Construction of Counter–models for the Modal Logic K4B *

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Abstract

The modal logic **K4B** is the smallest modal logic obtained by adding the axioms **4** : $\Box A \supset \Box\Box A$ and **B** : $A \supset \Box\Diamond A$ to the modal logic **K**, where $\Diamond$ is an abbreviation of $\neg\Box\neg$. We are going to give a constructive way to obtain a counter–model for **K4B** if a given formula is not provable in our sequent system for **K4B**.

1 Sequent system for K4B

We first introduce our sequent system $\mathcal{S}$**K4B** for the modal logic **K4B**, which was inspired by the sequent system proposed by Takano in [2]. Let small letters p, q etc. be propositional variables. Formulas are defined in the usual way with logical connectives $\wedge$, $\vee$, $\supset$, $\neg$ and $\Box$. Capital letters A, B etc. denote arbitrary formulas. Greek capital letters Γ, Δ etc. denote (finite, possibly empty) sets of formulas. Subscripts will be used if necessary. A sequent of $\mathcal{S}$**K4B** is an expression of the form $\Gamma \to \Delta$. Initial sequents of $\mathcal{S}$**K4B** are of the form $\Gamma, p \to p, \Delta$. Rules of $\mathcal{S}$**K4B** are given in Figure 1.

$$\frac{\Gamma, A, B \to \Delta}{\Gamma, A \wedge B \to \Delta}(\wedge \to) \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}(\to \vee) \qquad \frac{\Gamma \to \Delta, A}{\Gamma, \neg A \to \Delta}(\neg \to) \qquad \frac{\Gamma, A \to \Delta}{\Gamma \to \Delta, \neg A}(\to \neg)$$

$$\frac{\Gamma, A \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to\supset) \qquad \frac{A, \blacksquare A, \Gamma \to \Delta, \Box B}{\Box A, \Gamma \to \Delta, \Box B}(T)_{\Box} \qquad \frac{A, \blacksquare A, \Gamma \to \Delta, \blacksquare B}{\Box A, \Gamma \to \Delta, \blacksquare B}(T)_{\blacksquare}$$

$$\frac{\Gamma \to \Delta, A, B \quad \Gamma, B \to \Delta, A \quad \Gamma, A \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge) \qquad \frac{\Gamma, A, B \to \Delta \quad \Gamma, B \to \Delta, A \quad \Gamma, A \to \Delta, B}{\Gamma, A \vee B \to \Delta}(\vee \to)$$

$$\frac{\Gamma, A, B \to \Delta \quad \Gamma, B \to \Delta, A \quad \Gamma \to \Delta, A, B}{\Gamma, A \supset B \to \Delta}(\supset\to)$$

$$\frac{\Box\Gamma, \blacksquare A, \blacksquare\Phi, \Lambda \to \Box\Delta, \blacksquare\Psi, \Xi, A \quad A, \Box\Gamma, \blacksquare A, \blacksquare\Phi, \Lambda \to \Box\Delta, \blacksquare\Psi, \Xi}{\Box\Gamma, \Box A, \blacksquare\Phi, \Lambda \to \Box\Delta, \blacksquare\Psi, \Xi}(cut)_L$$

$$\frac{\Box\Gamma, \blacksquare\Phi, \Lambda \to \Box\Delta, \blacksquare A, \blacksquare\Psi, \Xi, A \quad A, \Box\Gamma, \blacksquare\Phi, \Lambda \to \Box\Delta, \blacksquare A, \blacksquare\Psi, \Xi}{\Box\Gamma, \blacksquare\Phi, \Lambda \to \Box\Delta, \Box A, \blacksquare\Psi, \Xi}(cut)_R$$

where Λ and Ξ denote sets of propositional variables.

$$\frac{\Box\Gamma, \Gamma \to \Box\Theta, A_1 \qquad \cdots \qquad \Box\Gamma, \Gamma \to \Box\Theta, A_m}{\blacksquare\Gamma, p_1, \cdots, p_n \to \blacksquare A_1, \cdots, \blacksquare A_m, q_1, \cdots, q_l}(\Box)_{K4B}$$

where $\Box\Theta = \{\Box A_1, \cdots, \Box A_m\}$.

Figure 1: Rules of $\mathcal{S}$**K4B**

Here, $\Box\Gamma$ denotes the set of formulas $\{\Box A_1, \cdots, \Box A_n\}$, when Γ is $\{A_1, \cdots, A_n\}$. The auxiliary modal operator $\blacksquare$ has the same semantics as that of $\Box$, but $\blacksquare$ plays a syntactical role different from $\Box$. In addition, $\blacksquare\Gamma$ is defined similarly to $\Box\Gamma$. In order to facilitate our discussion, each application of the

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rules is supposed to generate the upper sequent(s) from the lower sequent, though an usual application is proceeded conversely. The modal operator $\blacksquare$ is used to avoid redundant applications of the rules. Each upper sequent of $(\Box)_{K4B}$ must be regarded as 'or'–branch, which means that if one of the upper sequents of $(\Box)_{K4B}$ is provable then so is the lower sequent of it, while in other rules each upper sequent should be regarded as 'and'-branch. In order to emphasize 'or'–branch, double lines are used in $(\Box)_{\mathbf{K4B}}$.

Let $Sub(\Gamma)$ denote the set of the all subformulas of all formulas in Γ. The subformula property does not hold for $\mathcal{S}$**K4B** in the strict sense, but for a given sequent $\Gamma \to \Delta$, any formula occurring in all proofs of $\Gamma \to \Delta$ is in $(\Gamma \cup \Delta)^{\mathcal{S}\mathbf{K4B}*}$, where $(\Gamma \cup \Delta)^{\mathcal{S}\mathbf{K4B}*} = Sub(\Gamma \cup \Delta) \ \cup \ \{\blacksquare A \mid \Box A \in Sub(\Gamma \cup \Delta)\}$.

2 Model graphs for K4B

We construct **K4B**–models in which a given formula is not true if it is not provable in $\mathcal{S}$**K4B**. In order to do so, we first need some technical machinery as discussed in [1].

Suppose that $\Gamma \to \Delta$ is given. Then, let $a(\Gamma \to \Delta)$ and $s(\Gamma \to \Delta)$ denote the antecedent Γ and the succedent Δ, respectively. For sequents $\Gamma_1 \to \Delta_1$ and $\Gamma_2 \to \Delta_2$, we say that $\Gamma_1 \to \Delta_1$ *is in* $\Gamma_2 \to \Delta_2$ if $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$, and we write it as $\Gamma_1 \to \Delta_1 \ \subseteq \ \Gamma_2 \to \Delta_2$. A sequent $\Gamma \to \Delta$ is *closed with respect to a rule* (r) if whenever (an instance of) the lower sequent of (r) is in $\Gamma \to \Delta$, so is (a corresponding instance of) at least one of the upper sequents of (r). For instance, $\Gamma, A, B, A \wedge B \to \Delta$ is closed with respect to the rule $(\wedge \to)$, where $A \wedge B$ occurring in the left–hand side is the principal formula in the application of $(\wedge \to)$. A sequent $\Gamma \to \Delta$ is $\mathcal{S}$**K4B**–*saturated* if it is not provable in $\mathcal{S}$**K4B** and closed with respect to all rules except $(\Box)_{K4B}$. For the readability's sake, we omit $\mathcal{S}$**K4B**– from $\mathcal{S}$**K4B**–*saturated*.

In constructing counter–models, we associate saturated sequents with possible worlds. In order to generate a saturated sequent from a sequent $\Gamma \to \Delta$ unprovable in $\mathcal{S}$**K4B**, do as follows: Let $w = \Gamma \to \Delta$. While there is some rule (r), except $(\Box)_{K4B}$, with respect to which w is not closed, do the following: Let A be the principal formula of (r). Choose one of the corresponding instances of the upper sequents of (r), denoted by w', which is unprovable in $\mathcal{S}$**K4B**. Then, set $w = (a(w') \cup \{A\}) \to s(w')$ if A occurs in the left–hand side of the lower sequent of (r), otherwise set $w = a(w') \to (s(w') \cup \{A\})$. By the iteration, we can obtain a saturated sequent of $\Gamma \to \Delta$. Note that the saturated sequent is still unprovable in $\mathcal{S}$**K4B**.

Definition 2.1 (K4B–Model Graphs) *Let W be a nonempty set and R be a binary relation on W, that is $R \subseteq W \times W$. Then a* **K4B–model graph** *for a sequent $\Gamma \to \Delta$ is a finite* **K4B**–*frame (W, R) such that W consists of $\mathcal{S}$***K4B**–*saturated sequents w with $a(w),\ s(w) \subseteq (\Gamma \cup \Delta)^{\mathcal{S}\mathbf{K4B}*}$ and*

1. *$\Gamma \to \Delta \subseteq w_0$ for some $w_0 \in W$,*
2. *if $\blacksquare A \in s(w)$ then there exists some $w' \in W$ with wRw' and $A \in s(w')$,*
3. *if wRw' and $\blacksquare A \in a(w)$ then $A \in a(w')$.*

For the readability's sake, $\mathcal{S}$**K4B**–model graphs is written simply as model graphs. For a set $\Gamma = \{A_1, \cdots, A_n\}$ of formulas, let Γ_* and Γ^* denote $A_1 \wedge \cdots \wedge A_n$ and $A_1 \vee \cdots \vee A_n$, respectively.

Lemma 2.2 (Satisfiability Lemma) *If (W, R) is a* **K4B**–*model graph for $\Gamma \to \Delta$ then there exists a* **K4B**–*model $(W, R, \models)$ such that $w \not\models \Gamma_* \supset \Delta^*$ for some $w \in W$.*

This lemma can be proven by giving a valuation such that, for any $w \in W$, if a propositional variable p occurs in $a(w)$ then p is supposed to be true at w, that is $w \models p$ if and only if $p \in a(w)$. Once we construct a model graph, we can obtain a **K4B**–model from Lemma 2.2 immediately.

3 Construction of counter–models for K4B

Now, it remains to be shown how to construct model graphs. In this section, we show it through an example, in which we construct a model graph for $\Box\Box p \to \Box\neg p \vee \Box q$. In Figure 2, the failed proofs of it are shown. The lower part is the whole failed proofs. The failed proofs can be obtained by applying the rules of $\mathcal{S}$**K4B** repeatedly. In addition, all the top sequents are not initial sequents. We note that failed proofs are put together with the double line of $(\Box)_{K4B}$. Although $(\Box)_{K4B}$ is applicable to the top sequents shown explicitly in P_1 and P_2, we do not need to apply $(\Box)_{K4B}$ to them, because we can construct a model graph without doing it.

First of all, we generate possible worlds from sequents. Take the end sequent $\Box\Box p \to \Box\neg p \vee \Box q$ and the end sequents of P_1 and P_2, and saturate them. The failed proofs facilitate this saturation. Then we can obtain three possible worlds w_0, w_1 and w_2. Second of all, define the accessibility relation R. In Figure 2, arrows denote R. We note that R must be transitive and symmetric. Then we can obtain a **K4B**–frame, whose height is 1, or rather a model graph. Finally, we can obtain a counter–model of $\Box\Box p \to \Box\neg p \vee \Box q$ by giving a valuation as stated at the end of the previous section. We can see that $\Box\Box p \supset \Box\neg p \vee \Box q$ is not true at w_0.

P_1

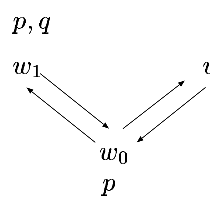
$$\dfrac{\dfrac{\dfrac{\vdots \qquad \dfrac{\dfrac{p,\, q,\, \blacksquare p,\, \blacksquare\Box p \to \blacksquare\neg p,\, \blacksquare q}{p,\, q,\, \blacksquare p,\, \blacksquare\Box p \to \blacksquare\neg p,\, \blacksquare q,\, \neg p}(\to\neg) \qquad \vdots}{p,\, q,\, \blacksquare p,\, \blacksquare\Box p \to \Box\neg p,\, \blacksquare q}(cut)_R}{p,\, \blacksquare p,\, \blacksquare\Box p \to \Box\neg p,\, \Box q}(cut)_R}{\dfrac{p,\, \Box p,\, \blacksquare\Box p \to \Box\neg p,\, \Box q}{\dfrac{p,\, \Box p,\, \Box\Box p \to \Box\neg p,\, \Box q}{p,\, \Box p,\, \Box\Box p \to \Box\neg p,\, \Box q,\, \neg p}(\to\neg)}(T)_{\Box}}(T)_{\Box}$$

P_2

$$\dfrac{\dfrac{\dfrac{\dfrac{p,\, \blacksquare p,\, \blacksquare\Box p \to \blacksquare\neg p,\, \blacksquare q,\, q}{p,\, \blacksquare p,\, \blacksquare\Box p \to \blacksquare\neg p,\, \blacksquare q,\, q,\, \neg p}(\to\neg) \qquad \vdots}{p,\, \blacksquare p,\, \blacksquare\Box p \to \Box\neg p,\, \blacksquare q,\, q}(cut)_R \qquad \vdots}{p,\, \blacksquare p,\, \blacksquare\Box p \to \Box\neg p,\, \Box q,\, q}(cut)_R$$

$$\dfrac{p,\, \blacksquare p,\, \blacksquare\Box p \to \Box\neg p,\, \Box q,\, q}{\dfrac{p,\, \Box p,\, \blacksquare\Box p \to \Box\neg p,\, \Box q,\, q}{p,\, \Box p,\, \Box\Box p \to \Box\neg p,\, \Box q,\, q}(T)_{\Box}}(T)_{\Box}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P_1 \qquad P_2}{p,\, \blacksquare p,\, \blacksquare\Box p \to \blacksquare\neg p,\, \blacksquare q,\, q}(\Box)_{\mathbf{K4B}}}{p,\, \blacksquare p,\, \blacksquare\Box p \to \blacksquare\neg p,\, \blacksquare p,\, q,\, \neg p}(\to\neg) \qquad \vdots}{p,\, \blacksquare p,\, \blacksquare\Box p \to \Box\neg p,\, \blacksquare q,\, q}(cut)_R \qquad \vdots}{\dfrac{p,\, \blacksquare p,\, \blacksquare\Box p \to \Box\neg p,\, \Box q}{\dfrac{\Box p,\, \blacksquare\Box p \to \Box\neg p,\, \Box q}{\dfrac{\Box\Box p \to \Box\neg p,\, \Box q}{\Box\Box p \to \Box\neg p \vee \Box q}(\to\vee)}(T)_{\Box}}(T)_{\Box}}(cut)_R$$



$w_0 = p, \Box p, \Box\Box p, \blacksquare p, \blacksquare\Box p \to \Box\neg p \vee \Box q, \Box\neg p, \Box q, \blacksquare\neg p, \blacksquare q, \neg p, q$

$w_1 = p, q, \Box p, \Box\Box p, \blacksquare p, \blacksquare\Box p \to \Box\neg p, \Box q, \blacksquare\neg p, \blacksquare q, \neg p$

$w_2 = p, \Box p, \Box\Box p, \blacksquare p, \blacksquare\Box p \to \Box\neg p, \Box q, \blacksquare\neg p, \blacksquare q, \neg p, q$

Figure 2: A counter–model of $\Box\Box p \supset \Box\neg p \vee \Box q$

In order to confirm that the above **K4B**–frame is a model graph, it is required to show the following two properties: For $i = 1, 2$, (1) $\blacksquare A \in a(w_i)$ implies $\blacksquare A \in a(w_0)$, and (2) $\blacksquare A \in s(w_i)$ implies $\blacksquare A \in s(w_0)$ From these properties, the properties of model graph in Definition 2.1 follow.

4 Concluding remarks

Some consequences can be obtained by constructing counter–models. First, we can obtain a proof of completeness of $\mathcal{S}$**K4B** by giving a counter–model if a given formula is not provable in $\mathcal{S}$**K4B**. Second, we can obtain the finite model property for **K4B**. Recall that the set of possible worlds of our model graphs is always finite. Also, the decidability for **K4B** follows immediately. Thus, we can regard $\mathcal{S}$**K4B** as a decision procedure for **K4B**, which gives us a proof if a given formula is provable.

References

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