Intuitionisitic Phase Spaces are Almost Classical (Abstract)

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Phase semantics is a standard semantics for linear logic that captures provability [Gir95]. Corresponding to two forms of linear logic, i.e. classical linear logic (**LL**) and intuitionistic linear logic (**ILL**), there are two forms of phase semantics: classical and intuitionistic phase semantics (see [Abr90, Tro92, Oka99]).

The aim of this work is to study the relationship between these two. Our main results are as follows:

(i) every intuitionistic phase space is a subspace of a classical phase space.

(ii) every intuitionistic phase space is phase isomorphic to a quasi-classical phase space. Here, a quasi-classical phase space is a phase space having a double-negation-like closure operator.

(iii) there is a syntactic embedding of propositional **ILL** into **LL**.

The result (i) on one hand establishes a relationship between classical and intuitionistic phase spaces, and on the other hand gives a new definition of intuitionistic closure operator $Cl(X) = X^{\perp\perp} \cap M$. A nice fact is that the definition is entirely first-order and it does not involve any "big" (second-order, impredicative) quantification, in contrast to [Abr90, Tro92, Oka99]). The result (ii) yields a completeness theorem for **ILL** with respect to a rather special class of phase models: quasi-classical phase models. Hence intuitionistic phase spaces may be considered "almost classical." The result (iii) is an application of our semantic considerations to a syntactic issue.

Let us begin with several definitions. The *formulas of* **ILL** are built from propositional variables and constants **1**, **0**, $\top$, $\perp$ by logical connectives $\otimes$, $\multimap$, &, $\oplus$ and !.

Definition 1 An *intuitionistic phase space* $(M,\cdot,\varepsilon,Cl)$ consists of a commutative monoid $(M,\cdot,\varepsilon)$ and a *closure operator* Cl, that is a mapping from the subsets of M to themselves such that for all $X,Y \subseteq M$:

$$\begin{array}{ll} \text{(Cl1) } X \subseteq Cl(X) & \text{(Cl2) } Cl(Cl(X)) \subseteq Cl(X) \\ \text{(Cl3) } X \subseteq Y \Longrightarrow Cl(X) \subseteq Cl(Y) & \text{(Cl4) } Cl(X)\cdot Cl(Y) \subseteq Cl(X\cdot Y) \end{array}$$

A set $X \subseteq M$ is said to be *closed* if $X = Cl(X)$.

We also define the following operations over sets $X,Y \subseteq M$:

$$\begin{array}{rcllcl} X \otimes Y & := & Cl(X\cdot Y) & \mathsf{1} & := & Cl(\{\varepsilon\}) \\ X \,\&\, Y & := & X \cap Y & \top & := & M \\ X \oplus Y & := & Cl(X \cup Y) & \mathsf{0} & := & Cl(\emptyset) \\ X \multimap Y & := & \multicolumn{4}{l}{\{z \in M \mid \forall x \in X (x \cdot z \in Y)\}} \\ !X & := & \multicolumn{4}{l}{Cl(X \cap I), \text{ where } I := \{x \in \mathsf{1} \mid x\cdot x = x\}.} \end{array}$$

An *intuitionistic phase model* $(M,\cdot,\varepsilon,Cl,v)$ is an intuitionistic phase space $(M,\cdot,\varepsilon,Cl)$ with a *valuation* v that maps each propositional variable to a closed subset of M, and, in addition, assigns a closed subset of M to constant $\perp$.

The valuation v is naturally extended to all formulas of **ILL**. A formula A is *satisfied* in model $(M,\cdot,\varepsilon,Cl,v)$ if and only if $\varepsilon \in v(A)$.

We have the following completeness theorem: a formula of **ILL** is satisfied in every intuitionistic phase model if and only if it is provable in **ILL**.

As to classical linear logic **LL**, the *formulas* are defined to be those of **ILL** as well as formulas of the form $A \parr B$, $?A$.

A *classical phase space* $(M, \cdot, \varepsilon, \bot)$ consists of a commutative monoid $(M, \cdot, \varepsilon)$ and a distinguished subset $\bot$ of M. A set $X \subseteq M$ is called a *closed set* (or a *fact*) if $X = X^{\bot\bot}$. Here, $X^{\bot}$ stands for $X \multimap \bot$.

Every classical phase space $(M, \cdot, \varepsilon, \bot)$ may be viewed as an intuitionistic phase space. Indeed, it is easily verified that the double negation operator $Cl(X) := X^{\bot\bot}$ satisfies the conditions (Cl1)—(Cl4).

Formulas of **LL** are again interpreted by closed sets, and we have a commpleteness result for **LL** with respect to classical phase models as before.

Definition 2 Let $\mathscr{M}_1 = (M_1, \cdot, \varepsilon, Cl_1)$ be an intuitionistic phase space. Then $\mathscr{M}_2 = (M_2, \cdot, \varepsilon, Cl_2)$ is called a *subspace* of $\mathscr{M}_1$ if

- $(M_2, \cdot, \varepsilon)$ is a submonoid of $(M_1, \cdot, \varepsilon)$, and
- $Cl_2(X) = Cl_1(X) \cap M_2$ for any $X \subseteq M_2$.

With this notion of subspace, it is straightforward to observe that every subspace of a classical phase space is an intuitionistic phase space. Namely, if $(M_c, \cdot, \varepsilon, \bot_c)$ is a classical phase space and M is a submonoid of M_c, then $(M, \cdot, \varepsilon, Cl)$ with $Cl(X) := X^{\bot_c \bot_c} \cap M$ is an intuitionistic phase space. The converse also holds. It is a corollary to our main theorem below.

We now consider a subclass of the intuitionistic phase spaces which are of special interest.

Definition 3 An intuitionistic phase space $(M, \cdot, \varepsilon, Cl)$ is called a *quasi-classical phase space* if it is a subspace of a classical phase space $(M_c, \cdot, \varepsilon, \bot_c)$ and $Cl(X) = X^{\bot_c \bot_c}$ for every $X \subseteq M$.

The closure operator Cl of a quasi-classical phase space $(M, \cdot, \varepsilon, Cl)$ looks almost like a classical double-negation operator, the only difference being that $\bot_c$ is not necessarily a subset of M. It is not known whether every intuitionistic phase space is quasi-classical or not. It is an open question. On the other hand, we know that every intuitionistic phase space is phase isomorphic to a quasi-classical one, with a natural notion of phase isomorphism. Roughly speaking, two phase spaces are *phase isomorphic* if they are equivalent as residuated lattice (enriched with a modal operator). This is also a corollary to our main theorem below.

Let us now state the main theorem, the *three-layered representation theorem*, that gives a good summary of the relationship between intuitionistic and classical phase semantics:

Theorem 4 (Three-Layered Representation) *Let $\mathscr{M} = (M, \cdot, \varepsilon, Cl)$ be an intuitionistic phase space. Then there exist a quasi-classical phase space $\mathscr{M}_q = (M_q, \cdot, \varepsilon, Cl_q)$ and a classical phase space $\mathscr{M}_c = (M_c, \cdot, \varepsilon, \bot_c)$ satisfying the following;*

(i) $\mathscr{M}$ is a subspace of $\mathscr{M}_q$, and $\mathscr{M}_q$ is a subspace of $\mathscr{M}_c$; specifically,

- *for any $X \subseteq M$: $Cl(X) = X^{\bot_c \bot_c} \cap M$, and*
- *for any $X_q \subseteq M_q$: $Cl_q(X_q) = X_q^{\bot_c \bot_c}$.*

(ii) There is a phase isomorphism from $\mathscr{M}_q$ to $\mathscr{M}$.

From this theorem, the following corollaries follow immediately:

Corollary 5 *Every intuitionistic phase space is a subspace of a classical phase space.*

Corollary 6 *Every intuitionistic phase space is phase isomorphic to a quasi-classical phase space. As a consequence, a formula A is provable in* **ILL** *if and only if A is satisfied in every quasi-classical phase model.*

Let us now move on to syntax. Our aim is to give a a syntactic embedding of **ILL** into **LL** based on the semantic insight above. First of all, note that **LL** is already conservative over **ILL** as far as the propositional formulas without **0** nor $\perp$ are concerned:

Theorem 7 ([Sch91]) *Let A be a formula of* **ILL** *which does not contain* **0** *nor* $\perp$. *Then A is provable in* **ILL** *if and only if it is provable in* **LL**.

Therefore, there is no need of translation for this fragment. On the other hand, in the presence of **0** or $\perp$ (or second order quantifiers), **LL** is not conservative over **ILL**, as witnessed by the following:

$$((p \multimap \perp) \multimap \perp) \multimap p; \quad (\top \multimap \mathbf{1}) \multimap ((p \multimap \mathbf{0}) \multimap \mathbf{0}) \multimap p.$$

These **ILL** formulas are provable in **LL** but not in **ILL**. Our embedding is intended to cover the full propositional logic **ILL** including **0** and $\perp$.

Definition 8 Let p_0 be a distinguished propositional variable. we define

$$\varphi(p_0) := (!p_0) \otimes !(p_0 \otimes p_0 \multimap p_0)$$

(The formula $\varphi(p_0)$ says, roughly, that "the interpretation of p_0 is a monoid.")

To each formula A of **ILL**, we associate another formula A° of **ILL** as follows;

$$\begin{array}{rcll}
q^\circ & := & q \& p_0, & \text{for a propositional variable } q \\
c^\circ & := & c \& p_0, & \text{for } c \in \{\top, \perp\} \\
d^\circ & := & d, & \text{for } d \in \{\mathbf{1}, \mathbf{0}\} \\
(A \multimap B)^\circ & := & (A^\circ \multimap B^\circ) \& p_0 & \\
(A \star B)^\circ & := & A^\circ \star B^\circ, & \text{for } \star \in \{\otimes, \&, \oplus\} \\
(!A)^\circ & := & !A^\circ. &
\end{array}$$

In the presence of the assumption $\varphi(p_0)$, the suffix $\& p_0$ behaves like an S4-modality $\Box$. In view of this analogy, the above translation can be considered as a linear analogue of Gödel's famous embedding of intuitionistic logic into modal logic S4. As a matter of fact, we have:

Theorem 9 *A formula A of* **ILL** *is provable in* **ILL** *iff* $\varphi(p_0) \multimap A^\circ$ *is provable in* **LL**.

To prove this theorem, we exploit phase semantics and the relationship between intuitionistic and classical phase models established above.

The details are given in [KOT04].

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