bound exists by proving that there is no primitive recursive decision algorithm for such logics.

Let us first introduce the intended 'expanding domain semantics' for the bimodal language with boxes $\boxminus$, $\boxbar$ and diamonds $\diamondminus$, $\diamondbar$. Suppose we have a ('horizontal') frame (W, R) and, for each $x \in W$, a ('vertical') frame (W_x, R_x) such that whenever xRx' then (W_x, R_x) is a subframe of $(W_{x'}, R_{x'})$ (that is, $W_x \subseteq W_{x'}$ and for all $u, v \in W_x$, we have uR_xv iff $uR_{x'}v$). Now let

$$U = \{(x, y) \mid x \in W,\ y \in W_x\}$$

and define the binary relations R_h and R_v on U by taking

$$\begin{array}{lll} (x,y)R_h(x',y') & \text{iff} & y = y' \text{ and } xRx' \\ (x,y)R_v(x',y') & \text{iff} & x = x' \text{ and } yR_xy'. \end{array}$$

Then the bimodal frame (U, R_h, R_v) is called an *expanding domain frame* (or simply an *e-frame*) obtained from (W, R) and (W_x, R_x) $(x \in W)$.

The *expanding domain product* of Kripke complete unimodal logics L_1 and L_2 is the normal bimodal logic $(L_1 \times L_2)^{\mathsf{e}}$ which is characterised by the class of e-frames (U, R_h, R_v) obtained from some horizontal component (W, R) that is a frame for L_1 and vertical components (W_x, R_x) $(x \in W)$ that are all frames for L_2. A logic $(L_1 \times L_2)^{\mathsf{e}}$ is said to have the *e-product fmp* if, for every formula $\varphi \notin (L_1 \times L_2)^{\mathsf{e}}$, there is a finite e-frame for $(L_1 \times L_2)^{\mathsf{e}}$ that refutes φ.

The main result of this section is the following:

Theorem 4. *Let $L_1, L_2 \in \{\mathbf{GL}, \mathbf{GL.3}\}$. Then the logic $(L_1 \times L_2)^{\mathsf{e}}$ has the e-product fmp and is decidable. However, $(L_1 \times L_2)^{\mathsf{e}}$ is not decidable in time bounded by a primitive recursive function.*

Proofs of all theorems above can be found in [3] and [2].

References

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[2] D. Gabelaia, A. Kurucz, F. Wolter, and M. Zakharyaschev. Non-primitive recursive decidability of products of modal logics with expanding domains. Manuscript, 2004. Available at `http://dcs.kcl.ac.uk/staff/kuag/publi/expand.ps`.

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