The Maximum Asymptotic Bias of Robust Regression Estimates over (c, γ)- Contamination Neighborhoods

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Abstract. When the observations may be contaminated in the linear model with the intercept, a certain large class of robust regression estimates including S-estimates, τ-estimates and CM-estimates is considered. The (c, γ)-contamination neighborhood, which is a generalization of the neighborhoods defined in terms of ε-contamination and total variation, is used for describing the contamination of the observations. Lower and upper bounds for the maximum asymptotic bias of the regression estimates over (c, γ)-contamination neighborhoods are derived without imposing elliptical regressors. As important special cases, the lower and upper bounds for S-estimates, τ-estimates and CM-estimates under the Gaussian model are obtained.

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1. Introduction

In the case where the observations may be contaminated in the location model, Huber (1964) introduced the maximum asymptotic bias $B_T(\varepsilon)$ of a location estimate T over the ε-contamination neighborhood. The $B_T(\varepsilon)$ is one of the most informative global quantitative measures to assess robustness of T, because $B_T(\varepsilon)$ shows the whole performance of T from $\varepsilon = 0$ (the central model distribution) to the breakdown point and its derivative $B'_T(0)$ equals the gross error sensitivity under some regularity conditions. Huber (1964) established that the median minimizes $B_T(\varepsilon)$ among translation equivariant location estimates. Martin and Zamer (1989, 1993) obtained minimax bias robust scale estimates.

As for the linear regression model, in the case of the zero-intercept and elliptical regressors, Martin, Yohai and Zamer (1989) obtained the minimax bias estimates in the respective classes of M-estimates with general scale and GM- estimates of regression. In particular, they showed that the least median of square estimate (LMS) introduced by Rousseeuw (1984) is nearly minimax. Yohai and Zamer (1993) extended this result to the larger class of residual admissible estimates. Berrendero and Zamer (2001) obtained maximum asymptotic bias of robust regression estimates in a broad class, which includes S-estimates and, τ-estimates, without requiring zero-intercept and/or elliptical regressors. Berrendero, Mendes and Tyler (2007) derived the

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maximum asymptotic bias of MM-estimates and constrained M-estimates (CM-estimates) of regression, and compared their estimates with S-estimates and τ -estimates in detail. All the authors mentioned above adopt the ε-contamination neighborhood to describe deviation from the central model.

On the other hand, in order to describe deviation from the central model Ando and Kimura (2003) introduced the (c, γ) - contamination neighborhood (the (c, γ) - neighborhood, for short) , which is a generalization of Rieder's (ε, δ) -neighborhood and includes the neighborhoods defined in terms of ε - contamination and total variation. They gave a characterization of the (c, γ) - neighborhood and applied it to bias-robustness study of estimates. Among their achievements, there are the extensions of Huber's (1964) and He and Simpson's (1993) results. The former states that the median minimizes the maximum asymptotic bias $B_T(c, \gamma)$ over (c, γ) - neighborhoods among translation equivariant location estimates. Ando and Kimura (2004) derived the lower and upper bounds for $B_S(c, \gamma)$ of regression S-estimates over (c, γ) - neighborhoods in the zero-intercept linear model with elliptical regressors, and showed that in the case of Rieder's (ε, δ) - neighborhood the lower and upper bounds coincide and is equal to $B_S(c, \gamma)$. Ando, Kakiuchi and Kimura (2009) gave the applications of the (c, γ) - neighborhoods to nonparametric confidence intervals and tests for the median.

In this paper, following Berrendero and Zamer (2001), without imposing the zero-intercept and/or elliptical regressors , we derive the lower and upper bounds for $B_T(c, \gamma)$ of estimates T in the large class. In the case of ε-contamination neighborhoods, the lower and upper bounds coincide and the results are reduced to Theorems 1 and 2 of Berrendero and Zamar (2001). As important special cases, we obtain the lower and upper bounds for the maximum asymptotic bias $B_S(c, \gamma)$, $B_\tau(c, \gamma)$ and $B_{CM}(c, \gamma)$ of S-estimates, τ-estimates and CM-estimates under the Gaussian model. We give some tables of the lower and upper bounds for $B_\tau(c, \gamma)$ of τ-estimates and $B_{CM}(c, \gamma)$ of CM-estimates based on Tukey's biweight function and also show two selective graphs to visualize the difference between the two bounds. We should emphasize that the characterization (Proposition 2.1) of the (c, γ)-neighborhoods is indispensable to the derivation of our results in the paper. The proofs of the results are collected in section 5.

2. Preliminaries

We consider the linear regression model

$$y \quad = \quad \alpha_0 + \boldsymbol{\theta}_0' \boldsymbol{x} + u, \tag{2.1}$$

where $\boldsymbol{x} = (x_1, \ldots, x_p)'$ is a random vector in R^p, $\boldsymbol{\theta}_0 = (\theta_{10}, \ldots, \theta_{p0})'$ is the vector in R^p of the true regression parameters, α_0 is the true intercept parameter in R and the error u is a random variable independent of $\boldsymbol{x}$. Let F_0 be the nominal distribution function of u and G_0 the nominal distribution function of $\boldsymbol{x}$. Then the nominal distribution function H_0 of $(y, \boldsymbol{x})$ is

$$H_0(y, \boldsymbol{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} F_0(y - \alpha_0 - \boldsymbol{\theta}_0' \boldsymbol{s}) dG_0(\boldsymbol{s}). \tag{2.2}$$

Let $\mathcal{M}$ be the set of all distribution functions H on $(R^{p+1}, \mathcal{B}^{p+1})$, where $\mathcal{B}^{p+1}$ is the Borel σ-field on R^{p+1}. As in Berrendero and Zamar (2001), we focus on the estimation of the slope parameter $\boldsymbol{\theta}_0$. Let $\boldsymbol{T}$ be a R^p-valued functional defined on $\mathcal{M}$. Given a sample of independent observations $(y_1, \boldsymbol{x}_1), \cdots, (y_n, \boldsymbol{x}_n)$ of size n from H, we define the corresponding estimate of $\boldsymbol{\theta}_0$ as $\boldsymbol{T}(H_n)$, where H_n is the empirical distribution of the sample. We assume that $\boldsymbol{T}$ is weakly continuous at H_0 and $\boldsymbol{T}(H_0) = \mathbf{0}$. Since we consider only regression and affine equivariant functionals $\boldsymbol{T}$, a natural invariant measure of the asymptotic bias of $\boldsymbol{T}$ at H is defined by

$$b_{\Sigma_0}(\boldsymbol{T}, H) = [(\boldsymbol{T}(H) - \boldsymbol{\theta}_0)'\boldsymbol{\Sigma}_0(\boldsymbol{T}(H) - \boldsymbol{\theta}_0)]^{\frac{1}{2}},$$

where $\boldsymbol{\Sigma}_0$ is an affine equivariant scatter matrix of $\boldsymbol{x}$ under G_0. In view of the equivariance of $\boldsymbol{T}$ and the invariance of $b_{\Sigma_0}(\boldsymbol{T}, H)$, we can assume without loss of generality that $\alpha_0 = 0,\ \boldsymbol{\theta}_0 = \mathbf{0}$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{I}_p$ (the identity matrix). Therefore the asymptotic bias of $\boldsymbol{T}$ at H is given by

$$b(\boldsymbol{T}, H) = \|\boldsymbol{T}(H)\|, \tag{2.3}$$

where $\|\cdot\|$ denotes the Euclidean norm.

In order to describe deviation from the nominal distribution H_0 we adopt the following neighborhood of H_0, which was introduced by Ando and Kimura (2003):

$$\mathcal{P}_{H_0}(c, \gamma) = \{H \in \mathcal{M}\ :\ H(B) \le c\, H_0(B) + \gamma,\ {}^{\forall}B \in \mathcal{B}^{p+1}\}, \tag{2.4}$$

where $0 \le \gamma < 1$ and $1 - \gamma \le c < \infty$. Note that $H_0(H)$ is used as both a distribution function and a probability measure for convenience. The neighborhood $\mathcal{P}_{H_0}(c, \gamma)$, which is called a (c, γ) -contamination neighborhood ((c, γ)-neighborhood, for short), is a generalization of Rieder's (1977) (ε, δ)-neighborhood and includes ε-contamination and total variation neighborhoods: Let ε and δ be some given constants such that $\varepsilon \ge 0, \delta \ge 0$ and $\varepsilon + \delta < 1$. Then we have the ε-contamination neighborhood $\mathcal{P}_{H_0}(1-\varepsilon, \varepsilon)$ for $c = 1 - \varepsilon$ and $\gamma = \varepsilon$, the total variation neighborhood $\mathcal{P}_{H_0}(1, \delta)$ for $c = 1$ and $\gamma = \delta$, and Rieder's (1977) (ε, δ)-neighborhood $\mathcal{P}_{H_0}(1 - \varepsilon,\ \varepsilon + \delta)$ for $c = 1 - \varepsilon$ and $\gamma = \varepsilon + \delta$. We can see that $\mathcal{P}_{H_0}(c, \gamma)$ is generated by a special capacity v defined as

$$v(B) = \begin{cases} \min\{cH_0(B) + \gamma,\ 1\} & \text{if } B \neq \phi, \quad B \in \mathcal{B}^{p+1}, \\ 0, & \text{if } B = \phi, \end{cases}$$

that is,

$$\mathcal{P}_{H_0}(c, \gamma) = \{H \in \mathcal{M}\ :\ H(B) \le v(B),\ {}^{\forall}B \in \mathcal{B}^{p+1}\}. \tag{2.5}$$

This means that the (c, γ)-neighborhood $\mathcal{P}_{H_0}(c, \gamma)$ has nice properties for developing minimax theory in robust inference (see Bednarski, 1981, Buja, 1986). Ando and Kimura (2003) gave the following useful characterization of $\mathcal{P}_{H_0}(c, \gamma)$.

Proposition 2.1. *For $0 \le \gamma < 1$ and $1 - \gamma \le c < \infty$ it holds that*

$$\mathcal{P}_{H_0}(c,\ \gamma) = \{H = c\,(H_0 - W) + \gamma\, K\ :\ W \in \mathcal{W}_{H_0,\lambda},\ K \in \mathcal{M}\},$$

where $\mathcal{W}_{H_0,\lambda}$ is the set of all measures W such that $W(B) \le H_0(B)$ holds for ${}^{\forall}B \in \mathcal{B}^{p+1}$ and $W(R^{p+1}) = \lambda = (c + \gamma - 1)/c$.

The maximum asymptotic bias of $\boldsymbol{T}$ over $\mathcal{P}_{H_0}(c, \gamma)$ is defined as

$$B_{\boldsymbol{T}}(c, \gamma) = \sup\{\|\boldsymbol{T}(H)\|\ :\ H \in \mathcal{P}_{H_0}(c, \gamma)\}. \tag{2.6}$$

We consider the following class of robust estimates defined as

$$(T_0(H), \boldsymbol{T}(H)) = \arg\min_{\alpha, \boldsymbol{\theta}} J(F_{H,\alpha,\boldsymbol{\theta}}), \tag{2.7}$$

where $J(\cdot)$ is a robust loss functional defined on the set of all distributions on the real line and $F_{H,\alpha,\theta}$ is the distribution of the absolute residual $|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}|$ under H. This class of estimates includes the well-known robust estimates such as S-estimates, τ-estimates, CM-estimates and R-estimates. We assume that J, F_0 and G_0 satisfy the same conditions A1 and A2 as in Berrendero and Zamar (2001).

A1. (a) If F and G are two distribution functions on $[0,\infty)$ such that $F(u) \leq G(u)$ for every $u \geq 0$, then $J(F) \geq J(G)$.

(b) Given two sequences of distribution functions on $[0,\infty)$, F_n and G_n, which are continuous on $(0,\infty)$ and such that $F_n(u) \rightarrow F(u)$ and $G_n \rightarrow G(u)$, where F and G are possibly sub-stochastic and continuous on $(0,\infty)$, with $G(\infty) \geq 1-\varepsilon$ and

$$G(u) \geq F(u) \qquad \text{for every } u > 0, \tag{2.8}$$

then

$$\lim_{n\to\infty} J(F_n) \geq \lim_{n\to\infty} J(G_n). \tag{2.9}$$

Moreover, if (2.8) holds strictly, then (2.9) also holds strictly.

(c) If F and G are two distribution functions on $[0,\infty)$ with F continuous, then

$$J((1-\gamma)\,F+\gamma\delta_\infty) = \lim_{n\to\infty} J((1-\gamma)F+\gamma U_n) \geq J((1-\gamma)F+\gamma G),$$

where U_n stands for the uniform distribution function on $[n-\frac{1}{n}, n+\frac{1}{n}]$.

A2. F_0 has an even and strictly unimodal density f_0 with $f_0(u) > 0$ for every $u \in R$, and $P_{G_0}(\boldsymbol{\theta}'\boldsymbol{x} = a) < 1$, for each $\boldsymbol{\theta} \in R^p$ ($\boldsymbol{\theta} \neq 0$) and $a \in R$.

3. Lower and upper bounds for maximum asymptotic bias

Let $\xi = \{W_{\alpha,\boldsymbol{\theta}} : \alpha \in R, \boldsymbol{\theta} \in R^p\}$ be any subset of $\mathcal{W}_{H_0,\lambda}$, where $\mathcal{W}_{H_0,\lambda}$ is given in Proposition 2.1, and let

$$F^{\xi}_{\alpha,\boldsymbol{\theta}}(u) = (H_0 - W_{\alpha,\boldsymbol{\theta}})(|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}| \leq u), \qquad {}^\forall u \geq 0. \tag{3.1}$$

We use $F^{\xi}_{\alpha,\boldsymbol{\theta}}$ as both a function on $[0,\infty)$ and a measure on $(R,\mathcal{B})$ such that $F^{\xi}_{\alpha,\boldsymbol{\theta}}(R) = 1-\lambda$ $(= (1-\gamma)/c)$. Let

$$d_\xi = J(c\,F^{\xi}_{0,\mathbf{0}} + \gamma\delta_\infty) \tag{3.2}$$

and

$$m_\xi(t) = \inf_{\|\boldsymbol{\theta}\|=t} \inf_{\alpha\in R} J(c\,F^{\xi}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0), \tag{3.3}$$

where δ_0 and δ_∞ are the point mass distributions at 0 and ∞, respectively.

Let $\mathcal{F}_\lambda$ be the set of all $\xi = \{W_{\alpha,\boldsymbol{\theta}} : \alpha \in R, \boldsymbol{\theta} \in R^p\} \subset \mathcal{W}_{H_0,\lambda}$ satisfying the following condition A3:

A3. (a) $F^{\xi}_{\alpha,\boldsymbol{\theta}}(u)$ is continuous in α and $\boldsymbol{\theta}$.

(b) $F^{\xi}_{k\alpha,k\boldsymbol{\theta}}(u)$ is strictly decreasing in $k > 0$ for $0 < F^{\xi}_{k\alpha,k\boldsymbol{\theta}}(k) < (1-\gamma)/c$.

We obtain the following two lemmas which correspond to Lemmas 4 and 6 in Berrendero and Zamar (2001).

Lemma 3.1. *Under A1(b) and A2, for any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ there exists $\alpha(\boldsymbol{\theta}) \in R$ such that*

$$J(c\,F^{\xi}_{\alpha(\boldsymbol{\theta}),\boldsymbol{\theta}} + \gamma\delta_0) = \inf_{\alpha\in R} J(c\,F^{\xi}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0).$$

Moreover, for any $t > 0$ there exists $K_t > 0$ such that $|\alpha(\boldsymbol{\theta})| \le K_t$ for every $\boldsymbol{\theta} \in \{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| = t\}$.

Lemma 3.2. *Let $m_\xi(t)$ be as in (3.3). Then, under A1(b) and A2, for any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ the following results hold:*

(a) *There exist $\boldsymbol{\theta}_t \in R^p$ and $\alpha(\boldsymbol{\theta}_t) \in R$ such that $\|\boldsymbol{\theta}_t\| = t$ and*

$$m_\xi(t) = J(c\,F^{\xi}_{\alpha(\boldsymbol{\theta}_t),\boldsymbol{\theta}_t} + \gamma\delta_0).$$

(b) *$m_\xi(t)$ is strictly increasing.*

We consider two special ξ's which play important roles. Let $\hat{\xi} = \{\hat{W}_{\alpha,\boldsymbol{\theta}}\}$ and $\xi^* = \{W^*_{\alpha,\boldsymbol{\theta}}\}$ be defined as follows:

$$\hat{W}_{\alpha,\boldsymbol{\theta}}(B) = H_0\left(B \cap \left\{\left|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}\right| \ge a_{\alpha,\boldsymbol{\theta}}\left(\tfrac{c+\gamma-1}{c}\right)\right\}\right), \quad {}^\forall B \in \mathcal{B}^{p+1} \tag{3.4}$$

$$W^*_{\alpha,\boldsymbol{\theta}}(B) = H_0\left(B \cap \left\{\left|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}\right| \le a_{\alpha,\boldsymbol{\theta}}\left(\tfrac{1-\gamma}{c}\right)\right\}\right), \quad {}^\forall B \in \mathcal{B}^{p+1}, \tag{3.5}$$

where $a_{\alpha,\boldsymbol{\theta}}(\eta)$ $(0 \le \eta < 1)$ denotes the upper $100\eta\%$ point of the distribution of $|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}|$ under H_0, that is,

$$F_{H_0,\alpha,\boldsymbol{\theta}}(a_{\alpha,\boldsymbol{\theta}}(\eta)) = 1-\eta.$$

We note that $\hat{\xi}$ and ξ^* belong to $\mathcal{F}_\lambda$. Then we obtain the following lemma which correspond to Lemma 5 in Berrendero and Zamar (2001).

Lemma 3.3. *Under A2, $F^{\hat{\xi}}_{k\alpha,k\boldsymbol{\theta}}(u)$ and $F^{\xi^*}_{k\alpha,k\boldsymbol{\theta}}(u)$ are strictly decreasing in $k > 0$ for $0 < F^{\hat{\xi}}_{k\alpha,k\boldsymbol{\theta}}(u) < (1-\gamma)/c$ and $0 < F^{\xi^*}_{k\alpha,k\boldsymbol{\theta}}(u) < (1-\gamma)/c$, respectively.*

Using these lemmas, we can derive the following theorem which gives lower and upper bounds for the maximum asymptotic bias $B_{\boldsymbol{T}}(c,\gamma)$ of $\boldsymbol{T}$.

Theorem 3.1. *Let $\boldsymbol{T}$ be a regression estimate defined by (2.7). Assume that A1 and A2 hold. Then it holds that*

$$\underline{B}_{\boldsymbol{T}}(c,\,\gamma) \leq B_{\boldsymbol{T}}(c,\,\gamma) \leq \overline{B}_{\boldsymbol{T}}(c,\,\gamma), \tag{3.6}$$

where

$$\underline{B}_{\boldsymbol{T}}(c,\,\gamma) = \sup_{\xi \in \mathcal{F}_\lambda} m_\xi^{-1}(d_\xi) \quad \textit{and} \quad \overline{B}_{\boldsymbol{T}}(c,\,\gamma) = m_{\hat{\xi}}^{-1}(d_{\xi^*}).$$

Remark 3.1 When $c = 1-\varepsilon$ and $\gamma = \varepsilon$ (i.e., the ε-contamination case) , Theorems 3.1 reduces to Theorem 1 of Berrendero and Zamar (2001), respectively. In this case, we have $\lambda = 0$ and $\hat{\xi} = \xi^*$, and hence $\overline{B}_{\boldsymbol{T}}(c,\,\gamma) = \underline{B}_{\boldsymbol{T}}(c,\,\gamma)$.

4. S-, τ-and CM-estimates under the Gaussian model

As important special cases, we consider S-estimates, τ-estimates and CM-estimates in the case that H_0 is the multivariate normal distribution $N(\mathbf{0}, \boldsymbol{I}_{p+1})$, where $\boldsymbol{I}_{p+1}$ is the $(p+1)\times(p+1)$ identity matrix. We denote by ϕ the density of the standard normal distribution $N(0,\,1)$. For any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ let $\varphi^\xi_{\alpha,\boldsymbol{\theta}}$ denote the density of $F^\xi_{\alpha,\boldsymbol{\theta}}$. Let $\mathcal{F}^\circ_\lambda$ be the set of all $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ satisfying the following condition A4:

A4. (a) $\varphi^\xi_{0,\boldsymbol{\theta}}$ is expressed in the form of

$$\varphi^\xi_{0,\boldsymbol{\theta}}(u) = \frac{1}{\sqrt{1+\|\boldsymbol{\theta}\|^2}}\phi_\xi\left(\frac{u}{\sqrt{1+\|\boldsymbol{\theta}\|^2}}\right), \quad {}^\forall u \geq 0, \tag{4.1}$$

where ϕ_ξ $(0 \leq \phi_\xi \leq 2\phi)$ is some measurable function defined on $[0,\infty)$ such that

$$\int_0^\infty \phi_\xi(u)du = \frac{1-\gamma}{c}.$$

(b) $F^\xi_{\alpha,\boldsymbol{\theta}}(u) \leq F^\xi_{0,\boldsymbol{\theta}}(u), \quad {}^\forall u > 0$

Then, for any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}^\circ_\lambda$ we have

$$\inf_{\alpha \in R} J(c\,F^\xi_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0) = J(c\,F^\xi_{0,\boldsymbol{\theta}} + \gamma\delta_0) = m_\xi(\|\boldsymbol{\theta}\|). \tag{4.2}$$

We can easily see that $\hat{\xi} = \{\hat{W}_{\alpha,\boldsymbol{\theta}}\}$ and $\xi^* = \{W^*_{\alpha,\boldsymbol{\theta}}\}$ belong to $\mathcal{F}^\circ_\lambda$.

Let ρ_1 and ρ_2 be functions satisfying the following conditions A5:

A5. (a) The functions ρ_1 and ρ_2 are even, bounded, monotone on $[0,\infty)$, continuous at 0 with $0=\rho_i(0)<\rho_i(\infty)=1,\ i=1,2$ and with at most a finite number of discontinuities.

(b) The function ρ_2 is differentiable with $2\rho_2(u)-\rho_2'(u)u\geq 0$.

The S-estimate (Rousseeuw and Yohai,1984) is defined with $J(F)=S(F)$, where

$$S(F) = \inf\left\{s>0 \ :\ E_F\left[\rho_1\left(\frac{u}{s}\right)\right]\leq b\right\},\quad 0<b<1. \tag{4.3}$$

For any $\xi=\{W_{\alpha,\boldsymbol{\theta}}\}\in\mathcal{F}_\lambda$ let

$$g_{\xi,i}(s)=E_{F^{\xi}_{0,\mathbf{0}}}\left[\rho_i\left(\frac{u}{s}\right)\right]=\int_0^\infty \rho_i\left(\frac{u}{s}\right)\varphi^{\xi}_{0,\mathbf{0}}(u)du,\quad i=1,2.$$

The following theorem gives the lower and upper bounds for the maximum asymptotic bias $B_S(c,\gamma)$ of S-estimates based on ρ_1.

Theorem 4.1. *Assume that H_0 is the multivariate normal distribution $N(\mathbf{0},\boldsymbol{I}_{p+1})$. Then*

$$\begin{aligned}&\underline{B}_S(c,\gamma)\leq B_S(c,\gamma)\leq\overline{B}_S(c,\gamma), && \text{if}\quad \gamma<\min(b,1-b),\\ &B_S(c,\gamma)=\infty, && \text{if}\quad \gamma\geq\min(b,1-b),\end{aligned}$$

where

$$\underline{B}_S(c,\gamma) = \sup_{\xi\in\mathcal{F}^\circ_\lambda}\left(\left\{g^{-1}_{\xi,1}\left(\frac{b-\gamma}{c}\right)\Big/ g^{-1}_{\xi,1}\left(\frac{b}{c}\right)\right\}^2-1\right)^{1/2} \tag{4.4}$$

and

$$\overline{B}_S(c,\gamma) = \left(\left\{g^{-1}_{\xi^*,1}\left(\frac{b-\gamma}{c}\right)\Big/ g^{-1}_{\hat{\xi},1}\left(\frac{b}{c}\right)\right\}^2-1\right)^{1/2}. \tag{4.5}$$

The τ-estimate (Yohai and Zamar, 1988) is defined with $J(F)=\tau^2(F)$, where

$$\tau^2(F)=S^2(F)E_F\left[\rho_2\left(\frac{u}{S(F)}\right)\right]. \tag{4.6}$$

As shown in Yohai and Zamar (1988), τ-estimates inherit the breakdown point of the initial S-estimate defined by ρ_1 and their efficiencies are mainly determined by ρ_2. The following theorem gives the lower and upper bounds for the maximum asymptotic bias $B_\tau(c,\gamma)$ of τ-estimates, which shows how $B_\tau(c,\gamma)$ relates to $B_S(c,\gamma)$ of the initial S-estimates based on ρ_1.

Theorem 4.2. *Assume that* H_0 *is the multivariate normal distribution* $N(\mathbf{0}, \boldsymbol{I}_{p+1})$. *Then*

$$\underline{B}_\tau(c,\gamma) \leq B_\tau(c,\gamma) \leq \overline{B}_\tau(c,\gamma),$$

where

$$\underline{B}_\tau(c,\gamma) = \sup_{\xi\in\mathcal{F}_\lambda^\circ}\left\{\left[\frac{g_{\xi,1}^{-1}\left(\frac{b-\gamma}{c}\right)}{g_{\xi,1}^{-1}\left(\frac{b}{c}\right)}\right]^2 H_{\xi,\xi}(c,\gamma)-1\right\}^{1/2}, \tag{4.7}$$

$$\overline{B}_\tau(c,\gamma) = \{[1+\overline{B}_S^2(c,\gamma)]H_{\xi^*,\hat{\xi}}(c,\gamma)-1\}^{1/2}, \tag{4.8}$$

$$H_{\xi_1,\xi_2}(c,\gamma) = \left[\overline{g}_{\xi_1}\left(\frac{b-\gamma}{c}\right)+\frac{\gamma}{c}\right]/\overline{g}_{\xi_2}\left(\frac{b}{c}\right) \quad \textit{and} \quad \overline{g}_\xi(t) = g_{\xi,2}[g_{\xi,1}^{-1}(t)].$$

The CM estimate (Mendes and Tyler, 1996) is defined by $J(F) = CM(F)$, where

$$CM(F) = \inf_{s\geq S(F)}\left\{aE_F\left[\rho_1\left(\frac{u}{s}\right)\right]+\log s\right\}, \tag{4.9}$$

a is a tuning constant, and $S(F)$ is given by (4.3). We let

$$\kappa_{\xi,c,\gamma} = g_{\xi,1}^{-1}\left(\frac{b-\gamma}{c}\right), \qquad \eta_{\xi,c,\gamma} = g_{\xi,1}^{-1}\left(\frac{b}{c}\right).$$

and

$$A_{a,c,\gamma}^{\xi}(s) = acg_{\xi,1}(s)+\log s. \tag{4.10}$$

The following theorem gives the lower and upper bounds for the maximum asymptotic bias $B_{CM}(c,\gamma)$ of CM-estimates based on ρ_1.

Theorem 4.3. *Assume that* H_0 *is the multivariate normal distribution* $N(\mathbf{0}, \boldsymbol{I}_{p+1})$. *Then*

$$\underline{B}_{CM}(c,\gamma) \leq B_{CM}(c,\gamma) \leq \overline{B}_{CM}(c,\gamma),$$

where

$$\underline{B}_{CM}(c,\gamma) = \sup_{\xi\in\mathcal{F}_\lambda^\circ}\{\exp[2a\gamma+2D_{\xi,\xi,a}(c,\gamma)]-1\}^{1/2}, \tag{4.11}$$

$$\overline{B}_{CM}(c,\gamma) = \{\exp[2a\gamma+2D_{\xi^*,\hat{\xi},a}(c,\gamma)]-1\}^{1/2}, \tag{4.12}$$

$$D_{\xi_1,\xi_2,a}(c,\gamma) = \inf_{s\geq\kappa_{\xi_1,c,\gamma}} A_{a,c,\gamma}^{\xi_1}(s) - \inf_{s\geq\eta_{\xi_2,c,\gamma}} A_{a,c,\gamma}^{\xi_2}(s).$$

Remark 4.1. When $c = 1 - \varepsilon$ and $\gamma = \varepsilon$, Theorems 4.1, 4.2 and 4.3 are reduced to (3.24) of Martin et al. (1989), Theorem 3 of Berrendero and Zamar (2001) and Theorem 4.1 of Bererendero et al. (2007), respectively.

Remark 4.2. The upper bound $\overline{B}_S(c,\gamma)$ in (4.5) is the same as (4.7) in Ando and Kimura (2004). Note that $h_\xi(\tau)$ in the equality (4.7) satisfies the relation $h_\xi(\tau) = g_{\xi,1}(\frac{1}{\tau})$. We also notice that when ρ_1 is a jump function, $\overline{B}_S(c,\gamma) = B_S(c,\gamma)$ holds for $c \leq 1$ (see Theorem 4.1 of Ando and Kimura, 2004).

Remark 4.3. The arguments concerning the intercept estimates can be seen in Section 7 of Berrendero and Zamar (2001). Here, we should point out that the same arguments also hold for our (c,γ)-neighborhood case.

We consider the lower and upper bounds of $B_S(c,\gamma)$, $B_\tau(c,\gamma)$, and $B_{CM}(c,\gamma)$ based on Tukey's biweight ρ-function, which is defined as

$$\rho(u) = \begin{cases} 3u^2 - 3u^4 + u^6 & \text{if} \quad 0 \leq |u| \leq 1 \\ 1 & \text{if} \quad |u| > 1. \end{cases} \tag{4.13}$$

We let

$$\rho_1(u) = \rho\left(\frac{u}{k_1}\right) \qquad \text{and} \qquad \rho_2(u) = \rho\left(\frac{u}{k_2}\right). \tag{4.14}$$

Tables 1, 3 and 5 exhibit the upper bounds $\overline{B}_S(c,\gamma)$, $\overline{B}_\tau(c,\gamma)$ and $\overline{B}_{CM}(c,\gamma)$, respectively. The constants k_1, k_2 and a are chosen so that the three estimates have 0.5 breakdown point (i.e., b=0.5), and that τ- and CM-estimates have 95% efficiency. In this case we have $k_1 = 1.548$, $k_2 = 6.039$, and $a = 4.835$, and the S-estimate is 28.7% efficiency. We note $\overline{B}_\tau(1-\gamma,\gamma) = B_\tau(1-\gamma,\gamma)$ for γ -contamination case. When $\rho_1 = \rho_2$, τ-estimates reduce to S-estimates. On the other hand, it is difficult to find the exact values of the lower bounds $\underline{B}_S(c,\gamma), \underline{B}_\tau(c,\gamma)$and $\underline{B}_{CM}(c,\gamma)$. In order to obtain their good approximate values we need to find $\xi \in \mathcal{F}_\lambda^\circ$ which makes the inside of the supremum in (4.4), (4.7) and (4.11) as large as possible. Such ξ depends on c and γ. Tables 2, 4, 6 present lower bounds of $\underline{B}_S(c,\gamma), \underline{B}_\tau(c,\gamma)$ and $\underline{B}_{CM}(c,\gamma)$, respectively. We obtained their bounds using the set $\{\xi_0, \xi_1, \cdots, \xi_{22}\}$ which consists of various types of ξ. Acoording to (4.1), we define $\phi_{\xi_0}, \phi_{\xi_1}, \cdots, \phi_{\xi_{22}}$ as follows: Let

$$\phi_{\xi_i}(u) = \begin{cases} 2\phi(u) & \text{if} \quad 0 \leq u < a_i \\ 0 & \text{if} \quad a_i \leq u < b_i \\ 2\phi(u) & \text{if} \quad b_i \leq u < \infty \end{cases} \tag{4.15}$$

where

$$a_i = id, \qquad b_i = \Phi^{-1}\left(\Phi(id) + \frac{c+\gamma-1}{2c}\right), \qquad i = 0,\ 1, \cdots,\ 20,$$

and

$$d = \frac{1}{20}\Phi^{-1}\left(\frac{c-\gamma+1}{2c}\right). \tag{4.16}$$

and Φ denotes the distribution function of $N(0,1)$. Let

$$\phi_{\xi_{21}}(u) = \begin{cases} 2\phi(a) & \text{if} \quad 0 \le u < k \\ 2\phi(u) & \text{if} \quad k \le u < \infty, \end{cases} \tag{4.17}$$

where k is the constant such that

$$\Phi(k) - \phi(k) = \frac{2c + \gamma - 1}{2c}, \tag{4.18}$$

and let $\phi_{\xi_{22}}(u) = \dfrac{2(1-\gamma)}{c}\phi(u),\ u \ge 0$. The lower bounds in Tables 2, 4 and 6 were obtained by

$$\max_{0\le i\le 22} \left\{ \left[\frac{g^{-1}_{\xi_i,1}\left(\frac{b-\gamma}{c}\right)}{g^{-1}_{\xi_i,1}\left(\frac{b}{c}\right)} \right]^2 - 1 \right\}^{1/2}, \tag{4.19}$$

$$\max_{0\le i\le 22} \left\{ \left[\frac{g^{-1}_{\xi_i,1}\left(\frac{b-\gamma}{c}\right)}{g^{-1}_{\xi_i,1}\left(\frac{b}{c}\right)} \right]^2 H_{\xi_i,\xi_i}(c,\gamma) - 1 \right\}^{1/2} \tag{4.20}$$

$$\max_{0\le i\le 22} \{\exp[2a\gamma + 2D_{\xi_i,\xi_i,a}(c,\gamma)] - 1\}^{1/2}, \tag{4.21}$$

where $a = 4.835$ and $b = 0.5$. We note that $\xi_0 = \xi^*$ and $\xi_{20} = \hat{\xi}$. For the purpose of getting good approximated values, we included different types of ξ as candidates in taking the maximum values. Figures 1, 2 and 3 give graphs of $\overline{B}_S(c,\gamma)$, $\overline{B}_\tau(c,\gamma)$ and $\bar{B}_{CM}(c,\gamma)$, and the maximum values of (4.19) , (4.20) and (4.21) for c=1.2. The graphs show that the derived lower and upper bounds are useful. In particular, the difference of two bounds is small in the range between 0 and 0.1, which corresponds to a realistic situation.

Table 1: $\underline{B}_S(c,\ \gamma)$ (Tukey's biweight ρ-function)

$c\backslash\gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	21.83
0.65	—	—	—	—	—	—	—	—	—	4.69	28.06
0.75	—	—	—	—	—	—	—	—	2.23	5.81	34.63
0.80	—	—	—	—	—	—	—	1.65	2.43	6.38	38.01
0.85	—	—	—	—	—	—	1.23	1.78	2.66	6.97	41.43
0.90	—	—	—	—	—	0.88	1.31	1.93	2.89	7.58	44.89
0.95	—	—	—	—	0.56	0.93	1.4	2.09	3.12	8.19	48.39
0.97	—	—	—	0.42	0.57	0.95	1.44	2.15	3.22	8.44	49.81
0.98	—	—	0.33	0.42	0.57	0.96	1.46	2.18	3.27	8.56	50.52
0.99	—	0.23	0.34	0.42	0.58	0.97	1.48	2.22	3.32	8.69	51.23
1.00	0.00	0.23	0.34	0.43	0.58	0.98	1.50	2.25	3.37	8.81	51.94
1.10	0.00	0.25	0.37	0.47	0.65	1.12	1.73	2.58	3.88	10.07	59.14
1.20	0.00	0.28	0.41	0.52	0.72	1.27	1.97	2.92	4.40	11.35	66.48
1.50	0.00	0.36	0.54	0.69	0.99	1.78	2.74	4.03	5.99	15.29	89.23
2.00	0.00	0.55	0.84	1.08	1.53	2.67	4.08	5.98	8.75	22.17	129.07
3.00	0.00	1.02	1.48	1.88	2.62	4.55	6.92	10.08	14.61	36.77	213.81

Table 2: $\overline{B}_S(c,\ \gamma)$ (Tukey's biweight ρ-function)

$c\backslash\gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	21.83
0.65	—	—	—	—	—	—	—	—	—	4.69	28.29
0.75	—	—	—	—	—	—	—	—	2.23	6.04	35.13
0.80	—	—	—	—	—	—	—	1.65	2.58	6.74	38.67
0.85	—	—	—	—	—	—	1.23	1.94	2.94	7.45	42.26
0.90	—	—	—	—	—	0.88	1.48	2.23	3.29	8.17	45.92
0.95	—	—	—	—	0.56	1.13	1.73	2.52	3.65	8.91	49.63
0.97	—	—	—	0.42	0.67	1.22	1.83	2.63	3.80	9.20	51.13
0.98	—	—	0.33	0.48	0.72	1.26	1.88	2.69	3.87	9.35	51.88
0.99	—	0.23	0.41	0.54	0.77	1.31	1.93	2.75	3.94	9.50	52.64
1.00	0.00	0.33	0.48	0.60	0.82	1.35	1.98	2.80	4.01	9.65	53.39
1.10	0.77	0.87	0.96	1.06	1.25	1.79	2.46	3.38	4.74	11.16	61.05
1.20	1.16	1.25	1.34	1.44	1.63	2.20	2.94	3.96	5.49	12.70	68.87
1.50	2.16	2.26	2.36	2.48	2.72	3.43	4.39	5.74	7.78	17.49	93.18
2.00	3.73	3.87	4.01	4.17	4.50	5.51	6.89	8.84	11.79	25.90	135.86
3.00	6.95	7.18	7.43	7.68	8.23	9.90	12.18	15.43	20.35	43.88	226.89

5 Proofs

Lemmas 3.1 and 3.2 are verified in the same way as the proofs of Lemmas 4 and 6 in Berrendero and Zamar (2001), respectively (replace $(1-\varepsilon)F_{H_0,\alpha,\boldsymbol{\theta}} + \varepsilon\delta_0$ with $cF^{\hat{\xi}}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0$).

Proof of Lemma 3.3. We note that $F^{\hat{\xi}}_{k\alpha,k\boldsymbol{\theta}}$ and $F^{\xi^*}_{k\alpha,k\boldsymbol{\theta}}$ are expressed in the form of

$$F^{\hat{\xi}}_{k\alpha,k\boldsymbol{\theta}}(u) = \min\left(F_{H_0,k\alpha,k\boldsymbol{\theta}}(u),\ \frac{1-\gamma}{c}\right),\quad {}^{\forall}u \geq 0,$$

Table 3: $\overline{B}_\tau(c,\ \gamma)$ (Tukey's biweight ρ-function)

$c\backslash\gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	24.38
0.65	—	—	—	—	—	—	—	—	—	6.41	31.69
0.75	—	—	—	—	—	—	—	—	3.42	8.71	39.44
0.80	—	—	—	—	—	—	—	2.61	4.14	.9.87	43.44
0.85	—	—	—	—	—	—	1.97	3.19	4.81	11.04	47.51
0.90	—	—	—	—	—	1.43	2.45	3.72	5.46	12.22	51.65
0.95	—	—	—	—	0.91	1.82	2.86	4.22	6.11	13.41	55.86
0.97	—	—	—	0.68	1.06	1.95	3.02	4.43	6.38	13.89	57.55
0.98	—	—	0.54	0.76	1.13	2.02	3.10	4.53	6.51	14.13	58.40
0.99	—	0.38	0.63	0.83	1.19	2.08	3.18	4.63	6.64	14.37	59.26
1.00	0.00	0.48	0.71	0.89	1.24	2.15	3.26	4.73	6.77	14.61	60.11
1.10	0.82	1.02	1.20	1.38	1.75	2.77	4.05	5.74	8.08	17.05	68.78
1.20	1.19	1.39	1.59	1.79	2.20	3.37	4.83	6.75	9.41	19.54	77.63
1.50	2.08	2.37	2.65	2.93	3.52	5.16	7.20	9.86	13.5	27.22	105.13
2.00	3.48	3.92	4.37	4.81	5.72	8.23	11.31	15.27	20.64	40.67	153.40
3.00	6.33	7.14	7.93	8.73	10.34	14.74	20.03	26.77	35.83	69.32	256.32

Table 4: $\underline{B}_\tau(c,\ \gamma)$ (Tukey's biweight ρ-function)

$c\backslash\gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	24.38
0.65	—	—	—	—	—	—	—	—	—	6.41	30.18
0.75	—	—	—	—	—	—	—	—	3.42	6.99	36.56
0.80	—	—	—	—	—	—	—	2.61	3.49	7.32	39.88
0.85	—	—	—	—	—	—	1.97	2.70	3.54	7.68	43.25
0.90	—	—	—	—	—	1.43	2.06	2.73	3.62	8.07	46.69
0.95	—	—	—	—	0.91	1.50	2.09	2.75	3.71	8.50	50.17
0.97	—	—	—	0.68	0.93	1.51	2.10	2.75	3.74	8.69	51.57
0.98	—	—	0.54	0.69	0.94	1.52	2.10	2.75	3.76	8.78	52.28
0.99	—	0.38	0.55	0.70	0.95	1.52	2.11	2.76	3.78	8.87	52.99
1.00	0.00	0.38	0.56	0.70	0.96	1.53	2.11	2.76	3.80	8.97	53.69
1.10	0.00	0.41	0.59	0.74	0.99	1.56	2.13	2.83	4.02	10.00	60.87
1.20	0.00	0.42	0.60	0.75	1.01	1.58	2.15	2.95	4.27	11.14	68.18
1.50	0.00	0.44	0.63	0.79	1.06	1.63	2.30	3.43	5.24	14.81	90.89
2.00	0.00	0.46	0.66	0.83	1.10	1.78	2.88	4.62	7.43	21.28	130.72
3.00	0.00	0.48	0.69	0.89	1.30	2.71	4.80	7.77	12.38	35.09	215.21

and

$$F^{\xi^*}_{k\alpha,k\boldsymbol{\theta}}(u) = \max\left(F_{H_0,k\alpha,k\boldsymbol{\theta}}(u) - \frac{c+\gamma-1}{c}, 0\right), \quad {}^\forall u \geq 0,$$

where $F_{H_0,k\alpha,k\boldsymbol{\theta}}(u)$ is the distribution function of $|y - k\alpha - k\boldsymbol{\theta}'\boldsymbol{x}|$ under H_0. By Lemma 5 of Berrendero and Zamar (2001), $F_{H_0,k\alpha,k\boldsymbol{\theta}}(u)$ is strictly decreasing in $k > 0$. Therefore, $F^{\hat{\xi}}_{\alpha,\boldsymbol{\theta}}(u)$ and $F^{\xi^*}_{\alpha,\boldsymbol{\theta}}(u)$ are strictly decreasing in $k > 0$. $\square$

Proof of Theorem 3.1. Let t^* be such that $d_{\xi^*} = m_{\hat{\xi}}(t^*)$. First, we show $B_{\boldsymbol{T}}(c,\gamma) \leq t^*$. Let $\tilde{\boldsymbol{\theta}} \in R^p$ be such that $\|\tilde{\boldsymbol{\theta}}\| = t > t^*$. It is enough to show that for any $H \in \mathcal{P}_{H_0}(c,\gamma)$ and any

Table 5: $\overline{B}_{CM}(c,\ \gamma)$ (Tukey's biweight ρ-function)

$c\backslash\gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	21.83
0.65	—	—	—	—	—	—	—	—	—	5.34	28.29
0.75	—	—	—	—	—	—	—	—	3.20	7.63	35.13
0.80	—	—	—	—	—	—	—	2.43	3.88	8.71	38.67
0.85	—	—	—	—	—	—	1.81	2.96	4.48	9.77	42.26
0.90	—	—	—	—	—	1.28	2.22	3.41	5.06	10.85	45.92
0.95	—	—	—	—	0.79	1.61	2.57	3.84	5.64	11.93	49.63
0.97	—	—	—	0.58	0.93	1.73	2.70	4.02	5.86	12.36	51.13
0.98	—	—	0.46	0.66	0.99	1.78	2.77	4.10	5.98	12.58	51.88
0.99	—	0.32	0.55	0.73	1.04	1.84	2.83	4.19	6.09	12.79	52.64
1.00	0.00	0.43	0.63	0.79	1.09	1.89	2.90	4.27	6.21	13.01	53.39
1.10	0.82	0.96	1.10	1.24	1.54	2.40	3.54	5.12	7.36	15.21	61.05
1.20	1.18	1.32	1.46	1.61	1.93	2.88	4.18	5.98	8.53	17.44	68.87
1.50	2.08	2.25	2.43	2.62	3.03	4.32	6.10	8.59	12.11	24.32	93.18
2.00	3.48	3.72	3.99	4.26	4.87	6.78	9.43	13.13	18.35	36.35	135.86
3.00	17.28	17.75	18.25	18.76	19.86	23.04	16.5	22.80	31.64	61.94	226.89

Table 6: $\underline{B}_{CM}(c,\ \gamma)$ (Tukey's biweight ρ-function)

$c\backslash\gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	21.83
0.65	—	—	—	—	—	—	—	—	—	5.34	28.06
0.75	—	—	—	—	—	—	—	—	3.20	5.80	34.63
0.80	—	—	—	—	—	—	—	2.43	3.20	6.36	38.01
0.85	—	—	—	—	—	—	1.81	2.43	3.20	6.95	41.43
0.90	—	—	—	—	—	1.28	1.81	2.43	3.20	7.55	44.89
0.95	—	—	—	—	0.79	1.28	1.81	2.43	3.20	8.15	48.39
0.97	—	—	—	0.58	0.79	1.28	1.81	2.43	3.20	8.40	49.81
0.98	—	—	0.46	0.58	0.79	1.28	1.81	2.43	3.20	8.52	50.52
0.99	—	0.32	0.46	0.58	0.79	1.28	1.81	2.43	3.21	8.64	51.23
1.00	0.00	0.32	0.46	0.58	0.79	1.28	1.81	2.43	3.22	8.76	51.94
1.10	0.00	0.32	0.46	0.58	0.79	1.28	1.81	2.43	3.55	10.00	59.14
1.20	0.00	0.32	0.46	0.58	0.79	1.28	1.81	2.60	3.92	11.26	66.48
1.50	0.00	0.32	0.46	0.58	0.79	1.28	2.15	3.39	5.27	15.14	89.23
2.00	0.00	0.32	0.46	0.63	0.98	1.89	3.10	4.92	7.68	21.91	129.07
3.00	0.00	0.44	0.66	0.86	1.36	3.33	5.24	8.15	12.75	36.28	213.81

$\alpha \in R$ we have

$$J(F_{H,\alpha,\tilde{\boldsymbol{\theta}}}) > J(F_{H,0,\mathbf{0}}). \tag{5.1}$$

It is clear that for any $H = c\,(H_0 - W) + \gamma K \in \mathcal{P}_{H_0}(c,\gamma)$, $\alpha \in R$ and $u > 0$,

$$F_{H,\alpha,\tilde{\boldsymbol{\theta}}}(u) \;=\; c\,F^{\xi}_{\alpha,\tilde{\boldsymbol{\theta}}}(u) + \gamma F_{K,\alpha,\tilde{\boldsymbol{\theta}}}(u) \;\le c\,F^{\hat{\xi}}_{\alpha,\tilde{\boldsymbol{\theta}}}(u) + \gamma\delta_0(u), \tag{5.2}$$

where $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ is defined as $W_{\alpha,\boldsymbol{\theta}} = W$ for any $\alpha \in R$ and $\boldsymbol{\theta} \in R^p$. From (5.2), A1(a), the definition of $m_\xi(t)$ and Lemma 3.2(b) it follows that for any $H \in \mathcal{P}_{H_0}(c,\gamma)$

$$J(F_{H,\alpha,\tilde{\boldsymbol{\theta}}}) \;\ge\; J(c\,F^{\hat{\xi}}_{\alpha,\tilde{\boldsymbol{\theta}}} + \gamma\delta_0) \ge m_{\hat{\xi}}(t) > m_{\hat{\xi}}(t^*). \tag{5.3}$$

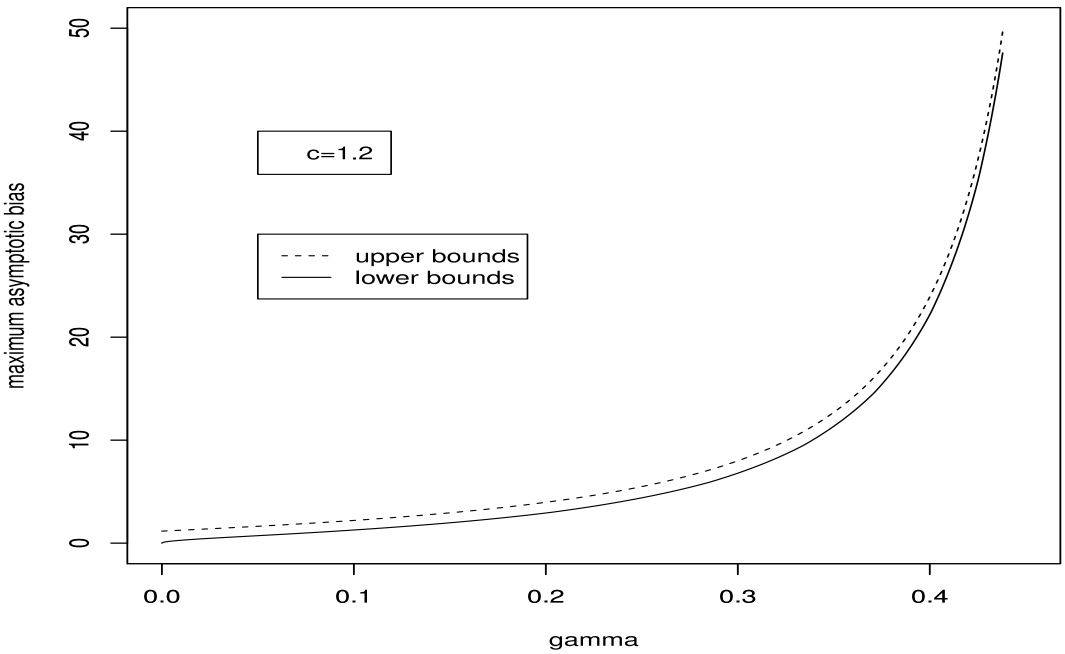

Figure 1: Lower and upper bounds of $B_S(c, \gamma)$ for c=1.2 (Tukey's biweight ρ-function)

The condition $d_{\xi^*} = m_{\hat{\xi}}(t^*)$ and A1(c) imply

$$m_{\hat{\xi}}(t^*) = \lim_{n\to\infty} J(c\,F^{\xi^*}_{0,\mathbf{0}} + \gamma U_n) \geq \lim_{n\to\infty} J(c\,F^{\xi}_{0,\mathbf{0}} + \gamma U_n) \geq J(F_{H,0,\mathbf{0}}). \tag{5.4}$$

Noting $t^* = m^{-1}_{\hat{\xi}}(d_{\xi^*})$, we obtain $B_{\boldsymbol{T}}(c, \gamma) \leq \overline{B}_{\boldsymbol{T}}(c, \gamma)$ from (5.3) and (5.4).

Next, we show $B_{\boldsymbol{T}}(c, \gamma) \geq m^{-1}_{\xi}(d_\xi)$, $^\forall\xi \in \mathcal{F}_\lambda$. Let $t_1 = m^{-1}_{\xi}(d_\xi)$ and let $t < t_1$. We find a distribution $H \in \mathcal{P}_{H_0}(c, \gamma)$ such that $\|\boldsymbol{T}(H)\| \geq t$. By Lemma 3.2(a), there exist $\boldsymbol{\theta}_t$ and $\alpha_t = \alpha(\boldsymbol{\theta}_t)$ such that $m_\xi(t) = J(c\,F^{\xi}_{\alpha_t,\boldsymbol{\theta}_t} + \gamma\delta_0)$. Define $\tilde{H}_n = \delta_{(y_n, \boldsymbol{x}_n)}$, where $\boldsymbol{x}_n = n\boldsymbol{\theta}_t$ and y_n is uniformly distributed on the interval $[\alpha_t + nt^2 - \frac{1}{n}, \alpha_t + nt^2 + \frac{1}{n}]$. If F_n is the uniform distribution function on $[-\frac{1}{n}, \frac{1}{n}]$, then for any $\boldsymbol{\beta} \in R^p$, $v > 0$ and $\alpha \in R$

$$F_{\tilde{H}_n,\alpha,\boldsymbol{\beta}}(u) \;=\; F_n(u + \alpha - \alpha_t - n(t^2 - \boldsymbol{\beta}'\boldsymbol{\theta}_t)) \tag{5.5}$$

$$-F_n(-u + \alpha - \alpha_t - n(t^2 - \boldsymbol{\beta}'\boldsymbol{\theta}_t)). \tag{5.6}$$

For any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ let $H^\xi_n(\alpha, \boldsymbol{\theta}) = c\,(H_0 - W_{\alpha,\boldsymbol{\theta}}) + \gamma\tilde{H}_n \in \mathcal{P}_{H_0}(c, \gamma)$. Suppose that $\sup_{n,\alpha,\boldsymbol{\theta}} \|\boldsymbol{T}(H^\xi_n(\alpha, \boldsymbol{\theta}))\| < t$ to find a contradiction. Then, for any $\alpha \in R$ and $\boldsymbol{\theta} \in R^p$ there exists a convergent subsequence, $\{\boldsymbol{T}(H^\xi_n(\alpha, \boldsymbol{\theta}))\}$, such that

$$\lim_{n\to\infty} \boldsymbol{T}(H^\xi_n(\alpha, \boldsymbol{\theta})) = \lim_{n\to\infty} \boldsymbol{\theta}^\xi_n(\alpha, \boldsymbol{\theta}) = \tilde{\boldsymbol{\theta}}^\xi(\alpha, \boldsymbol{\theta}), \quad where \quad \|\tilde{\boldsymbol{\theta}}^\xi(\alpha, \boldsymbol{\theta})\| = \tilde{t}^\xi(\alpha, \boldsymbol{\theta}) < t.$$

Since $t^2 - \boldsymbol{\theta}'_t\boldsymbol{\theta}_t = 0$, it follows from (5.6) that

$$\lim_{n\to\infty} F_{\tilde{H}_n,\alpha_t,\boldsymbol{\theta}_t}(u) = 1, \quad ^\forall u > 0. \tag{5.7}$$

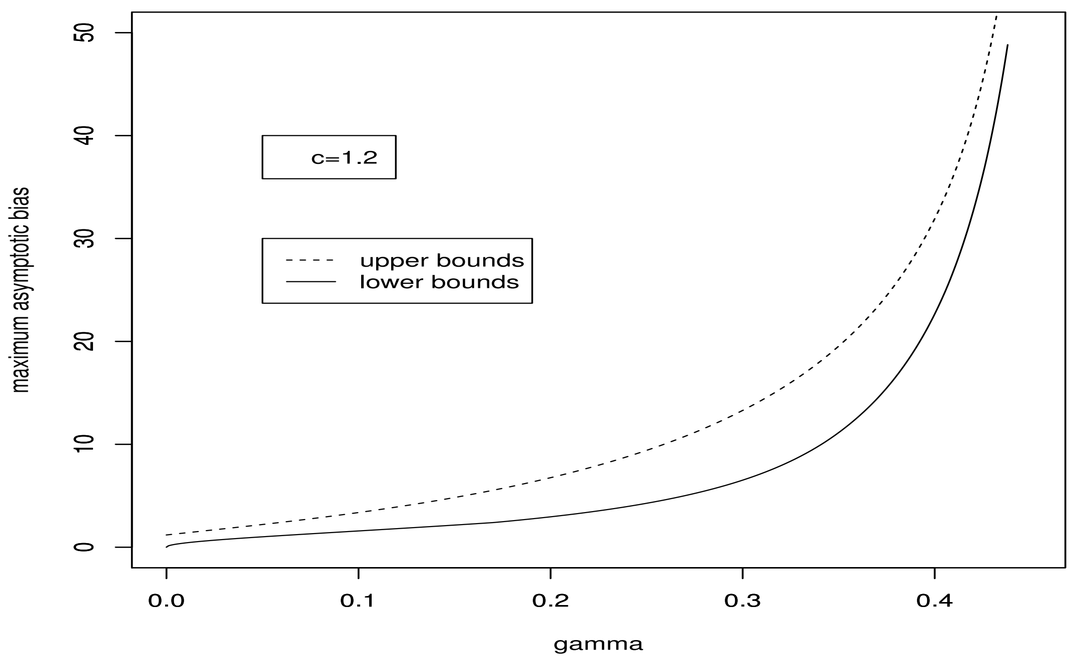

Figure 2: Lower and upper bounds of $B_\tau(c,\gamma)$ for c=1.2 (Tukey's biweight ρ-function)

We show that for any $\alpha \in R$ and $\boldsymbol{\theta} \in R^p$ the subsequence of intercepts corresponding to $\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})$, denoted by $\{T_0(H_n^\xi(\alpha,\boldsymbol{\theta}))\} = \{\alpha_n^\xi(\alpha,\boldsymbol{\theta})\}$ converges to a finite $\tilde{\alpha}^\xi(\alpha,\boldsymbol{\theta})$. To do this, assume $\lim_{n\to\infty} |\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*)| = \infty$ for some $\alpha^* \in R$ and $\boldsymbol{\theta}^* \in R^p$. Then, it follows from (5.7) that

$$\begin{aligned} &\lim_{n\to\infty} F_{H_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\boldsymbol{\theta}_n^\xi(\alpha^*,\boldsymbol{\theta}^*)}(u) \\ &= \gamma \lim_{n\to\infty} F_{\tilde{H}_n,\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\boldsymbol{\theta}_n^\xi(\alpha^*,\boldsymbol{\theta}^*)}(u) \qquad (5.8) \\ &< c\,F_{(H_0-W_{\alpha^*,\boldsymbol{\theta}^*}),\alpha_t,\boldsymbol{\theta}_t}(u) + \gamma\delta_0(u) \qquad (5.9) \\ &= \lim_{n\to\infty} F_{H_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\alpha_t,\boldsymbol{\theta}_t}(u), \quad {}^\forall u > 0. \end{aligned}$$

Hence, by A1(b) we have

$$J(F_{H_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\boldsymbol{\theta}_n^\xi(\alpha^*,\boldsymbol{\theta}^*)}) > J(F_{H_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\alpha_t,\boldsymbol{\theta}_t})$$

for large enough n. This fact contradicts the definition of $(\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\boldsymbol{\theta}_n^\xi(\alpha^*,\boldsymbol{\theta}^*))$. Therefore, for any α and $\boldsymbol{\theta}$ we have $\lim_{n\to\infty} |\alpha_n^\xi(\alpha,\boldsymbol{\theta})| = \tilde{\alpha}^\xi(\alpha,\boldsymbol{\theta}) < \infty$. Since $t^2 - |\boldsymbol{\theta}_t'\tilde{\boldsymbol{\theta}}^\xi(\alpha,\boldsymbol{\theta})| = t^2 - t\tilde{t}^\xi(\alpha,\boldsymbol{\theta}) > 0$, it follows from (5.6) that

$$\lim_{n\to\infty} F_{\tilde{H}_n,\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})}(u) = 0, \quad {}^\forall u > 0. \qquad (5.10)$$

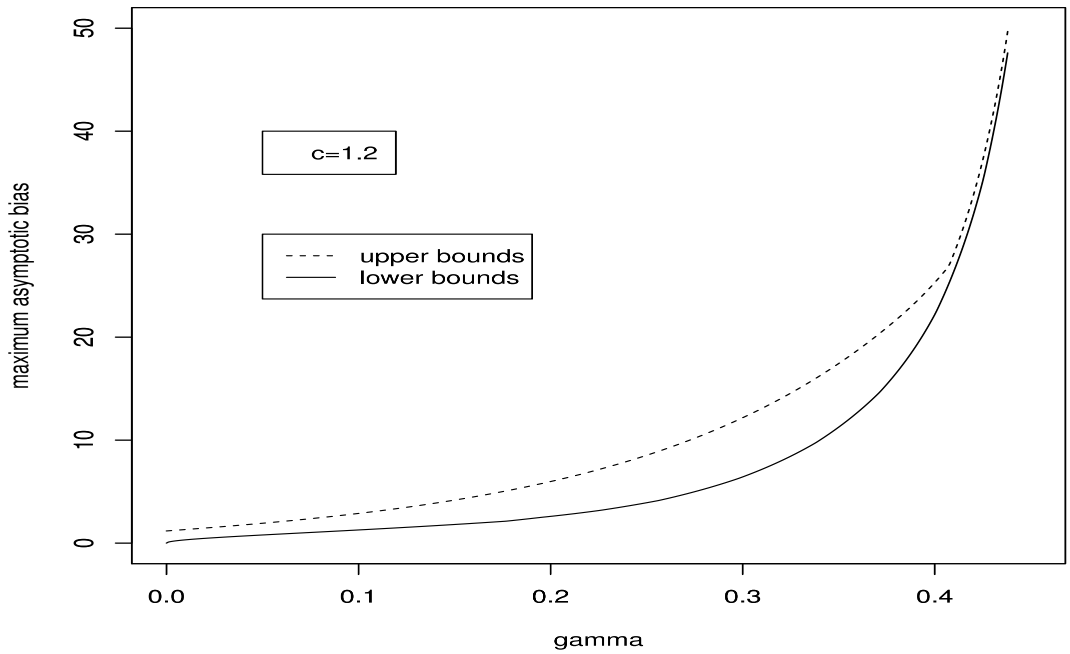

Figure 3: Lower and upper bounds of $B_{CM}(c,\gamma)$ for c=1.2 (Tukey's biweight ρ-function)

Hence, by (5.10) and $\xi \in \mathcal{F}_\lambda$ we have

$$\begin{aligned}\lim_{n\to\infty} F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})}(u) &= c\,F^\xi_{\tilde{\alpha}^\xi(\alpha,\boldsymbol{\theta}),\tilde{\boldsymbol{\theta}}^\xi(\alpha,\boldsymbol{\theta})}(u) \\ &\leq c\,F^\xi_{0,\mathbf{0}}(u) \qquad (5.11)\\ &= \lim_{n\to\infty}[c\,F^\xi_{0,\mathbf{0}}(u)+\gamma U_n(u)], \quad {}^\forall u>0.\end{aligned}$$

By A1(b) and A1(c) we have

$$\begin{aligned}\lim_{n\to\infty} J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})}) &\geq \lim_{n\to\infty} J(c\,F^\xi_{0,\mathbf{0}}+\gamma U_n) \qquad (5.12)\\ &= d_\xi = m_\xi(t_1).\end{aligned}$$

From (5.7) it follows that

$$\lim_{n\to\infty} F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_t,\boldsymbol{\theta}_t}(u) = c\,F^\xi_{\alpha_t,\boldsymbol{\theta}_t}(u)+\gamma\delta_0(u) \qquad (5.13)$$

The equation (5.13) and Lemma 3.2(b) imply

$$\lim_{n\to\infty} J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_t,\boldsymbol{\theta}_t}) = J(c\,F^\xi_{\alpha_t,\boldsymbol{\theta}_t}+\gamma\delta_0) = m_\xi(t) < m_\xi(t_1). \qquad (5.14)$$

By (5.12) and (5.14), we have

$$J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})}) > J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_t,\boldsymbol{\theta}_t})$$

for large enough n. This inequality is a contradiction because of

$$(\alpha_n^\xi(\alpha, \boldsymbol{\theta}), \boldsymbol{\theta}_n^\xi(\alpha, \boldsymbol{\theta})) = \arg\min_{\eta, \boldsymbol{\beta}} J(F_{H_n^\xi(\alpha, \boldsymbol{\theta}), \eta, \boldsymbol{\beta}}).$$

Thus, for any $t < t_1$ we obtain $\sup_{n, \alpha, \boldsymbol{\theta}} \|\boldsymbol{T}(H_n^\xi(\alpha, \boldsymbol{\theta}))\| \geq t$. This completes the proof. $\square$

Proof of Theorem 4.1. It follows from (4.3) that

$$d_{\xi^*} = S(c\, F_{0,\mathbf{0}}^{\xi^*} + \gamma\delta_\infty) = g_{\xi^*,1}^{-1}\left(\frac{b-\gamma}{c}\right)$$

and

$$m_{\hat{\xi},S}(\|\boldsymbol{\theta}\|) = S(c\, F_{0,\boldsymbol{\theta}}^{\hat{\xi}} + \gamma\delta_0) = \sqrt{1 + \|\boldsymbol{\theta}\|^2} g_{\hat{\xi},1}^{-1}\left(\frac{b}{c}\right).$$

Hence, solving $m_{\hat{\xi},S}(\|\boldsymbol{\theta}\|) = d_{\xi^*}$ in $\|\boldsymbol{\theta}\|$, we obtain (4.5). Similarly, we can obtain (4.4). Assume $b \leq 0.5$. Then we have $\min(b, 1-b) = b$,

$$\lim_{\gamma \uparrow b} g_{\xi^*,1}^{-1}\left(\frac{b-\gamma}{c}\right) = \infty \quad \text{and} \quad \lim_{\gamma \uparrow b} g_{\xi^\circ,1}^{-1}\left(\frac{b-\gamma}{c}\right) = \infty,$$

where $\xi^\circ = \{W_{\alpha,\boldsymbol{\theta}}^\circ\}$, $W_{\alpha,\boldsymbol{\theta}}^\circ = [(c + \gamma - 1)/c]\, H_0$. Therefore

$$\lim_{\gamma \uparrow b} \overline{B}_S(c, \gamma) = \lim_{\gamma \uparrow b} \underline{B}_S(c, \gamma) = \infty.$$

This completes the proof. $\square$

Proof of Theorem 4.2. It is seen from (4.6) that

$$\begin{aligned} d_{\xi^*} &= \tau^2(c\, F_{0,\mathbf{0}}^{\xi^*} + \gamma\delta_\infty) \\ &= \left[g_{\xi^*,1}^{-1}\left(\frac{b-\gamma}{c}\right)\right]^2 \left[c\, \overline{g}_{\xi^*}\left(\frac{b-\gamma}{c}\right) + \gamma\right] \end{aligned}$$

and that

$$\begin{aligned} m_{\hat{\xi},\tau}(\|\boldsymbol{\theta}\|) &= \tau^2(c\, F_{0,\boldsymbol{\theta}}^{\hat{\xi}} + \gamma\delta_0) \\ &= m_{\hat{\xi},S}^2(\|\boldsymbol{\theta}\|) \cdot c\, E_{F_{0,\boldsymbol{\theta}}^{\hat{\xi}}}\left[\rho_2\left(\frac{y - \boldsymbol{\theta}'\boldsymbol{x}}{m_{\hat{\xi},S}(\|\boldsymbol{\theta}\|)}\right)\right] \\ &= (1 + \|\boldsymbol{\theta}\|^2)\left[g_{\hat{\xi},1}^{-1}\left(\frac{b}{c}\right)\right]^2 c\, \overline{g}_{\hat{\xi}}\left(\frac{b}{c}\right). \end{aligned}$$

Solving $m_{\hat{\xi},\tau}(\|\boldsymbol{\theta}\|) = d_{\xi^*}$, we obtain.

$$\|\boldsymbol{\theta}\| = m_{\hat{\xi},\tau}^{-1}(d_{\xi^*}) = \{(1 + \overline{B}_S(c, \gamma)^2) H_{\xi^*,\hat{\xi}}(c, \gamma) - 1\}^{1/2}.$$

which implies (4.8). Similarly, we can obtain (4.7). $\square$

Proof of Theorem 4.3. It is easily seen from (4.9) that

$$\begin{aligned} d_{\xi^*} &= CM(c\,F_{0,\mathbf{0}}^{\xi^*}+\gamma\delta_\infty) \\ &= \inf_{s\geq\sigma(cF_{0,\mathbf{0}}^{\xi^*}+\gamma\delta_\infty)}\left\{aE_{cF_{0,\mathbf{0}}^{\xi^*}+\gamma\delta_\infty}\left[\rho_1\left(\frac{u}{s}\right)\right]+\log s\right\} \\ &= \inf_{s\geq\sigma_{b,c,\gamma}^{\xi^*}} A_{a,c,\gamma}^{\xi^*}(s)+a\gamma. \end{aligned}$$

We also see that

$$\begin{aligned} m_{\hat{\xi},CM}(\|\boldsymbol{\theta}\|) &= CM(c\,F_{0,\boldsymbol{\theta}}^{\hat{\xi}}+\gamma\delta_0) \\ &= \inf_{s\geq(1+\|\boldsymbol{\theta}\|^2)^{1/2}\eta_{b,c,\gamma}^{\hat{\xi}}}\left\{acg_{\hat{\xi},1}\left(\frac{s}{(1+\|\boldsymbol{\theta}\|^2)^{1/2}}\right)+\log s\right\} \\ &= \inf_{s\geq\eta_{b,c,\gamma}^{\hat{\xi}}} A_{a,c,\gamma}^{\hat{\xi}}(s)+\frac{1}{2}\log(1+\|\boldsymbol{\theta}\|^2). \end{aligned}$$

Therefore, it follows from $m_{\hat{\xi},CM}(\|\boldsymbol{\theta}\|)=d_{\xi^*}$ that

$$\|\boldsymbol{\theta}\|=\{\exp[2a\gamma+2D_{\xi^*,\hat{\xi},a}(c,\gamma)]-1\}^{1/2}.$$

which is (4.12).

Similarly, we obtain (4.11). $\square$

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