Control Systems Engineering

Aerospace Engineering

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The stable manifold approach for optimal swing up and stabilization of an inverted pendulum with input saturation

Ryu Fujimoto and Noboru Sakamoto

Nagoya University

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Outline

- **Background and problem formulation**
- **Nonlinear optimal control**
  - ✓ **Approximation methods for HJ equations**
  - ✓ **Stable manifold approach**
- **Simulation / Experiment results**
  - ✓ **Non-uniqueness of solution in HJ equations**
- **Concluding remarks**

Background/problem statement

Inverted pendulum

- ✓ Strong nonlinearity
- ✓ Under actuated system
- ✓ Benchmark problem of nonlinear controller design

Swing up and stabilization of inverted pendulum

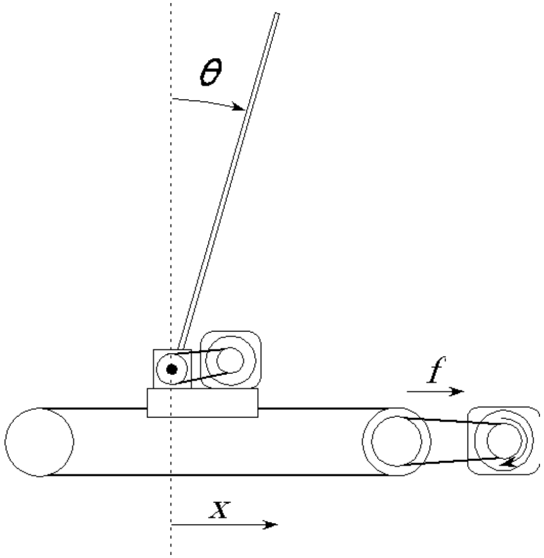
- ✓ Switching of swing up and stabilization controllers (Astrom & Furua ’00)
- ✓ Topological issues / (Angeli ‘01, Astrom et.al ‘08)

This talk:

- ✓ Single optimal state feedback control
- ✓ Enhancement of valid approximation range for HJ eq.

Inverted pendulum system

$$\dot{x} = f(x) + g(x)\cdot sat(u) \qquad x = [x_1, x_2, x_3, x_4]^T = [\theta, \dot{\theta}, x, \dot{x}]^T$$



$$f(x) = \begin{bmatrix} x_2 \\ \dfrac{-ml\cos x_1 C_1 x_4 - m^2 l^2 \sin x_1 \cos x_1 \cdot {x_2}^2 + (m+M)mgl\sin x_1}{J(M+m) + Mml^2 + m^2 l^2 \sin^2 x_1} \\ x_4 \\ \dfrac{(J+ml^2)C_1 x_4 + (J+ml^2)ml\sin x_1 \cdot {x_2}^2 - m^2 l^2 g \sin x_1 \cos x_1}{J(M+m) + Mml^2 + m^2 l^2 \sin^2 x_1} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ \dfrac{-ml\cos x_1 C_2}{J(M+m) + Mml^2 + m^2 l^2 \sin^2 x_1} \\ 0 \\ \dfrac{(J+ml^2)C_2}{J(M+m) + Mml^2 + m^2 l^2 \sin^2 x_1} \end{bmatrix}$$

Input limitation:

$$u_{\max} = 18[V]$$
$$u_{\min} = -18[V]$$

Hamilton-Jacobi:

$$\left(\frac{\partial V}{\partial x}\right)\left\{ f(x) + g(x)\cdot \mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right)\right\} + x^T Q x$$
$$+\,\mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right)^T \cdot R \cdot \mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right) = 0$$

Approximate solutions for HJ eq

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- Taylor approximation method:

  (×) Cannot handle non-analytic nonlinearities

  (×) Computationally inefficient

  (×) Small domain of convergence

- Stable manifold approach ： (N.Sakamoto and A.J.van der Schaft, IEEE TAC, 2008)

  (○) $\left(\frac{\partial V}{\partial x}\right)$ is directly computed

  (○) iterative method suitable for computer implementation

  (○) Larger domain of convergence etc…

Stable manifold method for HJ eq.

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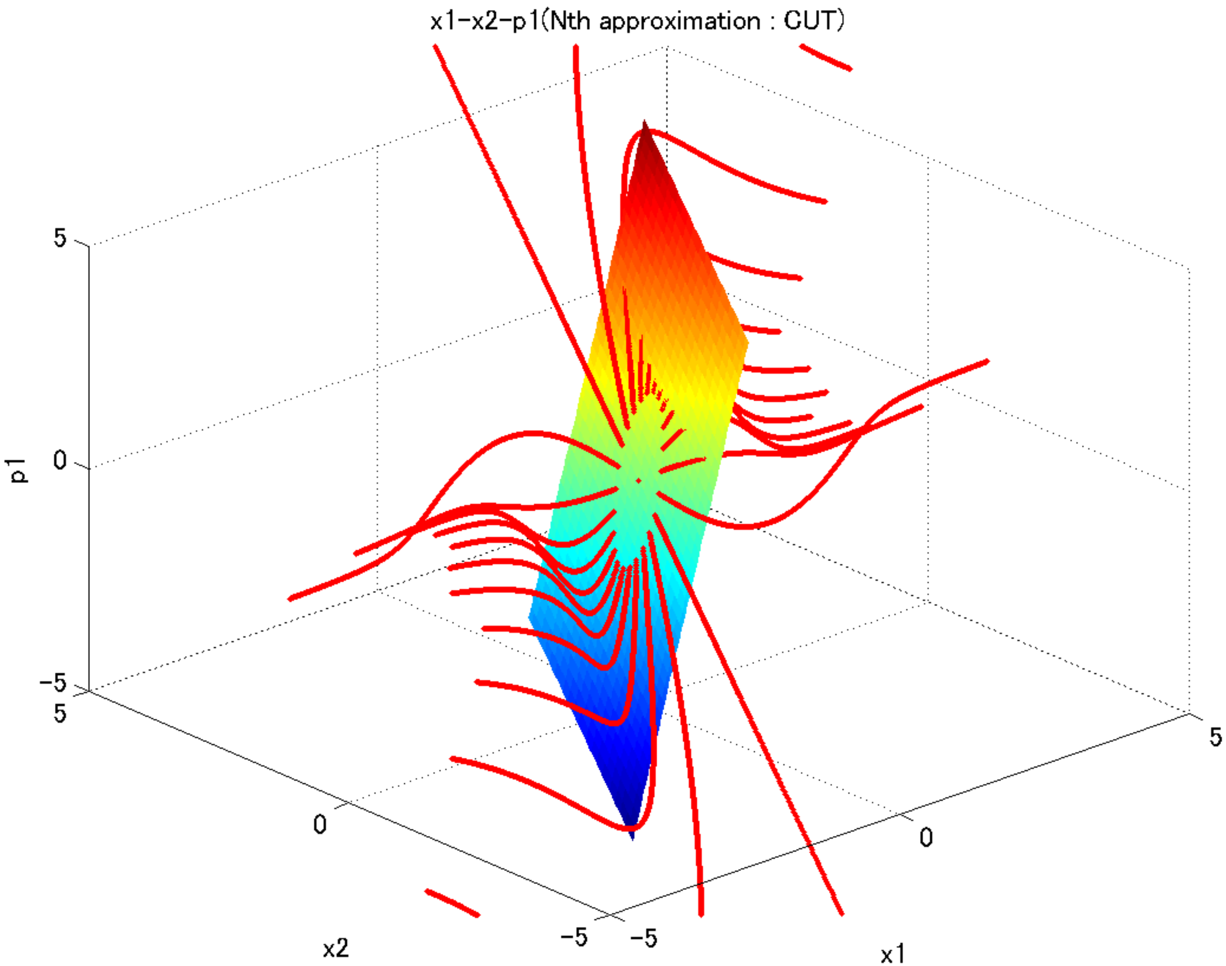

Stable manifold method for HJ eq.

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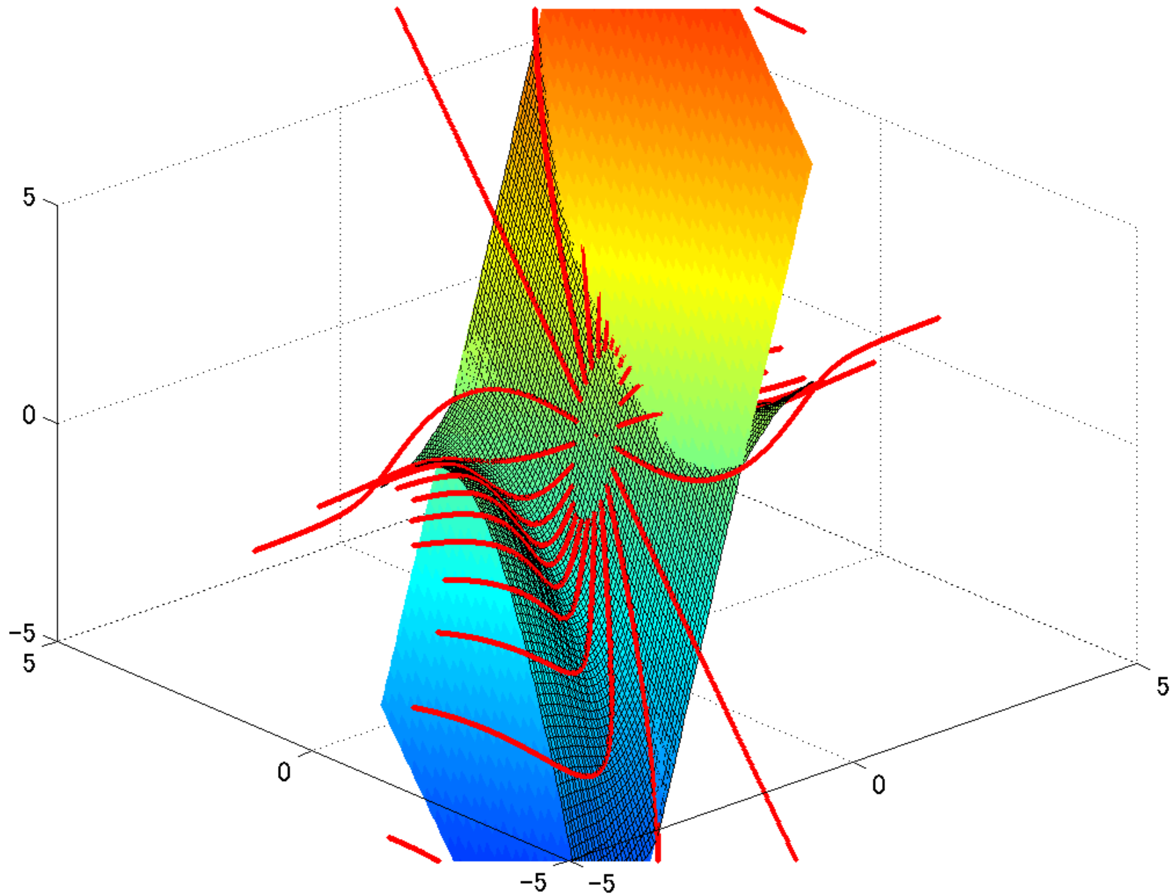

Computational result

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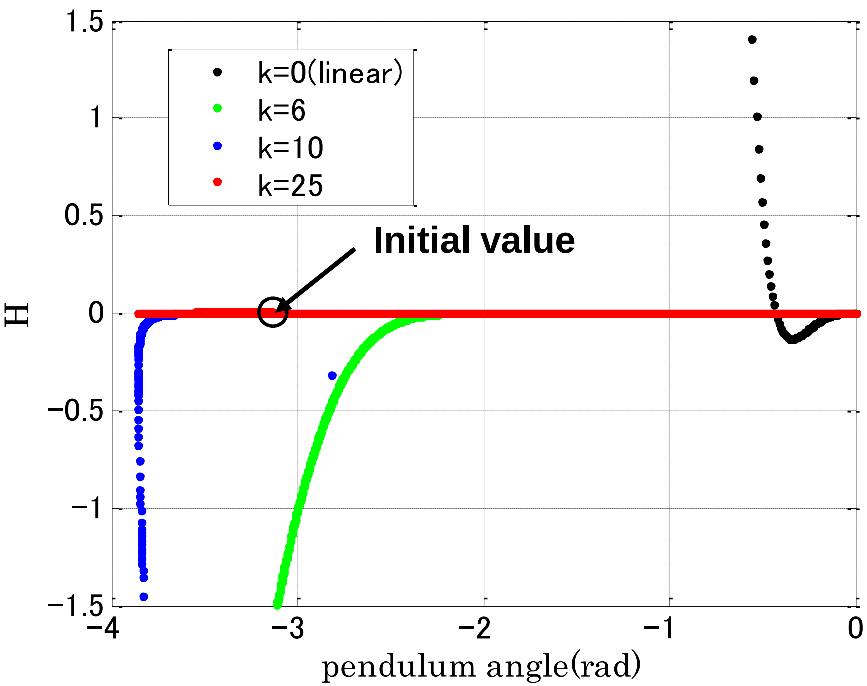
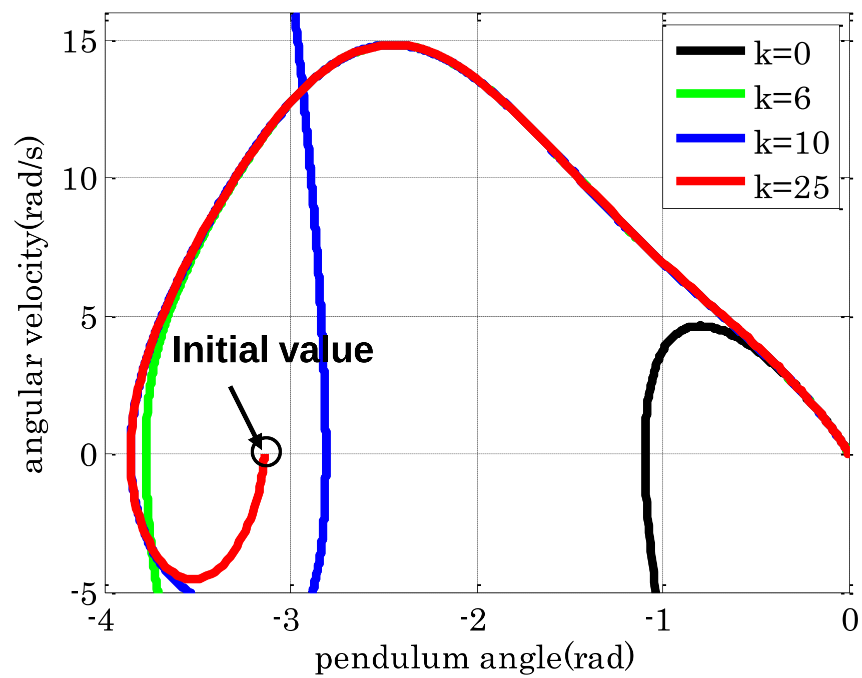
Closed loop trajectories for different iteration number

$$H = p(x)^T \left\{ f(x) + g(x) \cdot \mathrm{sat}\left(-\frac{1}{2} R^{-1} g(x)^T p(x) \right) \right\} + x^T Q x + \mathrm{sat}\left(-\frac{1}{2} R^{-1} g(x)^T p(x) \right)^T \cdot R \cdot \mathrm{sat}\left(-\frac{1}{2} R^{-1} g(x)^T p(x) \right) = 0$$

(HJ eq.)

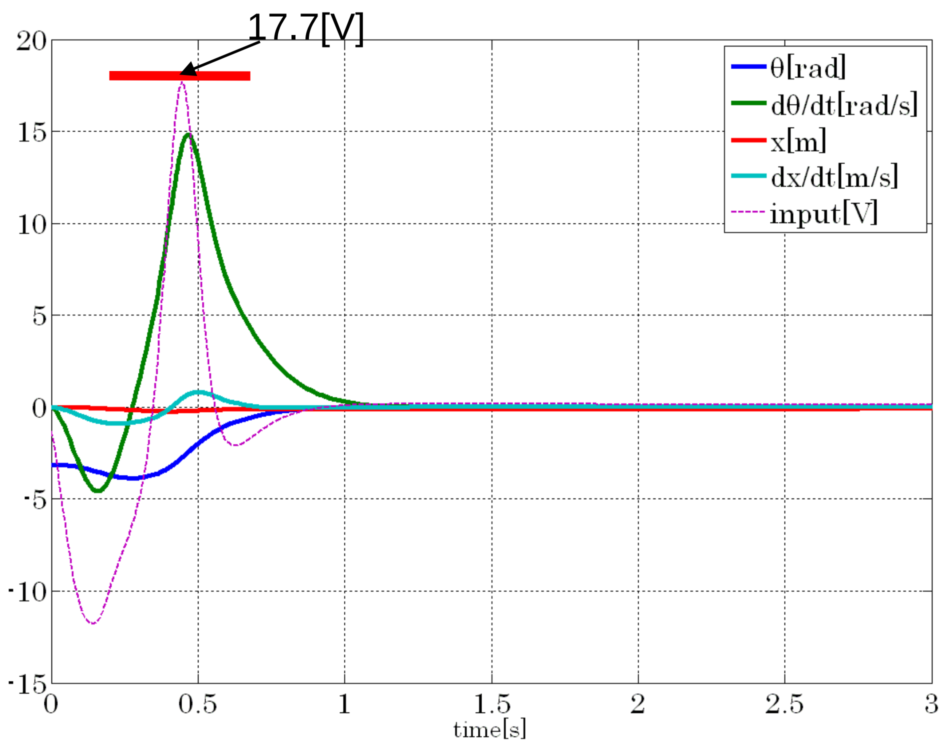


Hamiltonian



Trajectories in the (θ,dθ/dt)-space

Simulation/Experiment results



Responses (simulation)

- ✓ Input voltage is under the limitation 18[V]
- ✓ Swing up with 2 swings → Movie1

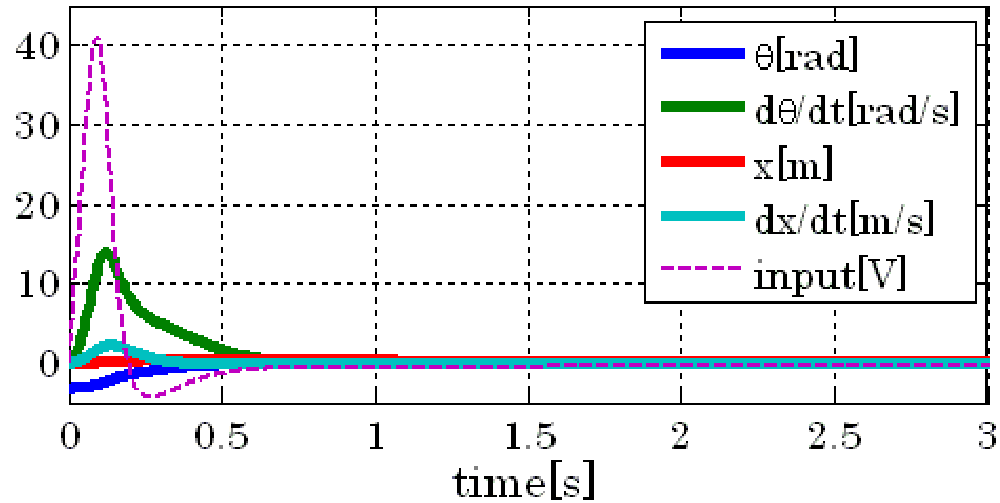
- ✓ Robustness for parameter variations 20% → Movie2

Effect of saturation function

What do saturation functions do?

Simulation by HJ eq without saturation function

- Swing up control with 1 swing
- Input voltage is far beyond 18[V]



Responses (simulation)

Hamilton-Jacobi eq with saturations

$$\left(\frac{\partial V}{\partial x}\right)\left\{f(x)+g(x)\cdot \mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right)\right\}+x^TQx$$
$$+\mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right)^T\cdot R\cdot \mathrm{sat}\left(-\frac{1}{2}R^{-1}g(x)^T\left(\frac{\partial V}{\partial x}\right)^T\right)=0$$

The solution solves the original HJ eq as well??

Saturation is an identity function inside of limitation

Uniqueness of solution

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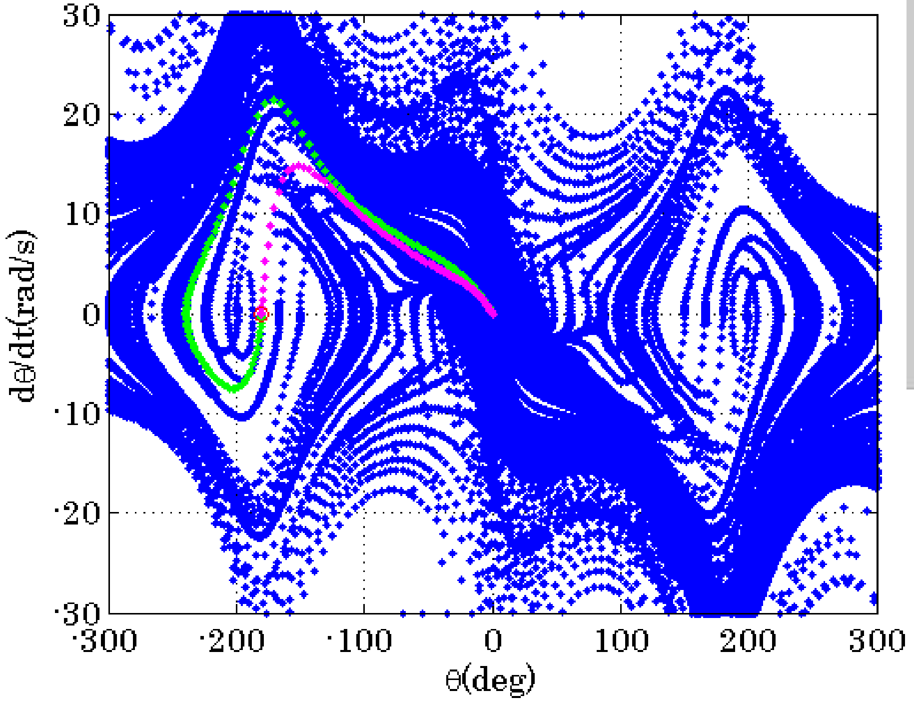
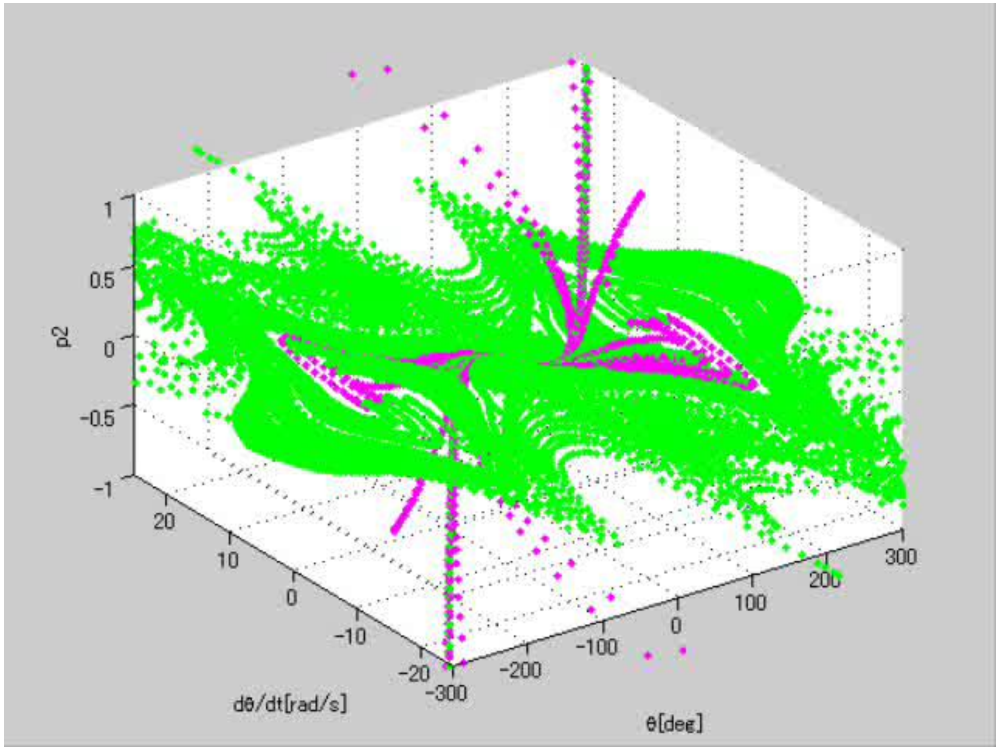
Analysis for a 2-dimensional model

Computed stable manifold without saturations



Closed loop trajectories



Two layers of stable manifolds

Uniqueness of solution

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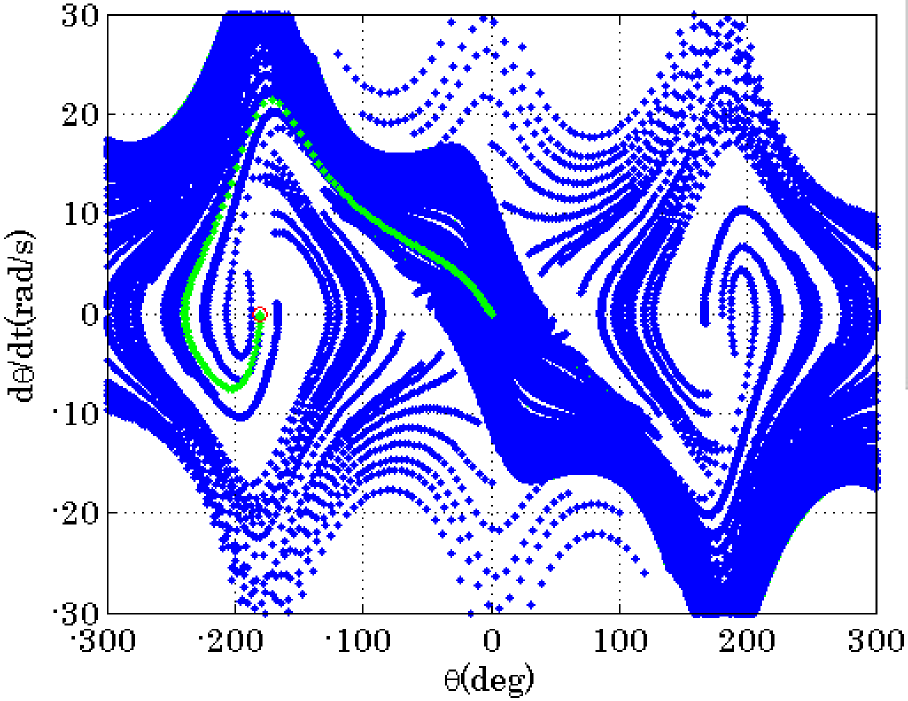
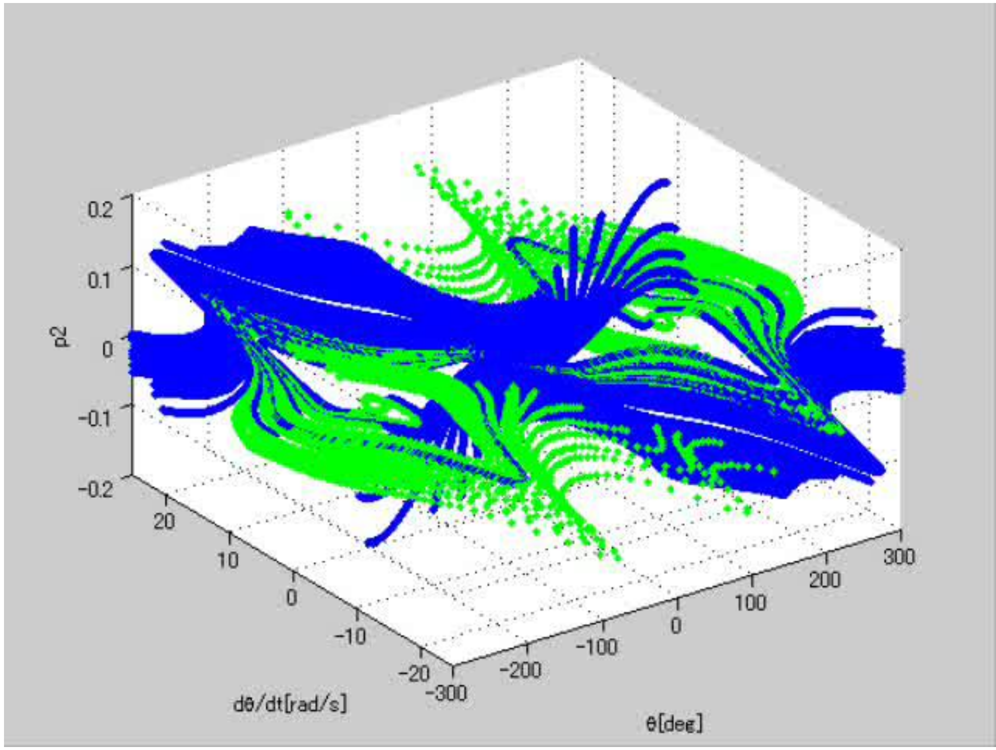
- Analysis for a 2-dimensional model

Computed stable manifold with saturations



Closed loop trajectories



One stable manifold is filtered out

With smaller input constraint･･･

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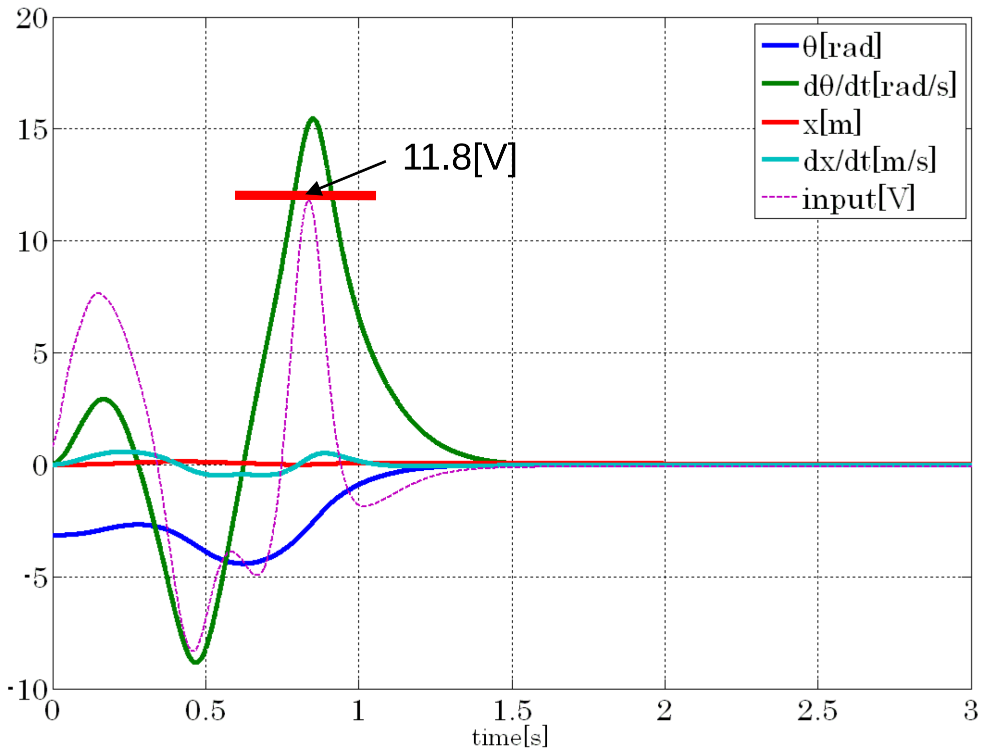
$$u_{\max} = 18[V] \quad u_{\min} = -18[V] \quad \Rightarrow \quad u_{\max} = 12[V] \quad u_{\min} = -12[V]$$

Apply 40 iterations of the stable manifold algorithm

↓

Feedback function is expressed with polynomials



Responses (simulation)

- ✓ Swing up with 3 swings
- ✓ Efficient strategy with low voltage
- ✓ Third stable manifold → infinite layers

Movie3

Concluding remarks

- Optimal swing up and stabilization of inverted pendulum
- Single feedback by solving a Hamilton-Jacobi equation
- Large domain of validity to include the pending position
- Explicitly include saturation functions in the HJ equation
- 1 swing, 2 swing and 3 swing controllers by changing the value of input limitation
- An example of non-unique solutions to HJ equation

Thank you for your attention

Stable manifold algorithm

N.Sakamoto and A.J.van der Schaft, 2008

A Hamiltoinan system

$$\begin{cases} \dot{x}' = Fx' + \phi(x', p') \\ \dot{p}' = -F^T p' + \varphi(x', p') \end{cases} \quad \cdots (※)$$

Stable *F*, smooth nonlinearities $\phi(x', p'), \varphi(x', p')$

$$x_0'(t,\xi) = e^{At}\xi \qquad p_0'(t,\xi) = 0$$

$$\begin{cases} x_{k+1}'(t,\xi) = e^{Ft}\xi + \int_0^t e^{F(t-s)}\phi(x_k'(s), p_k'(s))ds & (k = 0,1,2,\cdots) \\ p_{k+1}'(t,\xi) = -\int_t^\infty e^{-F^T(t-s)}\varphi(x_k'(s), p_k'(s))ds & (k = 0,1,2,\cdots) \end{cases}$$

$x_k'(t,\xi), p_k'(t,\xi)$ converge to a solution on $[0,\infty), k \to \infty$

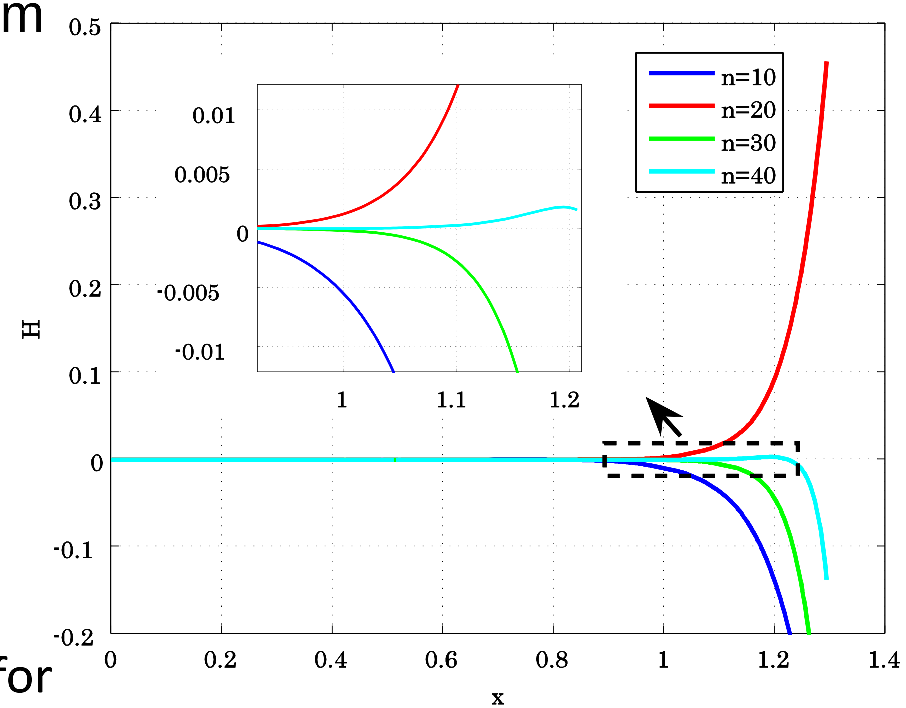
Limitation of the Taylor method

1-dimensional optimal control problem

$$\dot{x} = x - x^3 + u, \qquad J = \int_0^\infty x^2 + u^2 dt$$

$$H = p(x - x^3) - \frac{1}{2}p^2 + \frac{1}{2}x^2 = 0$$

$$p = x - x^3 + x\sqrt{(x^2-1)^2+1}$$



Series solutions cannot approximate for

$x > 1.19$

$(z^2-1)^2+1$ has complex zeros at $|z| = 1.19$

Inverted pendulum setup