Amalgamation property of Commutative Residuated Lattices

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1 Introduction

The aim of this paper is to show the variety of all commutative residuated lattices ($\mathcal{CRL}$) has the amalgamation property (AP). Moreover, we show that if $\mathbf{L}$ is a logic which is an extension of $\mathbf{FL}_e$ with the Craig's interpolation property (CIP) and $\mathcal{K}$ be the variety corresponding to $\mathbf{L}$, then $\mathcal{K}$ has the AP.

2 Amalgamation property

First, we define the amalgamation property (AP).

Definition 1 (Amalgamation property). A class of algebra $\mathcal{K}$ has the *amalgamation property* iff if $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, $f_1 : \mathbf{A} \to \mathbf{B}_1, f_2 : \mathbf{A} \to \mathbf{B}_2$ are embeddings then there exist an algebra $\mathbf{C} \in \mathcal{K}$ and embeddings $g_1 : \mathbf{B}_1 \to \mathbf{C}, g_2 : \mathbf{B}_2 \to \mathbf{C}$ such that $g_1 f_1 = g_2 f_2$.

The AP of algebraic structures related to some fragments of substructural logics were studied by many researchers. For instance, K. Iseki proved the AP for the class of all BCK-algebras [3]. A. Wroński proved that an algebraic version of interpolation theorem holds for the class of all BCK-algebras and also proved that the strong amalgamation property for the class of all BCK-algebras [12]. The concept of the connection between the AP and Craig's interpolation property was first studied by B. Jónsson and A. Daigneault independently. This connection was further studied by D. Pigozzi [9] and many other researchers. It is shown by L. Maksimova that a normal modal logic with a single unary modality has the Craig's interpolation property iff the corresponding class of algebras has the super-amalgamation property [7], and also that intuitionictic logic has the CIP iff the variety of Heyting algebras has the super-amalgamation property [6].

Next, we define an algebraic version of the interpolation property which is called *equational interpolation property.*

Definition 2 (Equational interpolation property). A variety $\mathcal{V}$ has the equational interpolation property (EIP) iff for all finite sets $\Sigma, \Gamma \cup \{\delta\}$ of identities in the language of $\mathcal{V}$ the following holds: if $\mathcal{V}$ satisfies the quasi-identity $\bigwedge(\Sigma \cup \Gamma) \Rightarrow \delta$ and the set of terms over $V(\Sigma) \cap V(\Gamma \cup \{\delta\})$ is non-empty, then there exists a finite set Δ of identities over $V(\Sigma) \cap V(\Gamma \cup \{\delta\})$ such that: (1) $\mathcal{V} \models \bigwedge \Sigma \Rightarrow \bigwedge \Delta$, (2) $\mathcal{V} \models \bigwedge(\Delta \cup \Gamma) \Rightarrow \delta$.

The EIP was introduced by A. Wroński and the similar concepts were investigated by B. Jónsson, D. Pigozzi and P. D. Bacsich. The following theorem is crucial to prove the AP in our argument.

Theorem 1 (Wroński). *A variety $\mathcal{V}$ has the EIP iff $\mathcal{V}$ has both congruence extension property* (CEP) *and amalgamation property* (AP)*.*

3 Craig's interpolation property

A logic $\mathbf{L}$ has the Craig's interpolation property (CIP) if the following statement holds for $\mathbf{L}$: If $A \to B$ is provable in $\mathbf{L}$ then there exists of a formula C such that both $A \to C$ and $C \to B$ are provable, and every propositional variables in C appears both A and B.

There are many way to prove the CIP. Original proof of Craig is obtained by using a semantical method. In 1960's, S. Maehara succeeded to show the CIP for classical logic follows from the cut elimination theorem [5]. This technique is called now Maehara's method.

Now, we can show that the CIP holds for $\mathbf{FL}_e^-$ ($\mathbf{FL}_e$ with only constant 1) by using Mehara's methods. Note that algebras for $\mathbf{FL}_e^-$ are precisely commutative residuated lattices, we will give a precise definition of commutative residuaed lattices in section 4.

Theorem 2. *Craig's interpolation theorem holds for* $\mathbf{FL}_e^-$.

4 Amalgamation property of $\mathcal{CRL}$

In [4], T. Kowalski showed that the AP for the variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_{ew}$-algebras, The result is obtained by showing that

(1) the logical system $\mathbf{FL}_{ew}$ has the CIP,
(2) showing that the variety of $\mathcal{FL}_{ew}$ has the EIP.

We will show that his proof of the AP also works well the variety of $\mathcal{CRL}$.

Here, we give a precise difinition of commutative residuated lattices. An algebraic structure $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \rightarrow \rangle$ is called a commuative residuated latticeif it satisfies the following conditions: (1) $\langle A, \wedge, \vee \rangle$ is a lattice, (2) $\langle A, \cdot, 1 \rangle$ is a commutative monoid, (3) $\rightarrow$ is binary operations called residuation which satisfy that $a \cdot b \leq c \Leftrightarrow a \leq b \rightarrow c$ holds for all a, b, c in A.

It is easy to see that for any identity $s = t$ can be translated $\tau \wedge 1 = 1$ for some term τ, and also a finite set of identities into a single identity. In the following we use an inequality $\tau \geq 1$ instead of an identity $\tau \wedge 1 = 1$ for simplicity.

Lemma 3. *The variety $\mathcal{CRL}$ of all commutative residuated lattices satisfies the quasi-identity $\tau \geq 1 \& \sigma \geq 1 \Rightarrow \delta \geq 1$ iff there exist natural numbers n and m with $n + m \geq 1$ such that $\mathcal{CRL}$ satisfies the identity $(\tau \wedge 1)^n (\sigma \wedge 1)^m \rightarrow \delta \geq 1$.*

Theorem 4. *The variety $\mathcal{CRL}$ of all commutative residuated lattices has the EIP. Therefore, $\mathcal{CRL}$ has the AP.*

Proof. Suppose $\mathcal{CRL}$ satisfies the quasi-identity $\bigwedge(\Sigma \cup \Gamma) \Rightarrow \epsilon$. Since we have a constant 1, the set of terms over $V(\Sigma) \cap V(\Gamma \cup \{\epsilon\})$ is non-empty. It is necessary to show that there exists a finite set of identities Δ which satisfies (1) and (2) of the definition of the EIP. We write τ_Σ, τ_Γ for the set of identities Σ and Γ. In particular, we assume that ϵ is of the form $\sigma \wedge 1 = 1$. Then, $\mathcal{CRL}$ satisfies the quasi-identity $(\tau_\Sigma \wedge 1 = 1) \& (\tau_\Gamma \wedge 1 = 1) \Rightarrow (\sigma \wedge 1 = 1)$. By Lemma 3, we have $\mathcal{CRL} \models (\tau_\Sigma \wedge 1)^n (\tau_\Gamma \wedge 1)^m \rightarrow \sigma \geq 1$ for some $n + m \geq 1$. It is also equivalent that $\mathcal{CRL} \models (\tau_\Sigma \wedge 1)^n \rightarrow [(\tau_\Gamma \wedge 1)^m \rightarrow \sigma] \geq 1$.

Since the logic $\mathbf{FL}_e^-$ has the CIP. And algebras for $\mathbf{FL}_e^-$ are exactly equal to commutative residuated lattices, there exists a term δ over $V((\tau_\Sigma \wedge 1)^n) \cap V((\tau_\Gamma \wedge 1)^m \rightarrow \sigma)$ such that (i) $\mathcal{CRL} \models (\tau_\Sigma \wedge 1)^n \rightarrow \delta \geq 1$, (ii) $\mathcal{CRL} \models \delta \rightarrow ((\tau_\Gamma \wedge 1)^m \rightarrow \sigma) \geq 1$. By Lemma 3, in the case $m = 0$, (i) is equivalent to $\mathcal{CRL} \models \tau_\Sigma \geq 1 \Rightarrow \delta \geq 1$. This is also equivalent to $\mathcal{CRL} \models \bigwedge \Sigma \Rightarrow \delta \geq 1$. This proves (1) of the definition of the EIP with Δ being $\{\delta \wedge 1 = 1\}$. Next, we will show (2) of the EIP. Let $\mathbf{A}$ be any algebra from $\mathcal{CRL}$ and $\vec{a}$ a vector from $\mathbf{A}$ such that $\mathbf{A} \models \bigwedge(\Delta \cup \Gamma)[\vec{a}]$. This means that $(\delta)[\vec{a}], (\tau_\Gamma)[\vec{a}] \geq 1$ by considering the representation for Δ and Γ. By (ii), we have $(\delta \wedge 1) \rightarrow ((\tau_\Gamma \wedge 1)^m \rightarrow \sigma) \geq 1$. Thus, $1 \rightarrow (1 \rightarrow \sigma) \geq 1$ and so $\sigma \wedge 1 = 1$. Hence we have $\mathbf{A} \models (\sigma \wedge 1)[\vec{a}]$. This shows that $\mathbf{A} \models \bigwedge(\Delta \cup \Gamma) \Rightarrow \sigma \geq 1$. This completes the proof.

Moreover, by considering filters of residuated lattices, we can show that the following result.

Theorem 5. *Let $\mathbf{L}$ be a logic which is an extension of $\mathbf{FL}_e$ with the CIP and $\mathcal{K}$ be the variety correspoding to $\mathbf{L}$. Then $\mathcal{K}$ has the EIP. Therefore, $\mathcal{K}$ has the AP.*

By using the fact that Craig's interpolation theorem holds for substructural logics in [8], we can show the following important classes of commutative residuated lattices have the EIP, and hence these have the AP.

(1) The variety $\mathcal{FL}_e$ of all $\mathbf{FL}_e$-algebras,
(2) The variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_{ew}$-algebras,
(3) The variety $\mathcal{FL}_{ec}$ of all $\mathbf{FL}_{ec}$-algebras,
(4) The variety $\mathcal{CFL}_e$ of all $\mathbf{CFL}_e$-algebras,
(5) The variety $\mathcal{CFL}_{ew}$ of all $\mathbf{CFL}_{ew}$-algebras,
(6) The variety $\mathcal{CFL}_{ec}$ of all $\mathbf{CFL}_{ec}$-algebras,
(7) The variety $\mathcal{CRL}^2$ of all increasing-idempotent commutative residuated lattices,
(8) The variety $\mathcal{CCRL}$ of all classical commutative residuated lattices,
(9) The variety $\mathcal{CIRL}$ of all commutative integral residuated lattices,
(10) The variety $\mathcal{CCIRL}$ of all classical commutative integral residuated lattices,
(11) The variety $\mathcal{CCRL}^2$ of all increasing-idempotent classical commutative residuated lattices.

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