Iterative methods to compute center and center-stable manifolds with application to the optimal output regulation problem

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December 14, 2011
CDC 2011

Outline of the talk

- To present a new method for computing center and center-stable manifolds
  - Iterative method
  - Suitable for computer implementation

- This method will be applied to the optimal output regulation
  - A constructive design method
  - No need to solve *regulator equation*
  - Center algorithm solves the regulator equation
  - Center-stable algorithm computes optimal controller

Setting of the problem

Consider the following system of ordinary differential equations:

$$\begin{aligned} \dot{x} &= Ax + X(x, y, z) \\ \dot{y} &= By + Y(x, y, z) \\ \dot{z} &= Cz + Z(x, y, z) \end{aligned}$$

- $A \in R^{n_x \times n_x}$, $\mathrm{Re}\lambda(A) < 0$
- $B \in R^{n_y \times n_y}$, $\mathrm{Re}\lambda(B) = 0$
- $C \in R^{n_z \times n_z}$, $\mathrm{Re}\lambda(C) > 0$
- The functions X, Y, Z are continuously differentiable
- X, Y, Z together with all of their first derivatives vanish at the origin.

Integral equations (Kelly 1966)

The center manifold integral equation:

$$x(t) = \int_{-\infty}^{t} e^{A(t-s)} X(x(s), y(s), z(s)) ds$$

$$y(t) = e^{Bt} y_0 + \int_{0}^{t} e^{B(t-s)} Y(x(s), y(s), z(s)) ds$$

$$z(t) = -\int_{t}^{\infty} e^{C(t-s)} Z(x(s), y(s), z(s)) ds$$

These functions move on $x = \varphi_1(y)$, $z = \varphi_2(y)$
The center-stable manifold integral equation:

$$x(t) = e^{At} x_0 + \int_{0}^{t} e^{A(t-s)} X(x(s), y(s), z(s)) ds$$

$$y(t) = e^{Bt} y_0 + \int_{0}^{t} e^{B(t-s)} Y(x(s), y(s), z(s)) ds$$

$$z(t) = -\int_{t}^{\infty} e^{C(t-s)} Z(x(s), y(s), z(s)) ds$$

These functions move on $z = \psi(x, y)$

Iterative algorithms

- In the center-stable manifold case:

$$x_1(t) = e^{At}x_0, y_1(t) = e^{Bt}y_0, z_1(t) = 0$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{pmatrix}(t, x_0, y_0) = \begin{pmatrix} e^{At}x_0 + \int_0^t e^{A(t-s)}X(x_k(s), y_k(s), z_k(s))\, ds \\ e^{Bt}y_0 + \int_0^t e^{B(t-s)}Y(x_k(s), y_k(s), z_k(s))\, ds \\ -\int_t^\infty e^{C(t-s)}Z(x_k(s), y_k(s), z_k(s))\, ds \end{pmatrix}$$

- In the center manifold case

$$x_1(t) = 0, y_1(t) = e^{Bt}y_0, z_1(t) = 0$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{pmatrix}(t, y_0) = \begin{pmatrix} \int_{-\infty}^t e^{A(t-s)}X(x_k(s), y_k(s), z_k(s))\, ds \\ e^{Bt}y_0 + \int_0^t e^{B(t-s)}Y(x_k(s), y_k(s), z_k(s))\, ds \\ -\int_t^\infty e^{C(t-s)}Z(x_k(s), y_k(s), z_k(s))\, ds \end{pmatrix}$$

Existence of limit

Theorem

i) There exists $\delta_c > 0$ such that for all $y_0, \ |y_0| < \delta_c$ the sequence converges pointwise to the solutions on $x = \varphi_1(y)$, $z = \varphi_2(y)$ (convergence to the center manifold).

ii) There exists $\delta_{cs} > 0$ such that for all $(x_0, y_0), \ |(x_0, y_0)| < \delta_{cs}$ the sequence converges pointwise to the solutions on $z = \psi(x, y)$ (convergence to the center-stable manifold).

Optimal output regulation - equations

System: $\dot{x} = f(x) + g(x)u, \quad x(t) \in \mathbb{R}^n,\ f(0) = 0$

Exosystem: $\dot{w} = s(w), \quad w(t) \in \mathbb{R}^p,\ s(0) = 0$

Error (output) equation: $e = h(x, w)$

Denote

$$A = \frac{\partial f}{\partial x}(0), \quad B = g(0), \quad C = \frac{\partial h}{\partial x}(0,0),$$
$$S = \frac{\partial s}{\partial w}(0), \quad Q = \frac{\partial h}{\partial w}(0,0).$$

Regulator equation:

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))\sigma(w), \quad h(\pi(w), w) = 0$$

Optimal output regulation - assumptions

- The exosystem is Lyapunov stable at $w = 0$, Poisson stable around $w = 0$ and all eigenvalues of S are purely imaginary.
- $(A,\ B)$ is stabilizable, (C, A) is detectable
- The number of inputs $\geqslant$ the number of outputs (square)
- The system has well-defined relative degree (rel.deg. =1, $\det L_g h(0,0) \neq 0$)

Optimal output regulation

$$\text{Cost function:} \quad J = \frac{1}{2}\int_0^\infty |e|^2 + |\dot{e}|^2 \, dt$$

The Hamiltonian H_D:

$$H_D = p_x^T(f + gu) + p_w^T s(w) + \frac{1}{2}\left|h(x,w)\right|^2 + \frac{1}{2}\left|L_f h(x,w) + (L_g h(x,w))u + L_s h(x,w)\right|^2,$$

The control vector $\bar{u}$ minimizing H_D:

$$\bar{u} = -(L_g h)^{-1}\left\{(L_g h)^{-T} g(x)^T p_x + L_f h + L_s h\right\}.$$

The *Hamilton-Jacobi equation* is then

$$p_x^T\left\{f - g(L_g h)^{-1}(L_f h + L_s h)\right\} + p_w^T s(w) - \frac{1}{2}p_x^T g(L_g h)^{-1}(L_g h)^{-T} g^T p_x + \frac{1}{2}|h(x,w)|^2 = 0$$

Associated Hamiltonian System

The Hamiltonian system is

$$\begin{aligned}
\dot{x} =& (A - B(CB)^{-1}CA)x - B(CB)^{-1}QSw \\
& - B(B^TC^TCB)^{-1}B^Tp_x + N_1(x, w, p_x) \\
\dot{w} =& Sw + N_2(w) \\
\dot{p}_x =& C^TCx - C^TQw \\
& - (A - B(CB)^{-1}CA)^Tp_x + N_3(x, w, p_x) \\
\dot{p}_w =& - Q^TCx - Q^TQw + S^TQ^T(B^TC^T)^{-1}B^Tp_x \\
& - S^Tp_w + N_4(x, w, p_x, p_w).
\end{aligned}$$

Define the Hamiltonian matrix H as

$$\frac{d}{dt}[x, w, p_x, p_w]^T = H[x, w, p_x, p_w]^T + [N_1, N_2, N_3, N_4]^T$$

Block diagonalization

The linear regulator equation

$$\Pi S = A\Pi + B\Sigma, \quad C\Pi + Q = 0$$

A Riccati equation

$$P\bar{A} + \bar{A}^T P - PR_B P + C^T C = 0;$$
$$\bar{A} = A - B(CB)^{-1}CA, \quad R_B = B(B^T C^T CB)^{-1}B^T$$

Lyapunov equation

$$VA_c + A_c^T V = B(B^T C^T CB)^{-1}B^T$$
$$\text{where} \quad A_c = A - B(CB)^{-1}CA - B(B^T C^T CB)^{-1}B^T P$$

Linear symplectic coordinate transformations

$$T_1 = \begin{pmatrix} I & \Pi & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & -\Pi^T & I \end{pmatrix}, \quad T_2 = \begin{pmatrix} I & 0 & V & 0 \\ 0 & I & 0 & 0 \\ P & 0 & PV + I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

New Hamiltonian system

$$[x''^T, w''^T, p_x''^T, p_w''^T]^T = T_2^{-1} T_1^{-1} [x^T, w^T, p_x{}^T, p_w{}^T]^T$$

Hamiltonian system with block-diagonalized linear part:

$$\begin{cases} \dot{x}'' = A_c x'' + \bar{N}_1(x'', w'', p_x'') & \textit{stable} \\ \dot{w}'' = S w'' + \bar{N}_2(w'') & \textit{center1} \\ \dot{p}_x'' = -A_c^T p_x'' + \bar{N}_3(x'', w'', p_x'') & \textit{unstable} \\ \dot{p}_w'' = -S^T p_w'' + \bar{N}_4(x'', w'', p_x'', p_w'') & \textit{center2} \end{cases}$$

✓ Center manifolds: $x'' = \bar{\pi}_1(w')$, $p_x'' = \bar{\pi}_2(w')$
CM algorithm computes $\bar{\pi}_1, \bar{\pi}_2 \Rightarrow$ Regulator equation

✓ Center-stable manifold $p_x'' = \pi_3(x'', w'')$
C-SM algorithm computes $\pi_3 \Rightarrow$ Feedback controller $u = u(w, x)$

A numerical example

Consider the example with unstable linearization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ 2 & 1.6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1^3 \\ 2x_1^3 + x_2^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

exosystem: $\dot{w} = 0$

The goal is to design a feedback law $u = u(x, w)$ that achieves $x_1 = w$ as $t \to \infty$ in an optimal way:

$$J = \frac{1}{2} \int_0^{+\infty} (x_1 - w)^2 + (\dot{x}_1)^2 dt$$

A numerical example

Hamiltonian system:

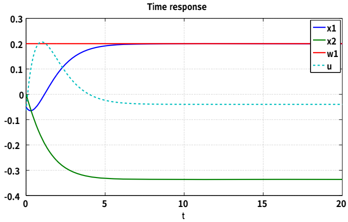
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{w} \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_w \end{pmatrix} = H \begin{pmatrix} x_1 \\ x_2 \\ w \\ p_1 \\ p_2 \\ p_w \end{pmatrix} + \begin{pmatrix} x_1^3 \\ 2x_1^3 + x_2^3 \\ 0 \\ -6x_1^2 p_2 \\ -3x_2^2 p_2 \\ 0 \end{pmatrix}; \quad H = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ 2 & 1.1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1.1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Block-diagonalization $\rightarrow$ center-stable manifold algorithm (Matlab)

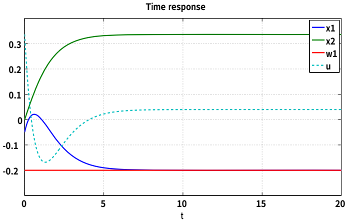
$\rightarrow$ back to the original coordinates

$\rightarrow$ feedback controller $u(x, w)$

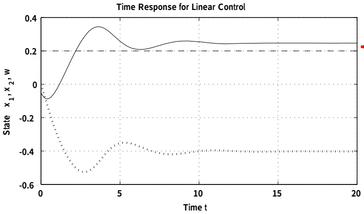
Simulation results



reference $(w(0)) = 0.2$



reference $(w(0)) = -0.2$



steady state error

Linear controller response

Conclusions

- New methods for computing center and center-stable manifolds
  - ✓ Iterative algorithms, Matlab codes
- Applications to the optimal output regulation problem
  - ✓ Center-stable algorithm directly computes feedback law
  - ✓ One does not need to solve the regulator equation

Controller expression

$$u = -(L_g h)^{-1} \left\{ (L_g h)^{-T} g(x)^T p_x + L_f h + L_s h \right\}$$

$$\begin{aligned} &p_{x1}(x, w) \\ =&1.2289x_1 - 7.2888e^{-5}x_1^2 + 0.35045x_1^3 + 0.48075x_2 - 6.5193e^{-5}x_1x_2 \\ &+ 0.32236x_1^2x_2 - 1.6075e^{-5}x_2^2 - 0.096271x_1x_2^2 + 0.37641x_2^3 - 0.35484w \\ &- 0.0002183x_1w + 0.59074x_1^2w - 0.00012334x_2w - 0.37257x_1x_2w \\ &+ 0.030812x_2^2w + 2.8843e^{-5}w^2 - 0.098178x_1w^2 + 0.25377x_2w^2 - 0.16055w^3 \end{aligned}$$

$$\begin{aligned} &p_{x2}(x, w) \\ =&0.48075x_1 + 0.00032451x_1^2 + 0.15632x_1^3 + 1.0096x_2 + 0.00051919x_1x_2 \\ &+ 0.062583x_1^2x_2 + 0.0001652x_2^2 + 0.93529x_1x_2^2 + 1.0773x_2^3 + 1.3548w \\ &+ 0.00068903x_1w + 0.0041476x_1^2w + 0.00030632x_2w + 0.079235x_1x_2w \\ &- 0.52794x_2^2w - 0.00011342w^2 + 0.4681x_1w^2 + 0.032968x_2w^2 + 0.17831w^3 \end{aligned}$$