The Banach-Steinhaus theorem for $(\mathcal{LF})$-spaces in constructive analysis

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1 Introduction

The space $\mathcal{D}(\mathbb{R})$ of test functions (infinitely differentiable functions with compact support) is an important example of a non-metrizable $(\mathcal{LF})$-space, and the domain of distributions (generalized functions). E. Bishop suggested in [1, Appendix A] and [2, Chapter 7, Notes] that the completeness of $\mathcal{D}(\mathbb{R})$ and the weak completeness of its dual space would not hold in Bishop's constructive mathematics. This matter had not be solved since Bishop referred it, and we first obtained the following consequence in [8, Theorem 4]: the completeness of $\mathcal{D}(\mathbb{R})$ is equivalent to a principle BD-$\mathbb{N}$, which can be proved in classical mathematics, intuitionistic mathematics of L. E. J. Brouwer and constructive recursive mathematics of A. A. Markov's school but cannot be in Bishop's framework (see [4] and [6] for more details). Therefore, in Bishop's framework, the completeness of a $\mathcal{D}(\mathbb{R})$ cannot be proved, and neither can that of a $(\mathcal{LF})$-space. On the other hand, we can prove the following version of the Banach-Steinhaus theorem for $\mathcal{D}(\mathbb{R})$ in Bishop's constructive mathematics, and therefore can in the others, since theorems in Bishop's framework belong to the others (see [3, Chapter 1]):

> [11, Theorem 4.10] *Let $\{u_k\}$ be a sequence of sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$ such that $\langle u, \phi \rangle := \lim_k \langle u_k, \phi \rangle$ exists for all ϕ in $\mathcal{D}(\mathbb{R})$. Then u is a sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$.*

This theorem implies the weak completeness of the dual space $\mathcal{D}^*(\mathbb{R})$ which consists of sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$. Thus the completeness of $\mathcal{D}(\mathbb{R})$ is not necessary for proving the weak completeness of $\mathcal{D}^*(\mathbb{R})$, although many classical proofs require the completeness for proving the weak completeness. In this paper, we discuss to generalize the theorem above to $(\mathcal{LF})$-spaces within Bishop's constructive mathematics. Although it is formulated in the setting of informal Bishop-style constructive mathematics, the proofs could easily be formalised in a system based on intuitionistic finite-type arithmetics $\mathbf{HA}^\omega$ (see [9, Chapter 9] for more details).

2 $(\mathcal{F})$-spaces

Let X be a vector space over $\mathbb{R}$. A mapping $p : X \to \mathbb{R}^{0+}$ is said to be a *seminorm* on X if it satisfies that for $x, y \in X$ and $\lambda \in \mathbb{R}$, (1) $p(x + y) \leq p(x) + p(y)$ and (2) $p(\lambda x) = |\lambda| p(x)$. Let $\{p_i\}$ be a class of seminorms on X. A pair $(X, \{p_i\})$ is said to be a *locally convex space over* $\mathbb{R}$ if for each x in X, whenever $p_i(x) = 0$ for all index i, then $x = 0$. Given a locally convex space $(X, \{p_n\})$ with countably many seminorms, we may assume that the seminorms $\{p_n\}$ is increasing. Note that a locally convex space X with countably many seminorms is metrizable. A sequence $\{x_n\}$ is said to *converge* to x in a locally convex space X if for each k in $\mathbb{N}$ and index i, there exists N in $\mathbb{N}$ such that $p_i(x - x_n) < 2^{-k}$ for all $n \geq N$. A sequence $\{x_n\}$ in X is said to be *a Cauchy sequence* if for each k in $\mathbb{N}$ and index i, there exists N in $\mathbb{N}$ such that $p_i(x_m - x_n) < 2^{-k}$ for all $m, n \geq N$. A locally convex space is *complete* if every Cauchy sequence converges. We call a complete locally convex space with countably many seminorms a $(\mathcal{F})$*-space*. For each k in $\mathbb{N}$, we let $\mathcal{D}_k(\mathbb{R})$ denote the space of test functions ϕ such that supp ϕ is contained in the closed interval $[-k, k]$, with the seminorms $\|\phi\|_m := \max_{l \leq m} \sup_{|x| \leq n} |\phi^{(l)}(x)|$ $(\phi \in \mathcal{D}_k(\mathbb{R}))$, where supp ϕ, which is called *support of* ϕ, is the closure of the set $\{x \in \mathbb{R} : |\phi(x)| > 0\}$. It is easy to show that $(\mathcal{D}_k(\mathbb{R}), \{\|\cdot\|_m\})$ is a $(\mathcal{F})$-space.

Let f be a mapping of a locally convex space X into a locally convex space Y. f is said to be *sequentially continuous* if for each sequence $\{x_n\}$ in X and x in X, if $\{x_n\}$ converges to x in X, then the sequence $\{f(x_n)\}$ converges to $f(x)$ in Y. f is also said to be *sequentially nondiscontinuous* if for each sequence $\{x_n\}$ converging to x, seminorm p'_j of Y and ε in $\mathbb{R}$, whenever $p'_j(f(x_n) - f(x)) \geq \varepsilon$ for all n, then $\varepsilon \leq 0$. Clearly, a sequentially continuous mapping on a locally convex space is sequentially nondiscontinuous. The converse for $(\mathcal{F})$-spaces can be proved in classical mathematics, intuitionistic mathematics and constructive recursive mathematics (see [6, Theorem 2 and Proposition 3] for details). We show the converse for linear mappings on a $(\mathcal{F})$-space in Bishop's framework. The following lemma can be now proved in a way similar to [7, Lemma 2] (see [10, Lemma 3.2.2] for details).

Lemma 1. *Let f be a linear mapping of a $(\mathcal{F})$-space X into a locally convex space Y, $\{x_n\}$ a sequence converging to 0 in X. a seminorm p'_j of Y, and a and b in $\mathbb{R}$ with $0 < a < b$. Then either $p'_j(f(x_n)) > a$ for infinitely many n or else $p'_j(f(x_n)) < b$ for all sufficiently large n.*

The above lemma immediately implies the conclusion.

Theorem 2. *A linear mapping of a $(\mathcal{F})$-space into a locally convex space is sequentially continuous if and only if it is sequentially nondiscontinuous.*

We then have the following version of the Banach-Steinhaus theorem for $(\mathcal{F})$-spaces.

Theorem 3. *Let $\{f_n\}$ be a sequence of sequentially continuous linear mappings of a $(\mathcal{F})$-space X into a locally convex space Y such that $f(x) := \lim_n f_n(x)$ exists for all x in X. Then f is a sequentially continuous linear mapping of X into Y.*

Proof. In view of Theorem 2, it is sufficient to show that f is sequentially nondiscontinuous. Then we can use the method of [5, Appendix A]; see [10, Theorem 3.2.4] for details. □

3 $(\mathcal{L}F)$-spaces

Let X be a locally convex space, $\{X_k\}$ a sequence of $(\mathcal{F})$-spaces, $\mathcal{B}_0$ a class of open balls for 0 in X, and $\mathcal{B}_0^k$ a class of open balls for 0 in X_k. Suppose that $X = \cup_k X_k$, $X_k \subset X_{k+1}$ for all k and that $\mathcal{B}_0^k$ is equivalent to the class $\{U \cap X_k : U \in \mathcal{B}_0^{k+1}\}$ for all k. Let $\mathcal{B}'_0$ be the class of all subsets U of X such that (1) for each s and t in $\mathbb{R}$, $s, t \geq 0$ and $s + t = 1$ imply $sU + tU \subset U$ (*convexity*), (2) for each r in $\mathbb{R}$ with $|r| \leq 1$, $rU \subset U$ (*circledness*), (3) for each x in X there exists $a > 0$ such that for each r in $\mathbb{R}$, if $|r| \leq a$ then $rx \in U$. (*absorbingness*), (4) for each k, $U \cap X_k$ is a neighbourhood of 0 in X_k; that is, there exists U^k in $\mathcal{B}_0^k$ such that $U^k \subset U \cap X_k$. Then X is said to be a *$(\mathcal{LF})$-space with sequence $\{X_k\}$* if $\mathcal{B}_0$ is equivalent to $\mathcal{B}'_0$. For instance, the space $\mathcal{D}(\mathbb{R})$ is a $(\mathcal{LF})$-space with the sequence $\{\mathcal{D}_k(\mathbb{R})\}$ (see [10, Section 4.5]). Note that for each k, $\mathcal{B}_0^k$ is equivalent to the class $\{U \cap X_k : U \in \mathcal{B}_0\}$.

Let $\{x_n\}$ be a sequence in a $(\mathcal{LF})$-space X with sequence $\{X_k\}$ of $(\mathcal{F})$-spaces, and x in X. We write $x_n \longrightarrow x$ if $\{x_n\}$ converges to x in some X_k. Note that $x_n \longrightarrow x$ implies that $\{x_n\}$ converges to x in X. On the other hand, the converse can be proved in classical mathematics, but cannot be in Bishop's framework. In fact, the case of $\mathcal{D}(\mathbb{R})$ is equivalent to BD-$\mathbb{N}$ (see [10, Corollary 4.3.9]). We say in this paper that a mapping f of a $(\mathcal{LF})$-space X into a locally convex space Y is *quasi-sequentially continuous* if for each sequence $\{x_n\}$ and x in X, whenever $x_n \longrightarrow x$, then the sequence $\{f(x_n)\}$ converges to $f(x)$ in Y. Clearly, a sequentially continuous mapping on a $(\mathcal{LF})$-space is quasi-sequentially continuous. It has not been known whether the converse can be proved constructively or not. However, the case of $\mathcal{D}(\mathbb{R})$ can be proved in Bishop's framework (see [12, Theorem 5.4]). We now obtain the following Banach-Steinhaus theorem with respect to quasi-sequentially continuous mappings on a $(\mathcal{LF})$-space, by Theorem 3.

Theorem 4. *Let X be a $(\mathcal{L}F)$-space, Y a locally convex space, and $\{f_n\}$ a sequence of quasi-sequentially continuous linear mappings of X into Y such that $f(x) := \lim_n f_n(x)$ exists for all x in X. Then f is a quasi-sequentially continuous linear mappings of X into Y.*

We here have the problem whether the Banach-Steinhaus theorem with respect to sequentially continuous linear mappings on a $(\mathcal{LF})$-space holds in Bishop's framework or not. The one for $\mathcal{D}(\mathbb{R})$, which is [11, Theorem 4.10] above, holds in this sense. The important point is to represent the convergence of a sequence in a $(\mathcal{LF})$-space, within the $(\mathcal{F})$-spaces. As noted above, the convergence in a $(\mathcal{LF})$-space is classically equivalent to that in some $(\mathcal{F})$-space. Also, the convergence of a sequence $\{\phi_n\}$ in $\mathcal{D}(\mathbb{R})$ is equivalent to the following conditions in Bishop's framework (see [11, Theorem 2.9]): (i) for each l, the sequence $\{\phi_n^{(l)}\}$ of l-th derivatives converges uniformly on $\mathbb{R}$ (or converges with respect to sup-norm), and (ii) $\cup_n \mathrm{supp}_{\mathbb{N}}\phi_n$ is pseudobounded. Here we write $\mathrm{supp}_{\mathbb{N}} f := \{0\} \cup \{n \in \mathbb{N} : \exists q \in \mathbb{Q}(|q| \geq n \wedge |f(q)| > 0)\}$, and a subset A of $\mathbb{N}$ is *pseudobounded* if for each sequence $\{a_n\}$ in A, $\lim_{n\to\infty} a_n/n = 0$. A bounded subset of $\mathbb{N}$ is pseudobounded. On the other hand, the converse for countable sets is called BD-$\mathbb{N}$; that is, the converse cannot be proved in Bishop's framework (see [4] and [6]). Thus the condition (ii) above means a boundedness property of the union of support of each ϕ_n, and implies classically that the set $\{k \in \mathbb{N} : \exists n[\phi_n \notin \mathcal{D}_k(\mathbb{R})]\}$ is bounded. It seems constructively difficult to derive such a property from the conditions of $(\mathcal{LF})$-spaces above. So, although detailed studies are needed, we would require to add such conditions as imply constructively that, letting $(X, \{p_i\})$ be a $(\mathcal{LF})$-space with sequence $\{X_k\}$, for each x in X and index i, $p_i(x_n - X_k) := \inf\{p_i(x_n - x) : x \in X\}$ exists and that, given a convergent sequence $\{x_n\}$ in X, the set $A := \{k \in \mathbb{N} : \exists n \exists i[p_i(x_n - X_k) > 0\}$ is pseudobounded, to the definition of a $(\mathcal{LF})$-space. Here we note that a subset S of $\mathbb{R}$ is *located* if for all r in $\mathbb{R}$, $\inf\{|r - s| : s \in \mathbb{R}\}$ exists and that it cannot be proved in the three frameworks of constructive mathematics that every subset of $\mathbb{R}$ is located (see [2, Chapter 4, Problem 8]). Then we also have to show that $\mathcal{D}(\mathbb{R})$ is located. Moreover, we would need such countable seminorms as are extensions of those of $(\mathcal{F})$-spaces $\{X_k\}_{k\in A}$ and as give a locally convex structure weaker than that of X in the space $\cup_{k\in A} X_k$.

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