Approximate solution method for the Hamilton-Jacobi equation based on stable manifold theory with applications

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Outline of talk

- General theory of 1st order PDEs

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- Stabilizing solution and stable manifold

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- Concluding remarks

Hamilton-Jacobi equation

$$\text{(HJ)} \qquad H(x,p) = p^T f(x) - \frac{1}{2} p^T R(x) p + q(x) = 0$$

$$\begin{cases} x_1, \cdots, x_n\text{: independent variables} \\ \qquad\qquad \cdots \text{ state space } X \\ p_j = \partial z / \partial x_j, \quad j = 1, \cdots, n \\ z\text{: unknown function} \end{cases}$$

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$$\begin{cases} f : \mathbb{R}^n \text{ valued function} \\ R : \mathbb{R}^{n \times n} \text{ valued function} \\ q : \text{ scalar function} \end{cases}$$

$$\left.\begin{array}{l} f(0) = 0 \\ R(x)^T = R(x) \\ q(0) = 0, \frac{\partial q}{\partial x}(0) = 0 \end{array}\right. \quad \text{: all } C^\infty$$

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$$\text{(RIC)} \quad x^T P A x + x^T A^T P x - x^T P R(0) P x + x^T Q x = 0$$

$$f(x) = Ax + O(|x|^2), \quad q(x) = \frac{1}{2} x^T Q x + O(|x|^3)$$

Theory of 1st order PDEs

FACT1

An n-dimensional invariant manifold of

$$\Sigma_H : \quad \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = f(x) - R(x)p \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -\dfrac{\partial f}{\partial x}(x)^T p + \dfrac{\partial (p^T R(x) p)}{\partial x}^T - \dfrac{\partial q}{\partial x}^T, \end{cases}$$

is a *Lagrangian* submanifold of T^*X.

Theory of 1st order PDEs

FACT2

$\Lambda \subset T^*X$: *L*-submanifold, $q \in \Lambda$. $\pi|_\Lambda : \Lambda \to X$(natural projection) is submersion, then, locally around q,

$$\Lambda = \left\{ (x, p) \,|\, p_j = \frac{\partial z}{\partial x_j}(x) \right\}$$

for a function $z(x)$ locally defined around q. (Graph of the derivative $\partial z / \partial x$)

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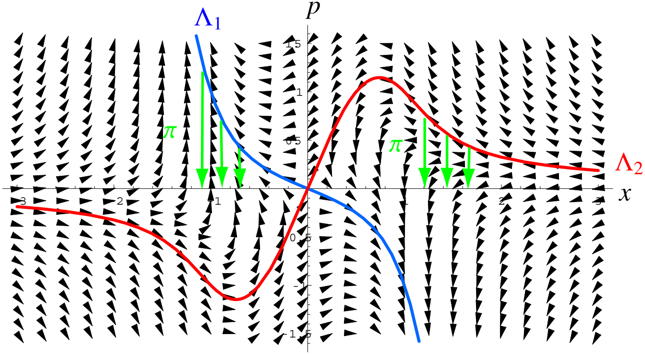
FACT1+FACT2 $\Longrightarrow$ $z(x)$ is a solution of

$$H(x, p) = p^T f(x) - \frac{1}{2} p^T R(x) p + q(x) = 0.$$

Example: Hamiltonian flow for $H = p(x - x^3) - \frac{1}{2}p^2 + \frac{1}{2}x^2$.

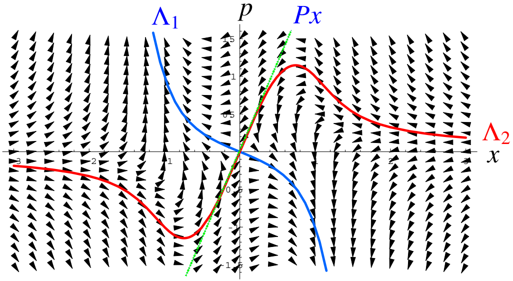


Stabilizing solution and stable manifold

$$\text{(HJ)} \quad H(p,x) = p^T f(x) - \frac{1}{2} p^T R(x) p + q(x) = 0,$$

$f(x) - R(x)p(x)$: asymp. stable

$$\text{(RIC)} \quad PA + A^T P - PR(0)P + Q = 0,$$ $A - R(0)P$: stable



Stable Lagrangian manifold and stabilizing solution

Theorem 1 (van der Schaft 91)
A hyperbolic stable manifold of a Hamiltonian system is a Lagrangian submanifold and a generating function (solution of HJ eq.) exists.

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How to compute it?
Modification of stable manifold theory

Approximation of stable manifold

$$\Sigma: \begin{cases} \dot{x} = Ax + f(t,x,y) \\ \dot{y} = -A^T y + g(t,x,y) \end{cases}$$

A: $n \times n$, $\mathrm{Re}\lambda_j(A) < 0$; $f(t,x,y)$, $g(t,x,y)$: smooth higher order terms

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Define sequences:

$$\begin{cases} x_{k+1} = e^{At}\xi + \displaystyle\int_0^t e^{A(t-s)} f(s, x_k(s), y_k(s))\, ds \\ y_{k+1} = -\displaystyle\int_t^\infty e^{-A^T(t-s)} g(s, x_k(s), y_k(s))\, ds \end{cases}$$

for $k = 0, 1, 2, \ldots$, and

$$\begin{cases} x_0 = e^{At}\xi \\ y_0 = 0 \end{cases}$$

where $\xi \in \mathbb{R}^n$ is arbitrary parameters.

$$\begin{cases} x_{k+1} = e^{At}\xi + \int_0^t e^{A(t-s)} f(s, x_k(s), y_k(s))\, ds \\ y_{k+1} = -\int_t^\infty e^{-A^T(t-s)} g(s, x_k(s), y_k(s))\, ds \end{cases} \qquad \begin{cases} x_0 = e^{At}\xi \\ y_0 = 0 \end{cases}$$

Theorem:

- $x_k(t, \xi)$, $y_k(t, \xi) \to 0$ as $t \to \infty$ for all $k = 0, 1, 2, \ldots$ for $|\xi| \ll 1$.

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- $x_k(t,\xi)$ and $y_k(t,\xi)$ are uniformly convergent to a solution of Σ on $[0, \infty)$ as $k \to \infty$.

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- $x_k(t,\xi)$ and $y_k(t,\xi)$ are uniformly convergent to a solution of Σ on $[0,\infty)$ as $k \to \infty$.
- $x(t,\xi) := \lim_{k\to\infty} x_k(t,\xi)$, $y(t,\xi) := \lim_{k\to\infty} y_k(t,\xi)$ satisfy $x(t,\xi)$, $y(t,\xi) \to 0$ as $t \to \infty$ and the solution on the stable manifold of Σ.

$$\begin{cases} x_{k+1} = e^{At}\xi + \int_0^t e^{A(t-s)} f(s, x_k(s), y_k(s))\, ds \\ y_{k+1} = -\int_t^\infty e^{-A^T(t-s)} g(s, x_k(s), y_k(s))\, ds \end{cases} \qquad \begin{cases} x_0 = e^{At}\xi \\ y_0 = 0 \end{cases}$$

$\{(x(t,\xi), y(t,\xi) \,|\, |\xi| \ll 1, t \in \mathbb{R}\}$: parametrization of the stable manifold!

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$\{(x_k(t,\xi), y_k(t,\xi) \,|\, |\xi| \ll 1, t \in \mathbb{R}\}$: approximation of the stable manifold!

Contraction mapping

$$x^*(t) = e^{At}\xi + \int e^{A(t-s)} f(x^*(t), y^*(t))\, ds$$

$$y^*(t) = -\int_t^\infty e^{-A^T(t-s)} g(x^*(t), y^*(t))\, ds$$

Integral equation

Contraction mapping

$$\left.\begin{aligned} x^*(t) &= e^{At}\xi + \int_0^t e^{A(t-s)} f(x^*(t), y^*(t))\,ds \\ y^*(t) &= -\int_t^\infty e^{-A^T(t-s)} g(x^*(t), y^*(t))\,ds \end{aligned}\right\} =: \mathcal{T}(x^*, y^*)(t)$$

$\mathcal{T}$ is a contraction mapping on a complete metric space of continuous functions.

(x^*, y^*) is a fixed point of $\mathcal{T}$

$$\Sigma_H : \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = f(x) - R(x)p \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -\dfrac{\partial f}{\partial x}(x)^T p + \dfrac{\partial (p^T R(x) p)}{\partial x}^T - \dfrac{\partial q}{\partial x}^T \end{cases}$$

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Assumption: The algebraic Riccati equation

$$PA + A^T P - PR(0)P + Q = 0$$

that approximates the HJ equation has a stabilizing solution $P = \Gamma$.

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Using a suitable coordinate transformation,

$$\Sigma_H' : \begin{pmatrix} \dot{x}' \\ \dot{p}' \end{pmatrix} = \begin{pmatrix} A - R(0)\Gamma & 0 \\ 0 & -(A - R(0)\Gamma)^T \end{pmatrix} \begin{pmatrix} x' \\ p' \end{pmatrix} + \text{higher order terms}$$

Now, the stable manifold approximation theorem can be applied.

Numerical example

$$\dot{x} = x - x^3 + u, \qquad J = \int_0^\infty \frac{q}{2}x^2 + \frac{r}{2}u^2 \, dt$$

The Hamilton-Jacobi equation for this problem:

$$H = p(x - x^3) - \frac{1}{2r}p^2 + \frac{q}{2}x^2 = 0, \quad p = \frac{dV}{dx}$$

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Hamilton's canonical equations:

$$\Sigma_H : \begin{cases} \dot{x} = x - x^3 - \dfrac{1}{r}p \\ \dot{p} = -(1 - 3x^2)p - qx. \end{cases}$$

Numerical example

$$\Sigma'_H : \begin{pmatrix} \dot{x}' \\ \dot{p}' \end{pmatrix} = \begin{pmatrix} -\sqrt{1+q/r}\, x' \\ \sqrt{1+q/r}\, p' \end{pmatrix} + \begin{pmatrix} -x(x',p')^3 \\ 3x(x',p')^2 p(x',p') \end{pmatrix},$$

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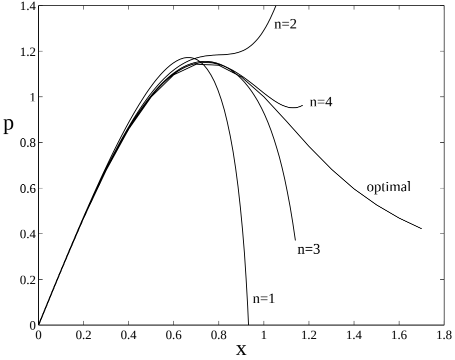

Properties of the algorithm

$$\begin{cases} x_{k+1} = e^{At}\xi + \int_0^t e^{A(t-s)} f(s, x_k(s), y_k(s))\, ds \\ y_{k+1} = -\int_t^\infty e^{-A^T(t-s)} g(s, x_k(s), y_k(s))\, ds \end{cases} \qquad \begin{cases} x_0 = e^{At}\xi \\ y_0 = 0 \end{cases}$$

- Recursive computation — Suitable for computer implementation

Properties of the algorithm

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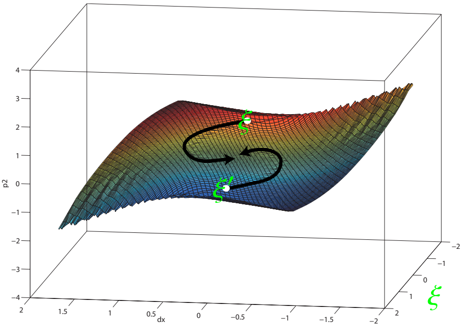
- Analytic expression — Analytic and numeric approaches $\int t^k \exp(-\lambda t) \sin \omega t \, dt$ (for polynomial nonlinearities)
- $\partial V / \partial x$ is obtained — No need to differentiate approximated solutions
- Coincide with Riccati solutions around the origin — Linear performance is guaranteed

Numerical approach



Construction of $p(x)$:

- MATLAB commands `griddatan` and `interpn`
- Polynomial fitting — better for higher dimensional systems

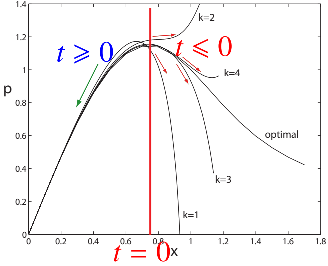
Important parameters

Radius of convergence:

- Taking small $|\xi|$ is essential for the convergence
- ξ determines the initial point ($t = 0$) of the solution curve on the stable manifold

Use of negative time:

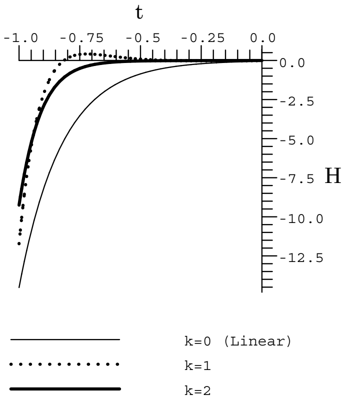
- The constructed surface is generally small if $t \geqslant 0$ is employed
- By using the invariance property of the stable manifold, the surface is enlarged by the negative time operation



Important parameters

Hamiltonian check

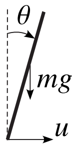
The number of iteration k and the maximum t_{neg} determine the size of effective domain for control and the computational time



Applications

- Swing-up and stabilization of a 2-dimensional inverted pendulum
- Optimal control of systems with input saturations (magnitude, rate)
- Optimal servo system design for a magnetic levitation system (with experiment)

Swing-up and stabilization of an inverted pendulum



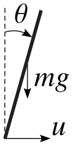
Introduce state variables $x = [x_1, x_2]^T = [\theta, \dot{\theta}]^T$

$$\dot{x} = f(x) + g(x)u$$

$$f(x) = \begin{pmatrix} x_2 \\ \frac{mgl\sin x_1 - ml^2 x_2^2 \sin x_1 \cos x_1}{J + ml^2 \sin^2 x_1} \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ \frac{-l\cos x_1}{J + ml^2 \sin^2 x_1} \end{pmatrix}.$$

$$J = \int_0^\infty x^T Q x + u^T R u \, dt; \quad Q = 0_{2\times 2}, \ R = 2$$

Swing-up and stabilization of an inverted pendulum



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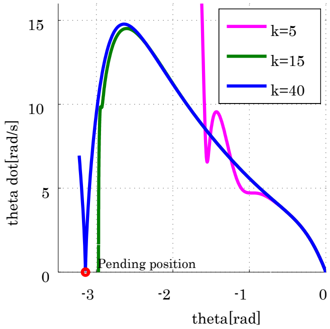
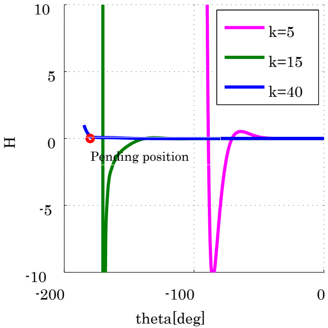
The problem of stability region enhancement with optimality

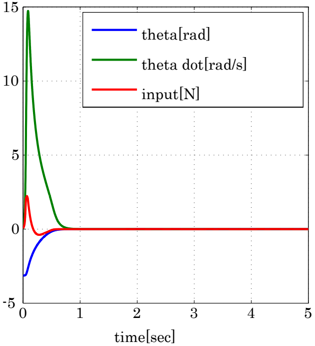


Trajectories of $x_k(t, \xi)$ with $\xi = [-0.0046, 0.0569]$ for $k = 5, 15, 40$.
The trajectory with $k = 40$ passes through the pending position.



Trajectories with $k = 40$ and different values of ξ will construct the right stabilizing feedback

The feedback function is constructed using 10th order polynomial fitting



Simulation result

Input magnitude saturations(1/3)

$$\dot{x} = f(x) + g(x)\,\mathrm{sat}(u)$$

$$J = \int_0^\infty x^T Q x + u^T R u \, dt \quad \text{with } R \text{ diagonal}$$

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The function sat is an m-channel saturation function defined by

$$\mathrm{sat}(u) = \begin{pmatrix} \mathrm{sat}_1(u_1) \\ \vdots \\ \mathrm{sat}_m(u_m) \end{pmatrix}, \quad \mathrm{sat}_j(z) = \begin{cases} \underline{\sigma}_j & (z < \underline{\sigma}_j) \\ z & (\underline{\sigma}_j \leqslant z \leqslant \bar{\sigma}_j) \\ \bar{\sigma}_j & (\bar{\sigma}_j < z) \end{cases}$$

with $\underline{\sigma}_j < 0 < \bar{\sigma}_j$ for $j = 1, \ldots, m$.

Input magnitude saturations(2/3)

The dynamic programming:

$$
\begin{aligned}
\bar{u}(x,p) =& \operatorname{argmin} \left\{ p^T (f(x) + g(x)\,\mathrm{sat}(u)) + x^T Q x + u^T R u \right\} \\
=& \mathrm{sat}\left(-\frac{1}{2} R^{-1} g(x)^T p \right).
\end{aligned}
$$

Input magnitude saturations(2/3)

The dynamic programming:

$$\begin{aligned}\bar{u}(x,p) =& \operatorname{argmin}\left\{p^T(f(x)+g(x)\,\mathrm{sat}(u))+x^TQx+u^TRu\right\}\\ =& \mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^Tp\right).\end{aligned}$$

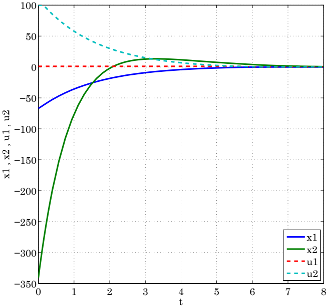
Hamilton-Jacobi equation

$$\begin{aligned}&\left(\frac{\partial V}{\partial x}\right)\left\{f(x)+g(x)\,\mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right)\right\}\\ &\quad+\mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right)^T R\,\mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right)\\ &\quad+x^TQx=0\end{aligned}$$

Input magnitude saturations(3/3)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{sat}_1(u_1) \\ \mathrm{sat}_2(u_2) \end{pmatrix},$$

where $\underline{\sigma}_1 = -1$, $\bar{\sigma}_1 = 1$, $\underline{\sigma}_2 = -100$ and $\bar{\sigma}_2 = 100$.



Linear control

Input magnitude saturations(3/3)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{sat}_1(u_1) \\ \mathrm{sat}_2(u_2) \end{pmatrix},$$

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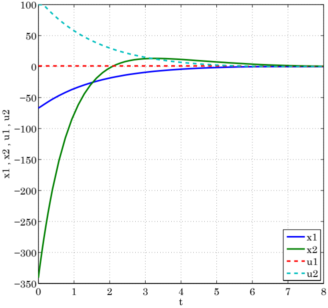
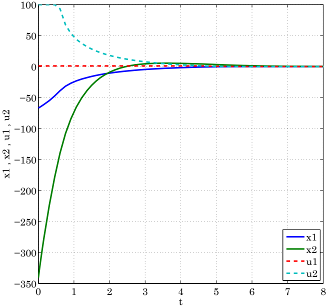


Linear control



Nonlinear control

Input rate saturation (1/4) — PIO

From NASA website

Input rate saturation (2/4)

$$\dot{x} = f(x) + g(x)\,\mathrm{RL}(u)$$

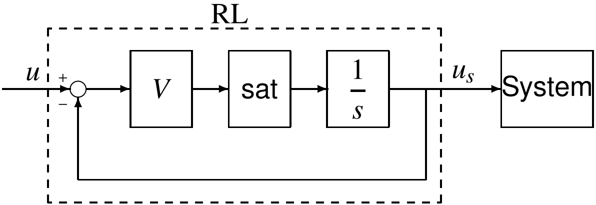
$$J = \int_0^\infty x^T Q x + u^T R u \, dt$$

Input rate saturation (2/4)

$$\dot{x} = f(x) + g(x)\,\mathrm{RL}(u)$$

$$J = \int_0^\infty x^T Q x + u^T R u \, dt$$

$$\dot{u}_s = \mathrm{sat}(V(u - u_s))$$



Approximation of rate limiter

Input rate saturation (3/4)

Augmented system

$$\begin{pmatrix} \dot{x} \\ \dot{u}_s \end{pmatrix} = \begin{pmatrix} f(x) + g(x)u_s \\ -Vu_s \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} \overline{\mathrm{sat}}(Vu_s, Vu), \quad \overline{\mathrm{sat}}(\xi, \eta) := \mathrm{sat}(\eta - \xi) + \xi$$

The dynamic programming is applied to get a HJ equation.

Input rate saturation (3/4)

Augmented system

$$\begin{pmatrix} \dot{x} \\ \dot{u}_s \end{pmatrix} = \begin{pmatrix} f(x) + g(x)u_s \\ -Vu_s \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} \overline{\mathrm{sat}}(Vu_s, Vu), \quad \overline{\mathrm{sat}}(\xi, \eta) := \mathrm{sat}(\eta - \xi) + \xi$$

The dynamic programming is applied to get a HJ equation.

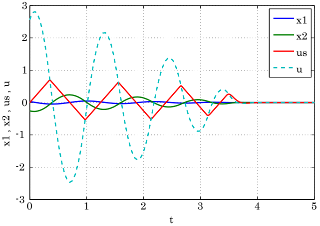
A numerical example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{RL}(u)$$

$$J = \int_0^\infty x^T Q x + u^T R u \, dt, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 0.1$$

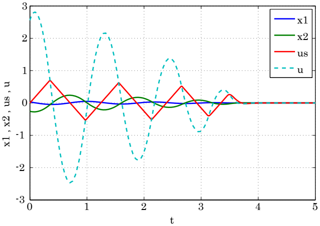
2-dimensinal system $\longrightarrow$ 3-dimensional augmented system

Input rate saturation (4/4) —Simulations

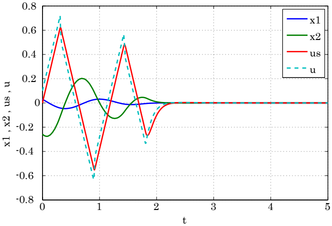


Linear, 2-dimensional

Input rate saturation (4/4) —Simulations

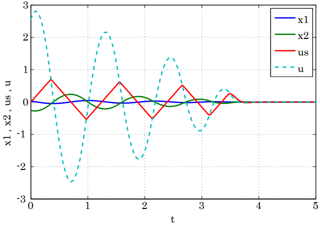


Linear, 2-dimensional

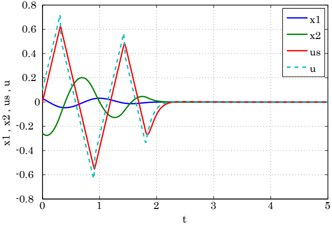


Linear, 3-dimensional

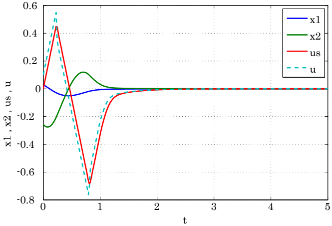
Input rate saturation (4/4) —Simulations



Linear, 2-dimensional

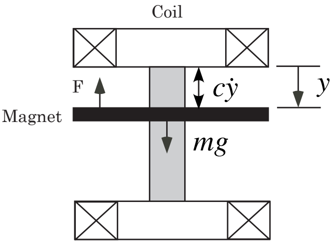


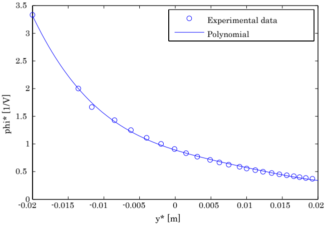
Linear, 3-dimensional



Nonlinear, 3-dimensional

Magnetic levitation system (1/6)





The equation of motion of the magnet is given by

$$m\ddot{y} + c\dot{y} = -mg\phi(y)u + mg$$

Magnetic levitation system (2/6)

$$\dot{x} = F(x) + G(x)u$$

$$F(x) = \begin{pmatrix} x_2 \\ -g\bar{u}(\phi(x_1) - q_5) - cx_2/m \end{pmatrix}, \quad G(x) = \begin{pmatrix} 0 \\ -g\phi(x_1) \end{pmatrix}$$

$$x = [x_1, x_2]^T = [y, \dot{y}]^T$$

Magnetic levitation system (2/6)

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$$x = [x_1, x_2]^T = [y, \dot{y}]^T$$

Output regulation problem: the magnet plate tracks a sinusoidal signal generated by

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} =: s(w) \quad \text{(exosystem)}$$

Magnetic levitation system (3/6)

Regulator equation:

$$\frac{\partial \pi}{\partial w} s(w) = F(\pi(w)) + G(\pi(w))c(w)$$

$$-\pi_1(w) + w_2 = 0$$

Magnetic levitation system (3/6)

Regulator equation:

$$\frac{\partial \pi}{\partial w} s(w) = F(\pi(w)) + G(\pi(w))c(w)$$
$$-\pi_1(w) + w_2 = 0$$

The solution:

$$\pi(w) = \begin{pmatrix} w_2 \\ \omega w_1 \end{pmatrix},$$
$$c(w) = \frac{-(gu(\phi(w_2) - q_5) + c\omega w_1/m - \omega^2 w_2)}{g\phi(w_2)}.$$

Magnetic levitation system (4/6)

Optimal stabilizing law:

$$J = \int_0^\infty x^T Q x + R u^2 \, dt,$$

$$Q = \begin{pmatrix} 10^5 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1$$

Magnetic levitation system (4/6)

Optimal stabilizing law:

$$J = \int_0^\infty x^T Q x + R u^2 \, dt,$$

$$Q = \begin{pmatrix} 10^5 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1$$

The HJ equation for this problem:

$$\left(\frac{\partial V}{\partial x}\right)\left\{F(x) - \frac{1}{4}G(x)G(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right\} + x^T Q x = 0$$

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Applying 5 iterations of the stable manifold algorithm, a stabilizing control is

$$k(x) = -\frac{1}{2}G(x)^T\left(\frac{\partial V}{\partial x}\right)^T$$

Magnetic levitation system (4/6)

Optimal stabilizing law:

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The HJ equation for this problem:

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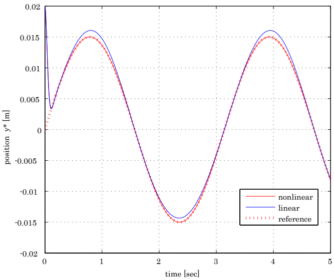
Applying 5 iterations of the stable manifold algorithm, a stabilizing control is

$$k(x) = -\frac{1}{2} G(x)^T \left(\frac{\partial V}{\partial x}\right)^T$$

The overall output regulation control is

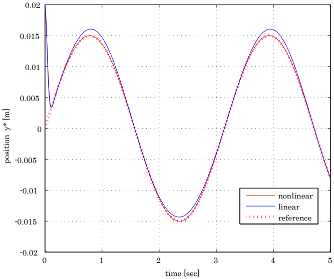
$$u = c(w) + k(x - \pi(w))$$

Magnetic levitation system (5/6)

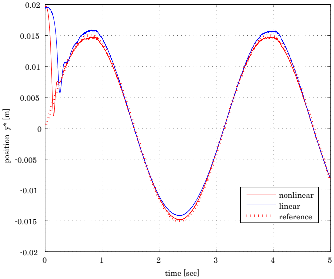


Simulation

Magnetic levitation system (5/6)



Simulation



Experiment

The performance of the nonlinear controller is closer to the simulation

Robustness test

Robustness test

Summary and future works

- Hamilton-Jacobi equation in nonlinear control theory (optimal, H^∞,...)

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- Stable manifold

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Matlab programme is available upon request:
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