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Abstract

By using normal forms and exact models, Sasaki (2010a) provided a detailed description of the mutual relation of formulas with finite propositional variables $p_1, \cdots, p_m$ in the modal logic **S4**. The description contains more information on **S4** than those given in Shehtman (1978) and Moss (2007). In the present paper, we extend most of the results in Sasaki (2010a) to normal modal logics containing the modal logic **K4**. Also, we point out the exact models listed in Sasaki (2010a) are only the exact **S4**-models.

1 Introduction

In the following three subsections, we introduce formulas, sequents, normal modal logics, and some types of Kripke models. Also, in subsection 1.4, we describe the purpose of the present paper.

1.1 Formulas and sequents

Formulas are constructed from $\bot$ (contradiction) and the propositional variables $p, q, p_1, p_2, \cdots$ by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and $\Box$ (necessitation). We use upper case Latin letters, $A, B, C, \cdots$, with or without subscripts, for formulas. We refer to $\neg A$ as $A \supset \bot$. Also, we use Greek letters, $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. The *depth* $d(A)$ of a formula A is defined as

$d(p_i) = d(\bot) = 0,$
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\},$
$d(\Box B) = d(B) + 1.$

The set of propositional variables $p_1, \cdots, p_m$ $(m \geq 1)$ is denoted by $\mathbf{V}$ and the set of formulas constructed from $\mathbf{V}$ and $\bot$ is denoted by $\mathbf{F}$. Also, for any $n = 0, 1, \cdots$, we define $\mathbf{F}(n)$ as $\mathbf{F}(n) = \{A \in \mathbf{F} \mid d(A) \leq n\}$. In the present paper, we mainly treat the sets $\mathbf{F}$ and $\mathbf{F}(n)$.

A *sequent* is the expression $(\Gamma \to \Delta)$. We often refer to

$$A_1, \cdots, A_i, \Gamma_1, \cdots, \Gamma_j \to \Delta_1, \cdots, \Delta_k, B_1, \cdots, B_\ell$$

as

$$(\{A_1, \cdots, A_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \cdots, B_\ell\}).$$

We use upper case Latin letters $X, Y, Z, \cdots$, with or without subscripts, for sequents. The *antecedent* $\mathbf{ant}(\Gamma \to \Delta)$ and the *succedent* $\mathbf{suc}(\Gamma \to \Delta)$ of a sequent $\Gamma \to \Delta$ are defined as

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \to \Delta) = \Delta,$$

respectively. Also, for a sequent X and a set S of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(S)$ as

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) \neq \emptyset \\ \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) = \emptyset \end{cases}$$

and

$$\mathbf{for}(S) = \{\mathbf{for}(X) \mid X \in S\}.$$

By the equivalence between $(A_1 \wedge \cdots \wedge A_k) \supset B$ and $\neg A_1 \vee \cdots \vee \neg A_k \vee C$ in the classical propositional logic $\mathbf{CL}$, the set of sequent

$$\mathbf{ED} = \mathbf{for}(\{\mathbf{V} - V \to V \mid V \subseteq \mathbf{V}\})$$

means the set of elementary disjunctions

$$\{p_1^* \vee \cdots \vee p_m^* \mid p_i \in \{p_i, \neg p_i\}\}$$

in $\mathbf{CL}$.

1.2 Normal modal logics

A normal modal logic is a set of formulas containing all tautologies and the axiom $\Box(p \supset q) \supset (\Box p \supset \Box q)$; and closed under modus ponens, substitution, and necessitation $(A/\Box A)$. By $\mathbf{K}$, we mean the smallest normal modal logic. For a normal modal logic L and a formula A, we refer to $L + A$ as the smallest normal modal logic including $L \cup \{A\}$. The normal modal logics $\mathbf{K4}$ and $\mathbf{S4}$ are defined as

$$\mathbf{K4} = \mathbf{K} + \Box p \supset \Box\Box p \quad \text{and} \quad \mathbf{S4} = \mathbf{K4} + \Box p \supset p.$$

For a normal modal logic L, we use $A \equiv_L B$ instead of $(A \supset B) \wedge (B \supset A) \in L$ and use $[A] \leq_L [B]$ instead of $A \to B \in L$. Thus, the structures

$$\langle \mathbf{F}(n)/ \equiv_L, \leq_L \rangle \quad \text{and} \quad \langle \mathbf{F}/ \equiv_L, \leq_L \rangle$$

express the mutual relation of formulas in $\mathbf{F}(n)$ and $\mathbf{F}$, respectively. In $\mathbf{CL}$, a conjunctive normal form $\bigwedge S$ for $S \in 2^{\mathbf{ED}}$ is a canonical representative of the quotient set. Specifically, the following two conditions hold:

- $\mathbf{F}(0)/ \equiv_{\mathbf{CL}} = \{[\bigwedge \mathbf{for}(S)] \mid S \subseteq \mathbf{ED}\}$,

- for any subsets S_1 and S_2 of **ED**, $[\bigwedge \mathbf{for}(S_1)] \leq_{\mathbf{CL}} [\bigwedge \mathbf{for}(S_2)]$ if and only if $S_2 \subseteq S_1$

In order to treat normal modal logics, we often use sequents. We say that a sequent X is provable in L, write $X \in L$, if $\mathbf{for}(X) \in L$. In this sense, every inference rule in the sequent system **LK** for **CL** given by Gentzen (1934-35) holds. Also, the rule

$$\frac{\Gamma, \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\Box)$$

holds in a normal modal logic containing **K4**.

1.3 Kripke models

Here, we introduce a Kripke model and an exact model, which are useful to investigate the structures $\langle \mathbf{F}(n)/\equiv_L, \leq_L \rangle$ and $\langle \mathbf{F}/\equiv_L, \leq_L \rangle$.

A *Kripke model* is a structure $\langle W, R, P \rangle$, where W is a non-empty set, R is a binary relation on W, and P is a mapping from the set of propositional variables to 2^W. We extend, as usual, the domain of P to include all formulas. We call a member of W a *world*. For a Kripke model $M = \langle W, R, P \rangle$, and for a world $\alpha \in W$, we often write $(M, \alpha) \models A$ and $M \models A$, instead of $\alpha \in P(A)$ and $P(A) = W$, respectively. For a set S of formulas, we say that a Kripke model M is a S-model, write $M \models S$, if $M \models B$ for any $B \in S$.

The following lemma is described in several articles (for example in Chagrov & Zakharyaschev (1997)):

Lemma 1.1 *$\to A \in$ **K4** if and only if $M \models A$ for any transitive Kripke model.*

Below, we define an exact model. This is an extension from the exact model introduced in de Bruijn (1975) in order to treat the cases that W is infinite. For details, one can consult subsection 1.3 in Sasaki (2010a).

Definition 1.2 *Let S be a set of formulas closed under $\supset$ and $\wedge$. We say that a Kripke model $M = \langle W, R, P \rangle$ is exact for S in a modal logic L if the following two conditions hold:*

(1) *for any $A \in S$, $M \models A$ if and only if $\to A \in L$,*

(2) *$\{\{\alpha\} \mid \alpha \in W\} \subseteq \{P(A) \mid A \in S\}$.*

The following lemmas is observed easily; therefore, exact models are useful to investigate the structures $\langle \mathbf{F}/\equiv, \leq_L \rangle$ and $\langle \mathbf{F}(n)/\equiv, \leq_L \rangle$.

Lemma 1.3 *Let S be a set of formulas closed under $\supset$ and $\wedge$ and let $\langle W, R, P \rangle$ be an exact model for S in L. Then the mapping P^* from $S/\equiv_L$ to $\{P(A) \mid A \in S\}$ defined as*

$$P^*([A]) = P(A)$$

is an isomorphism and the structure $\langle S/\equiv_L, \leq_L \rangle$ is isomorphic to the structure $\langle \{P(A) \mid A \in S\}, \subseteq \rangle$. Moreover, $\{P(A) \mid A \in S\} = 2^W$ if $S/\equiv_L$ is finite.

Lemma 1.3 can be achieved without using (2) in Definition 1.2 However, by (2), we observe that an exact model is one of the simplest Kripke models satisfying the condition in Lemma 1.3

We note that there may exist an exact model for $\mathbf{F}(n)$ in L, which is not L-model. However, Sasaki (2010a), in section 4, did not describe this existence and only treated exact $\mathbf{S4}$-models for $\mathbf{F}(n)$ in $\mathbf{S4}$. In other words, Sasaki (2010a) gave a method to list all exact $\mathbf{S4}$-models for $\mathbf{F}(n)$ in $\mathbf{S4}$, but did not give a method to list all exact models for $\mathbf{F}(n)$ in $\mathbf{S4}$.

1.4 The purpose

The purpose of the paper is to provide a detailed description of the mutual relation of formulas in $\mathbf{F}$ in a normal modal logic L containing $\mathbf{K4}$. In other words, we provide a detailed description of the structures $\langle \mathbf{F}(n)/\equiv, \leq_L\rangle$ and $\langle \mathbf{F}/\equiv, \leq_L\rangle$ by extending the results in Sasaki (2010a).

There are also many previous works on this topic. Finite structures like $\langle \mathbf{F}(n)/\equiv_L, \leq\rangle$ have been studied previously(e. g. Diego (1966), Urquhart (1974), de Bruijn (1975), Hendriks (1996), Sasaki (2001), and Moss (2007)). Infinite structures like $\langle \mathbf{F}/\equiv_L, \leq\rangle$ have also been studied in many articles(e. g. Rieger (1949), Nishimura (1960), Urquhart (1973), Esakia & Grigolia (1975), Esakia & Grigolia (1977), Shehtman (1978), Bellissima (1985), Ghilardi (1995)). We wrote about these works in Sasaki (2010a).

In Sasaki (2010a), we treated the case that $L = \mathbf{S4}$. Let us list the results in Sasaki (2010a).

(I) We gave a construction of a finite set $\mathbf{ED}(n)$ of sequents satisfying

- (I-1) $\mathbf{F}(n)/\equiv_{\mathbf{S4}} = \{[\bigwedge \mathbf{for}(S)] \mid S \subseteq \mathbf{ED}(n)\}$,
- (I-2) for any subsets S_1 and S_2 of $\mathbf{ED}(n)$, $[\bigwedge \mathbf{for}(S_1)] \leq_{\mathbf{S4}} [\bigwedge \mathbf{for}(S_2)]$ if and only if $S_1 \subseteq S_2$.

(II) We gave another construction of $\mathbf{ED}(n)$ without using $\mathbf{S4}$-provability(the construction in (I) depends on $\mathbf{S4}$-provability for a kind of sequents).

(III) By clarifying a relation between $\mathbf{ED}(k)$ and $\mathbf{ED}(k+1)$, we gave a finite method to find $S \in 2^{\mathbf{ED}(n)}$ satisfying $A \equiv_{\mathbf{S4}} \bigwedge \mathbf{for}(S)$ for each $A \in \mathbf{F}(n)$.

(IV) We define the sets $\mathbf{CNF}$ and $S\Downarrow$ for $S \in 2^{\mathbf{ED}(n)}$; and clarified $\langle \mathbf{F}/\equiv, \leq_L\rangle$ by showing two conditions corresponding to (I-1) and (I-2).

(V) We constructed the exact model $\langle W_{\mathbf{S4}}, R_{\mathbf{S4}}, P_{\mathbf{S4}}\rangle$ for $\mathbf{F}$ in $\mathbf{S4}$ and proved

- (V-1) $\{P_{\mathbf{S4}}(A) \mid A \in \mathbf{F}\} = \{W_{\mathbf{S4}} - S\Downarrow \mid S \in \mathbf{CNF}\}$.

(VI) We gave a method to list all exact $\mathbf{S4}$-models for $\mathbf{F}(n)$ in $\mathbf{S4}$.

(VII) By using each exact $\mathbf{S4}$-model for $\mathbf{F}(n)$ in $\mathbf{S4}$, we gave another finite method to find $S \in 2^{\mathbf{ED}(n)}$ satisfying $A \equiv_{\mathbf{S4}} \bigwedge \mathbf{for}(S)$ for each $A \in \mathbf{F}(n)$.

In the next section, we treat (I) and (III). In section 3, we treat (IV). In section 4, we treat (V); and in section 5, we treat (VI) and (VII). Here, we do not treat the construction in (II). Sasaki (2010b) gave such construction for **K4**, but not in general.

2 A construction of $\mathbf{ED}_L(n)$ and a clarification of $\langle \mathbf{F}(n)/\equiv_L, \leq_L \rangle$

In the present section, we extend (I) and (III) in subsection 1.4, to normal modal logics containing **K4**. In the rest of the present paper, we let L be a normal modal logic containing **K4**.

First, we construct $\mathbf{ED}_L(n)$.

Definition 2.1 *The sets $\mathbf{G}_L(n)$ and $\mathbf{G}^*_L(n)$ of sequents are defined inductively as follows.*

- $\mathbf{G}_L(0) = \{(\mathbf{V} - V \to V) \mid V \subseteq \mathbf{V}\}$,
- $\mathbf{G}^*_L(0) = \emptyset$,
- $\mathbf{G}_L(k+1) = \bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}_L(X)$,
- $\mathbf{G}^*_L(k+1) = \{X \in \mathbf{G}_L(k+1) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y))^\Box$ *implies* $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, *for any* $Y \in \mathbf{G}_L(k+1)\}$,

for any $X \in \mathbf{G}_L(k)$,

- $\mathbf{next}_L(X) = \{Y \in \mathbf{next}^+_L(X) \mid Y \notin L\}$,
- $\mathbf{next}^+_L(X) = \{\mathbf{n}_L(X, S) \mid S \subseteq \mathbf{G}_L(k)\}$,
- $\mathbf{n}_L(X, S) = (\Box\mathbf{for}(\mathbf{G}_L(n) - S), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(S))$.

Definition 2.2 *We define the sets $\mathbf{ED}_L(n)$ and $\mathbf{G}^+_L(n)$ as*

$$\mathbf{ED}_L(n) = \mathbf{G}_L(n) \cup \bigcup_{i=0}^{n-1} \mathbf{G}^*_L(i),$$

$$\mathbf{G}^+_L(n) = \begin{cases} \mathbf{G}_L(0) & \text{if } n = 0 \\ \bigcup_{X \in \mathbf{G}_L(n-1) - \mathbf{G}^*_L(n-1)} \mathbf{next}^+_L(X) & \text{if } n > 0. \end{cases}$$

If $L = \mathbf{S4}$, then the above construction is almost same as the construction in Sasaki (2010a). The only one difference is the definition of $\mathbf{next}^+_{\mathbf{S4}}(X)$. Specifically, in Sasaki (2010a), we defined it as $\{\mathbf{n}_{\mathbf{S4}}(X, S) \mid S \subseteq \mathbf{G}_{\mathbf{S4}}(k), X \in S\}$. However, every member in $\{\mathbf{n}_{\mathbf{S4}}(X, S) \mid S \subseteq \mathbf{G}_{\mathbf{S4}}(k), X \notin S\}$ is provable in **S4**, and thus, the set $\mathbf{next}_{\mathbf{S4}}(X)$ above and the corresponding set in Sasaki (2010a) are the same set. Hence, we can treat the above construction as an extension of the construction in Sasaki (2010a).

Also, in Sasaki (2010a), we wrote that the construction is based on the construction of the normal forms in Fine (1975) and that there are three differences between Fine's normal forms and $\mathbf{ED}_L(n)$. However, there is one more difference between them. The difference is basically the difference between $\mathbf{n}_L(X,S)$ and the sequent $(\Box\mathbf{for}(\mathbf{G}_L(n)-S), \mathbf{ant}(X)\cap\mathbf{V} \to \mathbf{suc}(X)\cap\mathbf{V}, \Box\mathbf{for}(S))$. This is also a difference between our construction and the construction in Moss (2007), and makes it easier to compare $\mathbf{ED}_L(n)$ with $\mathbf{ED}_L(n+1)$. From this, we obtain the idea to define **CNF** in the next section. Moreover, Sasaki (2010a) defined $\mathbf{Fine}(n)$ as the set of normal forms corresponding to Fine in a sequent style, but there is the same kind of difference between $\mathbf{Fine}(n)$ and the set of such normal forms in a sequent style.

In order to treat $\mathbf{ED}_L(n)$, it is convenient to distinguish two types of members:

$$(\cdots\Box\mathbf{for}(X), \mathbf{ant}(X) \to \mathbf{suc}(X)\cdots) \quad \text{and} \quad (\cdots\mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(X)\cdots).$$

To do so, we define $X_\ominus$, $\mathbf{G}^\circ_L(n)$, and $\mathbf{G}^\bullet_L(n)$ as follows.

Let X be a sequent in $\mathbf{G}^+_L(n+1)$. Then there exists only one sequent $Y \in \mathbf{G}_L(n) - \mathbf{G}^*_L(n)$ such that $X \in \mathbf{next}^+_L(Y)$. We refer to $X_\ominus$ as this sequent Y. We note that $X_\ominus \in \mathbf{G}_L(n) - \mathbf{G}^*_L(n)$ and $X \in \mathbf{next}_L(X_\ominus)$.

Definition 2.3 *We define the sets $\mathbf{G}^\circ_L(n)$ and $\mathbf{G}^\bullet_L(n)$ as*

$$\mathbf{G}^\circ_L(n) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{X \in \mathbf{G}^*_L(n) \mid \Box\mathbf{for}(X_\ominus) \in \mathbf{suc}(X)\} & \text{if } n > 0, \end{cases}$$

$$\mathbf{G}^\bullet_L(n) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{X \in \mathbf{G}^*_L(n) \mid \Box\mathbf{for}(X_\ominus) \in \mathbf{ant}(X)\} & \text{if } n > 0. \end{cases}$$

Example 2.4 *We list members of $\mathbf{G}_L(n)$, $\mathbf{G}^\bullet_L(n)$, and $\mathbf{G}^\circ_L(n)$ in the case that $L = \mathbf{K4}$, $m = 1$, and $n = 0, 1, 2$. We use $(\quad)^\bullet$ and $(\quad)^\circ$ for a sequent in $\mathbf{G}^\bullet_L(n)$ and $\mathbf{G}^\circ_L(n)$, respectively.*

$\mathbf{G}_L(0) = \{T, F\}$,

$\mathbf{next}^+_L(T) = \mathbf{next}_L(T) = \{T1^\bullet, T2, T3, T4\}$,

$\mathbf{G}_L(1) = \{T1^\bullet, T2, T3, T4, F1^\bullet, F2, F3, F4\}$,

$\mathbf{G}^\bullet_L(1) = \{T1^\bullet, F1^\bullet\}$, $\mathbf{G}^\circ_L(1) = \emptyset$,

$\mathbf{next}_L(T2) = \{T2.1^\bullet, T2.2^\bullet, T2.3\}$,

$\mathbf{next}_L(T3) = \{T3.1^\bullet, T3.2^\circ, T3.3\}$,

$\mathbf{next}_L(T4) \ni T4.1^\circ$,

$\mathbf{G}^\bullet_L(2) = \{T2.1^\bullet, T2.2^\bullet, T3.1^\bullet, F2.1^\bullet, F3.1^\bullet, F3.2^\bullet\}$,

$\mathbf{G}^\circ_L(2) = \{T3.2^\circ, T4.1^\circ, F2.2^\circ, F4.1^\circ\}$,

Table 1: Members of $\mathbf{G}_L(n)$ in the case that $L = \mathbf{K4}$, $m = 1$, and $n = 0, 1, 2$

$$
\begin{array}{lcllcl}
T & = & (p_1 \rightarrow), & F & = & (\rightarrow p_1), \\
T1^{\bullet} & = & \mathbf{n}_L(T, \emptyset), & F1^{\bullet} & = & \mathbf{n}_L(F, \emptyset), \\
T2 & = & \mathbf{n}_L(T, \{F\}), & F2 & = & \mathbf{n}_L(F, \{F\}), \\
T3 & = & \mathbf{n}_L(T, \{T\}), & F3 & = & \mathbf{n}_L(F, \{T\}), \\
T4 & = & \mathbf{n}_L(T, \{T, F\}), & F4 & = & \mathbf{n}_L(F, \{T, F\}), \\
T2.1^{\bullet} & = & \mathbf{n}_L(T2, \{F1^{\bullet}\}), & F2.1^{\bullet} & = & \mathbf{n}_L(F2, \{F1^{\bullet}\}), \\
T2.2^{\bullet} & = & \mathbf{n}_L(T2, \{F2\}), & F2.2^{\circ} & = & \mathbf{n}_L(F2, \{F2\}), \\
T2.3 & = & \mathbf{n}_L(T2, \{F1^{\bullet}, F2\}), & F2.3 & = & \mathbf{n}_L(F2, \{F1^{\bullet}, F2\}), \\
T3.1^{\bullet} & = & \mathbf{n}_L(T3, \{T1^{\bullet}\}), & F3.1^{\bullet} & = & \mathbf{n}_L(F3, \{T1^{\bullet}\}), \\
T3.2^{\circ} & = & \mathbf{n}_L(T3, \{T3\}), & F3.2^{\bullet} & = & \mathbf{n}_L(F3, \{T3\}), \\
T3.3 & = & \mathbf{n}_L(T3, \{T1^{\bullet}, T3\}), & F3.3 & = & \mathbf{n}_L(F3, \{T1^{\bullet}, T3\}), \\
T4.1^{\circ} & = & \mathbf{n}_L(T4, \{T4, F4\}), & F4.1^{\circ} & = & \mathbf{n}_L(F4, \{T4, F4\}).
\end{array}
$$

where $T, F, T1^{\bullet}, \cdots$ are the sequents in table 1.

By an induction on n, we can show the following lemma.

Lemma 2.5

(1) *None of the members in $\mathbf{G}_L(n)$ is provable in L.*

(2) *For any $X, Y \in \mathbf{ED}_L(n)$, $X \neq Y$ implies $\mathbf{for}(X) \vee \mathbf{for}(Y) \in L$.*

(3) *For any $X \in \mathbf{G}_L^{+}(n)$, $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}_L(i))$ and $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$.*

We use the above lemma without specific justification in the rest of the present paper.

The main purpose in the present section is to prove the following two theorems.

Theorem 2.6

(1) $\bot \equiv_L \bigwedge \mathbf{for}(\mathbf{ED}_L(n))$ *and* $p_i \equiv_L \bigwedge \mathbf{for}(\{X \in \mathbf{ED}_L(n) \mid p_i \in \mathbf{suc}(X)\})$.

(2) *For any subsets S_1 and S_2 of $\mathbf{ED}_L(n)$,*

- $\bigwedge \mathbf{for}(S_1) \wedge \bigwedge \mathbf{for}(S_2) \equiv_L \bigwedge \mathbf{for}(S_1 \cup S_2)$.

(3) *For any subset S of $\mathbf{ED}_L(k)$, $\Box \bigwedge \mathbf{for}(S) \equiv_L \bigwedge \mathbf{for}(S_1 \cup S_2)$, where*

- $S_1 = \bigcup_{X \in S} \{Y \in \mathbf{ED}_L(k+1) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y)\}$,

- $S_2 = \bigcup_{i=1}^{k} \bigcup_{X \in S \cap \mathbf{G}^{\circ}_L(i)} \{Y \in \mathbf{G}^*_L(i) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}\}$.

We note that $S_1 \cup S_2 \subseteq \mathbf{ED}_L(k+1)$ for S_1 and S_2 in the above (4).

Theorem 2.7

(1) $\mathbf{F}(n)/\equiv_L \; = \; \{[\bigwedge \mathbf{for}(S))] \mid S \subseteq \mathbf{ED}_L(n)\}$.

(2) *For subsets S_1 and S_2 of $\mathbf{ED}_L(n)$,*

$$S_1 \subseteq S_2 \textit{ if and only if } \bigwedge \mathbf{for}(S_2) \to \bigwedge \mathbf{for}(S_1) \in L.$$

By the above theorem, the conditions (I-1) and (I-2) in subsection 1.4 are shown. Theorem 2.6 provide a finite method described in (III) in subsection 1.4.

Theorem 2.7(1) and Theorem 2.7(2) can be shown by Theorem 2.6 and Lemma 2.5, respectively. In order to prove Theorem 2.6, especially (3), we need some lemmas.

Lemma 2.8 *Let Σ, Γ and Δ be finite sets of formulas. Then*

$$\{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L.$$

Proof. We use an induction on the number $\#(\Sigma)$ of elements in Σ. If $\Sigma = \emptyset$, then the lemma is clear. Suppose that $A \in \Sigma$. Then by the induction hypothesis,

$$\{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma - \{A\}, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L.$$

Therefore,

$$\Box A, \{\mathbf{for}(\Box\Phi, \Box A, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma - \{A\}, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L$$

and

$$\neg\Box A, \{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box A, \Box\Psi) \mid \Phi \cup \Psi = \Sigma - \{A\}, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L.$$

Using $\Box A \vee \neg\Box A \in L$, we obtain the lemma. ■

Corollary 2.9 *For any $X, Y \in \mathbf{G}_L(n)$,*

(1) $\mathbf{for}(\mathbf{next}_L(X)) \to \mathbf{for}(X) \in L$,

(2) $\bigwedge \mathbf{for}(\mathbf{next}_L(X)) \equiv_L \mathbf{for}(X)$,

(3) $\mathbf{for}(\{Z \in \mathbf{next}_L(X) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(Z)\}) \to \mathbf{for}(X), \Box\mathbf{for}(Y) \in L$.

Proof. We can show (3) by considering the case that

$$(\Sigma, \Gamma, \Delta) = (\mathbf{for}(\mathbf{G}_L(n) - \{Y\}), \mathbf{ant}(X), \mathbf{suc}(X) \cup \{\Box\mathbf{for}(Y)\})$$

in Lemma 2.8 (1) can be shown similarly. (2) is clear from (1). ■

Lemma 2.10 *Let X and Y be sequents in $\mathbf{G}_L(n)$. If $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$, then $(\to \mathbf{for}(X), \Box\mathbf{for}(Y)) \in L$.*

Proof. We can show the lemma as in the proof of Lemma 2.10 in Sasaki (2010a). ■

Lemma 2.11 *Let X and Y be sequents in $\mathbf{G}^*_L(n)$ and $\mathbf{G}^+_L(n)$, respectively. If $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$ and $\Box\mathbf{for}(Y_\ominus) \in \mathbf{suc}(Y)$, then $\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in L$ and $Y \in \mathbf{G}_L(n)$.*

Proof. By $\Box\mathbf{for}(Y_\ominus) \in \mathbf{suc}(Y)$, Corollary 2.9(1), Lemma 2.10 and $(\Box)$, we can show the lemma as in the proof of Lemma 2.11(2) in Sasaki (2010a). ■

Corollary 2.12 *Let X and Y be sequents in $\mathbf{G}_L(n)$ satisfying $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. Then*

(1) *$X \in \mathbf{G}^*_L(n)$ if and only if $Y \in \mathbf{G}^*_L(n)$,*

(2) *$Y \in \mathbf{G}^\circ_L(n)$ implies $\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in L$.*

Definition 2.13 *For any $X \in \mathbf{G}_L(n)$, we define the sets $\mathbf{pclus}_L(X)$ and $\mathbf{clus}_L(X)$ as*

$$\mathbf{pclus}_L(X) = \{Y \in \mathbf{G}_L(n) \mid (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box\},$$

$$\mathbf{clus}_L(X) = \begin{cases} \emptyset & \textit{if } n = 0 \\ \{Y \in \mathbf{pclus}_L(n) \mid \Box\mathbf{for}(Y_\ominus) \in \mathbf{suc}(Y)\} & \textit{if } n > 0. \end{cases}$$

By Corollary 2.12(1), if $X \in \mathbf{G}^*_L(n)$ $(n > 0)$, then

$$\mathbf{pclus}_L(X) \subseteq \mathbf{G}^*_L(n) \text{ and } \mathbf{clus}_L(X) \subseteq \mathbf{G}^\circ_L(n).$$

Also, as in Sasaki (2010a), if $X \in \mathbf{G}^\circ_L(n)$, then we will find that $\mathbf{clus}_L(X)$ is the cluster containing X of the Kripke model $\mathbf{EM}_L$ introduced in section 3.

Lemma 2.14 *For any $X \in \mathbf{G}_L(n)$,*

(1) *$X \notin \mathbf{G}^*_L(n)$ implies $\mathbf{G}_L(\subseteq, X) = \mathbf{G}^\triangle_L(\subseteq, X)$,*

(2) *$\mathbf{for}(\mathbf{G}_L(\subseteq, X)) \to \Box\mathbf{for}(X) \in L$,*

(3) *$X \notin \mathbf{G}^*_L(n)$ implies $(\mathbf{for}(\mathbf{G}^\triangle_L(\subseteq, X)) \to \Box\mathbf{for}(X)) \in L$,*

(4) *$X \in \mathbf{G}^*_L(n)$ implies $(\mathbf{for}(\mathbf{pclus}_L(X)), \mathbf{for}(\mathbf{G}^\triangle_L(\subseteq, X)) \to \Box\mathbf{for}(X)) \in L$,*

(5) *$X \in \mathbf{G}^\circ_L(n)$ implies $(\mathbf{for}(\mathbf{pclus}_L(X)), \mathbf{for}(\mathbf{G}^\triangle_L(\subseteq, X)) \to \Box\mathbf{for}(X)) \in L$,*

(6) *$X \in \mathbf{G}^\bullet_L(n)$ implies $(\mathbf{for}(\mathbf{G}^\triangle_L(\subseteq, X)) \to \Box\mathbf{for}(X)) \in L$,*

where

- $\mathbf{G}_L(\subseteq, X) = \{Y \in \mathbf{G}_L(n) \mid (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\}$,
- $\mathbf{G}^\triangle_L(\subseteq, X) = \mathbf{G}_L(\subseteq, X) - \mathbf{G}^*_L(n)$.

Proof. (1) can be shown by

$$X \notin \mathbf{G}^*_L(n) \text{ and } (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box \text{ imply } Y \notin \mathbf{G}^*_L(n).$$

(3) is clear from (1) and (2). (4) can be shown similarly to Lemma 2.14(4) in Sasaki (2010a). (5) is clear from (4). Therefore, we have only to show (2) and (6).

We show (2) by an induction on n. Basis ($n = 0$) is clear from $\mathbf{for}(\mathbf{G}_L(\subseteq, X)) \to \bot \in L$. We show Induction step ($n > 0$). By the induction hypothesis,

$$\mathbf{for}(\mathbf{G}_L(\subseteq, X_\ominus)) \to \Box\mathbf{for}(X_\ominus) \in L.$$

Using $X_\ominus \in \mathbf{G}_L(n-1) - \mathbf{G}^*_L(n-1)$, (1), and $\Box\mathbf{for}(X_\ominus) \to \mathbf{for}(X) \in L$, we have

$$\mathbf{for}(\mathbf{G}^\triangle_L(\subseteq, X_\ominus)) \to \Box\mathbf{for}(X) \in L. \tag{2.1}$$

In order to prove (2), we show

$$\Sigma, \mathbf{for}(\mathbf{G}_L(\subseteq, X)) \to \Box\mathbf{for}(X) \in L \tag{2.2}$$

for any subset Σ of $\mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1))$. In order to show (2.2), we use an induction on $\#(\mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)) - \Sigma)$. We show Basis ($\Sigma = \mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1))$) and Induction step ($\Sigma \subsetneq \mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1))$) simultaneously. We define the set Ψ as

$$\Psi = \bigcup_{Y' \in \mathbf{G}^\triangle_L(\subseteq, X_\ominus)} \mathbf{for}(\{Y \in \mathbf{next}^+(Y') \mid \mathbf{ant}(Y) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)) = \Sigma\}).$$

We note that $\Psi - L \subseteq \mathbf{for}(\mathbf{G}_L(\subseteq, X))$ and for any $\mathbf{for}(Y) \in \Psi$,

$$\begin{aligned} & \mathbf{suc}(Y) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)) \\ = & \Box\mathbf{for}(\mathbf{G}_L(n-1)) - \Sigma \\ = & ((\mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1))) \cup (\mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)))) - \Sigma \\ = & (\mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1))) \cup (\mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)) - \Sigma). \end{aligned} \tag{2.3}$$

We can easily observe that

$$A \to \Box\mathbf{for}(X) \in L \quad \text{for any } A \in \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)). \tag{2.4}$$

Also, by the induction hypothesis, we have

$$A, \Sigma, \mathbf{for}(\mathbf{G}_L(\subseteq, X)) \to \Box\mathbf{for}(X) \in L \quad \text{for any } A \in \mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)) - \Sigma. \tag{2.5}$$

By (2.1), (2.3), (2.4) and (2.5), we have

$$\Sigma, \mathbf{for}(\mathbf{G}_L(\subseteq, X)), \Psi \to \Box\mathbf{for}(X) \in L.$$

Using $\Psi - L \subseteq \mathbf{for}(\mathbf{G}_L(\subseteq, X))$, we have (2.2). Considering the case that $\Sigma = \emptyset$, we obtain (2).

We show (6). If $n = 0$, then (6) is clear. So, we assume that $n > 0$. Suppose that $Y \in \mathbf{pclus}_L(X)$. Then $\Box\mathbf{for}(X_\ominus) \in (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. Therefore,

$$\mathbf{ant}(Y) \to \mathbf{suc}(Y), \Box\mathbf{for}(X) \in L$$

and thus,

$$\to \mathbf{for}(Y) \vee \Box\mathbf{for}(X) \in L. \tag{6.1}$$

On the other hand, by (4),

$$\{\mathbf{for}(Y) \vee \Box\mathbf{for}(X) \mid Y \in \mathbf{pclus}_L(X)\}, \mathbf{for}(\mathbf{G}_L^\triangle(\subseteq, X)) \to \Box\mathbf{for}(X) \in L.$$

Using (6.1), we obtain (6). ■

Lemma 2.15 *For any* $X \in \mathbf{G}_L(n)$,

$$\Box\mathbf{for}(X) \equiv_L \begin{cases} \bigwedge \mathbf{for}(\mathbf{pclus}_L(X)) \wedge \bigwedge \Phi & \textit{if } X \in \mathbf{G}_L^\circ(n) \\ \bigwedge \Phi & \textit{if } X \notin \mathbf{G}_L^\circ(n). \end{cases}$$

where $\Phi = \{\mathbf{for}(Y) \mid Y \in \mathbf{G}_L(n+1), \Box\mathbf{for}(X) \in \mathbf{suc}(Y)\}$.

Proof. It is sufficient to show the following four conditions:

(1) $\Box\mathbf{for}(X) \to \bigwedge \Phi \in L$,

(2) $\Box\mathbf{for}(X) \to \bigwedge \mathbf{for}(\mathbf{pclus}_L(X)) \in L$ if $X \in \mathbf{G}_L^\circ(n)$,

(3) $\Phi \to \Box\mathbf{for}(X) \in L$ if $X \notin \mathbf{G}_L^\circ(n)$,

(4) $\mathbf{for}(\mathbf{pclus}_L(X)), \Phi \to \Box\mathbf{for}(X) \in L$ if $X \in \mathbf{G}_L^\circ(n)$.

(1) is clear. (2) is also clear from Corollary 2.12(2). We show (3) and (4). By Corollary 2.9(3), for any $Z \in \mathbf{G}_L(n) - \mathbf{G}_L^*(n)$,

$$\mathbf{for}(\{Z_\oplus \in \mathbf{next}_L(Z) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Z_\oplus)\}) \to \mathbf{for}(Z), \Box\mathbf{for}(X) \in L.$$

Therefore,

$$\Phi \to \bigwedge \mathbf{for}(\mathbf{G}_L(n) - \mathbf{G}_L^*(n)), \Box\mathbf{for}(X) \in L.$$

Using Lemma 2.14(3) and Lemma 2.14(6); and Lemma 2.14(5), we obtain (3); and (4), respectively. ■

By Corollary 2.9(2), we obtain Theorem 2.6(1). We can easily observe Theorem 2.6(2). By Lemma 2.15 and Corollary 2.12(1), we obtain Theorem 2.6 (3).

3 A construction of $\mathbf{CNF}_L$ and a clarification of $\langle \mathbf{F}/\equiv_L, \leq_L \rangle$

In the present section, we consider (IV) in subsection 1.4. However, we can directly extend the result (IV) in Sasaki (2010a) to our general cases. Thus, we show the extended results without proofs.

Definition 3.1 *We define the sets $\mathbf{CNF}_L(k)$ and $\mathbf{CNF}_L$ as follows.*

$$\mathbf{CNF}_L(k) = \begin{cases} 2^{\mathbf{ED}_L(0)} & \text{if } k = 0 \\ 2^{\mathbf{ED}_L(k)} - \{ \bigcup_{X \in S} \mathbf{next}_L(X) \mid S \subseteq \mathbf{G}_L(k{-}1) - \mathbf{G}^*_L(k{-}1) \} & \text{if } k > 0, \end{cases}$$

$$\mathbf{CNF}_L = \bigcup_{i=0}^{\infty} \mathbf{CNF}_L(i).$$

We note that each member of $\mathbf{CNF}_L$ is a finite set.

Example 3.2 *Using the sequents in Example 2.4 in the case that $L = \mathbf{K4}$, $m = 1$, and $n = 0, 1, 2$, we list some examples:*

- $\mathbf{CNF}_L(0) = \{\emptyset, \{T\}, \{F\}, \{T, F\}\}$,
- $\mathbf{CNF}_L(1) = 2^{\mathbf{ED}_L(1)} - \{\emptyset, \mathbf{next}_L(T), \mathbf{next}_L(F), \mathbf{next}_L(T) \cup \mathbf{next}_L(F)\}$,

Definition 3.3 *For any sequent X, we define $X(k)$ as*

$$X(k) = (\mathbf{ant}(X) \cap \mathbf{B}(k) \to \mathbf{suc}(X) \cap \mathbf{B}(k)),$$

where $\mathbf{B}(k) = \mathbf{V} \cup \Box\mathbf{F}(k-1)$.

Definition 3.4 *For any $X \in \mathbf{G}_L(n)$, we define the sets $X{\Downarrow_{k,L}}$ and $X{\Downarrow_L}$ as*

$$X{\Downarrow_{k,L}} = \{Y \in \mathbf{G}_L(k) \mid Y(n) = X\} \quad \text{and} \quad X{\Downarrow_L} = \bigcup_{i=1}^{\infty} X{\Downarrow_{i,L}}.$$

Also, for any $S \in 2^{\mathbf{ED}_L(n)}$, we define $S{\Downarrow_L}$ as $S{\Downarrow_L} = \bigcup_{X \in S} X{\Downarrow_L}$.

We can easily observe that for any $X \in \mathbf{G}_L(3)$,

- $X(4) = X(3) = X$, $X(2) = X_\ominus$, and $X(1) = (X_\ominus)_\ominus$,
- $X{\Downarrow_{2,L}} = \emptyset$, $X{\Downarrow_{3,L}} = \{X\}$, and $X{\Downarrow_{4,L}} = \begin{cases} \emptyset & \text{if } X \in \mathbf{G}^*_L(3) \\ \mathbf{next}_L(X) & \text{if } X \notin \mathbf{G}^*_L(3), \end{cases}$
- $X{\Downarrow_L} = \begin{cases} \{X\} & \text{if } X \in \mathbf{G}^*_L(3) \\ \{X\} \cup \mathbf{next}_L(X){\Downarrow_L} & \text{if } X \notin \mathbf{G}^*_L(3). \end{cases}$

Lemma 3.5 *Let X and Y be sequents in $\mathbf{G}_L(n)$ and $\mathbf{G}_L(k)$, respectively. If $n \geq k$, then*

$$\mathbf{ant}(Y) \subseteq \mathbf{ant}(X) \text{ and } \mathbf{suc}(Y) \subseteq \mathbf{suc}(X) \text{ if and only if } X \in Y{\Downarrow_L}.$$

Theorem 3.6

(1) $\mathbf{F}/\equiv_L \; = \; \{[\bigwedge \mathbf{for}(S)] \mid S \in \mathbf{CNF}_L\}$.

(2) *For any $S_1 \in \mathbf{CNF}_L(\ell)$ and for any $S_2 \in \mathbf{CNF}_L(k)$,*

   (2.1) $[\bigwedge \mathbf{for}(S_1)] \leq [\bigwedge \mathbf{for}(S_2)]$ *if and only if either $S_2 \subseteq S_1{\Downarrow_L}$ or both $S_2{\Downarrow_L} \cap \mathbf{ED}_L(\ell) \subseteq S_1$ and $k \leq \ell$,*

   (2.2) $\bigwedge \mathbf{for}(S_1) \equiv_L \bigwedge \mathbf{for}(S_2)$ *if and only if $S_1 = S_2$.*

4 The exact model for F

In the present section, we extend (V) in subsection 1.4 to a kind of normal modal logics. Specifically,

- we define a Kripke model $\mathbf{EM}_L = \langle W_L, R_L, P_L \rangle$,
- we introduce a finite model property of L in weakened form, which is equivalent to the condition

$$\mathbf{EM}_L \text{ is the exact } L\text{-model for } \mathbf{F} \text{ in } L, \tag{\dagger}$$

- for any L satisfying (†), we clarify the set $\{P_L(A) \mid A \in \mathbf{F}\}$.

Considering Lemma 1.3, the third task above is important. Also, as pointed out in Sasaki (2010a), the clarification of the task has not directly been described in previous works.

Definition 4.1 *The Kripke model $\mathbf{EM}_L$ is defined as*

$$\mathbf{EM}_L = \langle W_L, R_L, P_L \rangle,$$

where $W_L = \bigcup_{n=0}^{\infty} \mathbf{G}^*_L(n)$, $R_L = \{(X, Y) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(X) \text{ or } Y \in \mathbf{clus}_L(X)\}$, *and* $P_L(p_i) = \{X \mid p_i \in \mathbf{ant}(X)\}$.

Definition 4.2 *We say that L has the finite model property for $\mathbf{F}$ if for any $A \in \mathbf{F} - L$, there exists a finite transitive $L \cap \mathbf{F}$-model M such that $M \not\models A$.*

Theorem 4.3

(1) *The following three conditions are equivalent:*

   (1.1) $\mathbf{EM}_L$ *is the exact model for* $\mathbf{F}$ *in* L, (1.2) $\mathbf{EM}_L$ *is the exact L-model for* $\mathbf{F}$ *in* L,

(1.3) L *has the finite model property for* $\mathbf{F}$.

(2) *For any* $Y \in \mathbf{G}_L(n)$,

$$P_L(\mathbf{for}(Y)) = W_L - Y\Downarrow_L \text{ and } P_L(\neg\mathbf{for}(Y)) = W_L \cap Y\Downarrow_L.$$

(3) $\{P_L(A) \mid A \in \mathbf{F}\} = \{W_L - S\Downarrow_L \mid S \in \mathbf{CNF}_L\}$.

To prove the theorem above, we need some lemmas. The condition (1.1) $\Rightarrow$ (1.2) will be proved in Lemma 4.5 and the converse is clear. Three conditions (1.1) $\Rightarrow$ (1.3), (1.3) $\Rightarrow$ (1.1), and (2) will be proved in Lemma 4.7, Lemma 4.17, and Corollary 4.23 respectively. By (2), we have

$$P_L(\bigwedge \mathbf{for}(S)) = W_L - S\Downarrow_L, \text{ for any } S \in \mathbf{CNF}_L,$$

and using Theorem 3.6 we obtain (3).

Definition 4.4 *Let $M = \langle W, R, P\rangle$ be a Kripke model. For any subset S of W, we define the Kripke model $M|_S$ as*

$$M|_S = \langle S, R|_S, P|_S\rangle,$$

where $R|_S = R \cap S^2$ and $(P|_S)(p_i) = P(p_i) \cap S$. For any world $\alpha \in W$, we define $\alpha\uparrow$ as

$$\alpha\uparrow = \{\alpha\} \cup \{\beta \in W \mid \alpha R \beta\}.$$

Lemma 4.5 *If $\mathbf{EM}_L$ is the exact model for $\mathbf{F}$ in L, then $\mathbf{EM}_L$ is an L-model.*

Proof. We note that

for any $X \in W_L$, there exists a formula $f(X) \in \mathbf{F}$ such that $\{X\} = P(f(X))$.

Suppose that M is not L-model. Then there exist a world $X \in W_L$ and a formula $A \in L$ such that $(M, X) \not\models A$. We can easily observe finiteness of $\mathbf{EM}_L|_{X\uparrow}$ and $(\mathbf{EM}_L|_{X\uparrow}, X) \not\models A$. We define the S_i as

$$S_i = \{f(Y) \mid Y \in (P|_{X\uparrow})(p_i)\}$$

Then we can easily observe

$$(P|_{X\uparrow})(p_i) = (P|_{X\uparrow})(\bigvee S_i). \tag{1}$$

Let A' be the formula obtained from A be substituting p_i with $\bigvee S_i$ for any i. Then we have $A' \in L \cap \mathbf{F}$; and by (1), we have $(\mathbf{EM}_L, X) \not\models A'$. Hence, M is not exact model for $\mathbf{F}$ in L. ∎

Lemma 4.6 $\mathbf{EM}_L$ *is transitive.*

Proof. By Corollary 2.12(2). ■

Lemma 4.7 *If $\mathbf{EM}_L$ is the exact model for $\mathbf{F}$ in L, then L has the finite model property for $\mathbf{F}$.*

Proof. Suppose that $A \in \mathbf{F} - L$. Then there exists $X \in W_L$ such that $(\mathbf{EM}_L, X) \not\models A$. By Lemma 4.6 we can easily observe finiteness and transitivity of $\mathbf{EM}_L|_{X\uparrow}$, $\mathbf{EM}_L|_{X\uparrow} \models L \cap \mathbf{F}$, and $(\mathbf{EM}_L|_{X\uparrow}, X) \not\models A$. ■

Definition 4.8 *Let $M = \langle W, R, P\rangle$ be a Kripke model. For any $k \geq 0$ and for any $\alpha \in W$, we define the sequent $\mathbf{ed}_L(k, \alpha)$ as follows.*

$$\mathbf{ed}_L(0, \alpha) = (\{p_i \in \mathbf{V} \mid (M, \alpha) \models p_i\} \to \{p_i \in \mathbf{V} \mid (M, \alpha) \not\models p_i\}),$$

$$\mathbf{ed}_L(k+1, \alpha) = \begin{cases} \mathbf{ed}_L(k, \alpha) & \textit{if } \mathbf{ed}_L(k, \alpha) \in W_L \\ \mathbf{n}_L(\mathbf{ed}_L(k, \alpha), \{X \in \mathbf{G}_L(k) \mid (M, \alpha) \not\models \Box\mathbf{for}(X)\}) & \textit{o. w.} \end{cases}.$$

We can easily observe the following lemma.

Lemma 4.9 *Let $M = \langle W, R, P\rangle$ be an $L \cap \mathbf{F}$-model. Then for any $\alpha, \beta \in W$,*

(1) $\{X \in \mathbf{ED}_L(n) \mid (M, \alpha) \not\models \mathbf{for}(X)\} = \{\mathbf{ed}_L(n, \alpha)\}$,

(2) *if $k < n$, then* $(\mathbf{ed}_L(n, \alpha))(k) = \mathbf{ed}_L(k, \alpha)$,

(3) $\{X \in \mathbf{G}_L(k) \mid (M, \alpha) \not\models \Box\mathbf{for}(X)\} = \{\mathbf{ed}_L(k, \beta) \in \mathbf{G}_L(k) \mid \alpha R\beta\}$,

(4) *if M is transitive, $\alpha R\beta$, $\beta R\alpha$, and $\mathbf{ed}_L(k, \alpha), \mathbf{ed}_L(k, \beta) \in \mathbf{G}_L(k)$, then* $(\mathbf{ant}(\mathbf{ed}_L(k, \alpha)))^{\Box} = (\mathbf{ant}(\mathbf{ed}_L(k, \beta)))^{\Box}$.

Definition 4.10 *For any $X \in \mathbf{G}_L(n)$, we define the sets $\mathbf{G}_L(X, \subsetneq)$ and $\mathbf{G}_L(X, \subseteq)$ as follows:*

$$\mathbf{G}_L(X, \subsetneq) = \{Y \in \mathbf{G}_L(n) \mid (\mathbf{ant}(X))^{\Box} \subsetneq (\mathbf{ant}(Y))^{\Box}\},$$

$$\mathbf{G}_L(X, \subseteq) = \{Y \in \mathbf{G}_L(n) \mid (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box}\}.$$

Lemma 4.11 *Let $M = \langle W, R, P\rangle$ be a finite transitive $L \cap \mathbf{F}$-model and let α be a world in W. If*

$$\{\mathbf{ed}_L(n, \beta) \mid \beta \in W, \alpha R\beta, \beta \not R \alpha\} \subseteq W_L, \tag{$*$}$$

then there exists $k \leq \#(\mathbf{G}_L(\mathbf{ed}_L(n+1, \alpha), \subsetneq)) + n + 1$ such that $\mathbf{ed}_L(k, \alpha) \in W_L$.

Proof. For brevity, we refer to X as $\mathbf{ed}_L(n+1, \alpha)$. We note that $(M, \alpha) \not\models \mathbf{for}(X)$ and $X \in \mathbf{ED}_L(n+1)$. If $X \notin \mathbf{G}_L(n+1)$, then $X \in W_L$, and thus, the lemma is clear. So, we assume that $X \in \mathbf{G}_L(n+1)$.

We use an induction on $\#(\mathbf{G}_L(X, \subsetneq))$.

Basis($\#(\mathbf{G}_L(X, \subsetneq)) = 0$) is clear from $X \in W_L$.

Induction step($\#(\mathbf{G}_L(X,\subsetneq)) > 0$). We refer to $X_\oplus$ as $\mathbf{ed}_L(n+2,\alpha)$. By Lemma 4.9(1), we have $X_\oplus \notin L$. By $(*)$ and Lemma 4.9(3), we have

$$X_\oplus = \mathbf{n}_L(X, \{\mathbf{ed}_L(n+1,\beta) \in \mathbf{G}_L(n+1) \mid \alpha R\beta, \beta R\alpha\}).$$

Using Lemma 4.9(4), we have that for any $Y_1, Y_2 \in \mathbf{G}_L(n+1)$,

$$\Box\mathbf{for}(Y_1), \Box\mathbf{for}(Y_2) \in \mathbf{suc}(X_\oplus) \text{ implies } (\mathbf{ant}(Y_1))^\Box = (\mathbf{ant}(Y_2))^\Box. \tag{1}$$

Let Y be a sequent in $\mathbf{G}_L(X,\subseteq) - \mathbf{G}^*_L(n+1)$. We define $Y_\oplus$ as

$$Y_\oplus = \mathbf{n}_L(Y, \{Z \in \mathbf{G}_L(n+1) \mid \Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus), \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(Y)\}).$$

We show the following two conditions:

$$\mathbf{next}_L(Y) \cap \mathbf{G}_L(X_\oplus,\subsetneq) \subseteq \{Y_\oplus\}, \tag{2}$$

$$(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box \text{ implies } (\mathbf{ant}(X_\oplus))^\Box = (\mathbf{ant}(Y_\oplus))^\Box. \tag{3}$$

For (2). Let $Y'_\oplus$ be a sequent in $\mathbf{next}_L(Y) \cap \mathbf{G}_L(X_\oplus,\subsetneq)$. We note that

$$(\mathbf{ant}(X_\oplus))^\Box \subseteq (\mathbf{ant}(Y'_\oplus))^\Box, \tag{2.1}$$

Let Z be a sequent in $\mathbf{G}_L(n+1)$. Then by (2.1), we have

$$\Box\mathbf{for}(Z) \in \mathbf{ant}(X_\oplus) \text{ implies } \Box\mathbf{for}(Z) \in \mathbf{ant}(Y'_\oplus). \tag{2.2}$$

We also note that $Y'_\oplus \notin L$. Using $\Box\mathbf{for}(Z_\ominus) \to \Box\mathbf{for}(Z) \in L$, we have

$$\Box\mathbf{for}(Z_\ominus) \in \mathbf{ant}(Y) \text{ implies } \Box\mathbf{for}(Z) \in \mathbf{ant}(Y'_\oplus). \tag{2.3}$$

By (1) and (2.1), we have

$$\Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus) \text{ implies } \Box\mathbf{for}(\mathbf{next}_L(Z_\ominus) - \{Z\}) \subseteq (\mathbf{ant}(X_\oplus))^\Box \subseteq (\mathbf{ant}(Y'_\oplus))^\Box,$$

and using Corollary 2.9 and $Y'_\oplus \notin L$, we have

$$\Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus) \text{ and } \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(Y) \text{ imply } \Box\mathbf{for}(Z) \in \mathbf{suc}(Y'_\oplus). \tag{2.4}$$

By (2.2), (2.3), and (2.4), we obtain (2).

For (3). Suppose that $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. Then for any $Z \in \mathbf{G}_L(n+1)$, we have

$$\Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus) \text{ and } \Box\mathbf{for}(Z_\ominus) \in \mathbf{ant}(Y) \text{ imply } X_\oplus \in L,$$

using $X_\oplus \notin L$, we have

$$\Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus) \text{ implies } \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(Y).$$

Therefore, we have

$$Y_\oplus = \mathbf{n}_L(Y, \{Z \in \mathbf{G}_L(n+1) \mid \Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus)\}),$$

and hence, we obtain (3).

By (2) and (3), we have

$$\begin{array}{rcl}\mathbf{G}_L(X_\oplus, \subsetneq) & = & \{Y_\oplus \mid Y \in \mathbf{G}_L(X, \subseteq) - \mathbf{G}^*_L(n+1)\} \cap \mathbf{G}_L(X_\oplus, \subsetneq) \\ & = & \{Y_\oplus \mid Y \in \mathbf{G}_L(X, \subsetneq) - \mathbf{G}^*_L(n+1)\} \cap \mathbf{G}_L(X_\oplus, \subsetneq) \\ & \subseteq & \{Y_\oplus \mid Y \in \mathbf{G}_L(X, \subsetneq) - \mathbf{G}^*_L(n+1)\},\end{array}$$

and therefore,

$$\begin{array}{rcl}\#(\mathbf{G}_L(X_\oplus, \subsetneq)) & \leq & \#(\{Y_\oplus \mid Y \in \mathbf{G}_L(X, \subsetneq) - \mathbf{G}^*_L(n+1)\}) \\ & = & \#(\mathbf{G}_L(X, \subsetneq) - \mathbf{G}^*_L(n+1)).\end{array}$$

By the definition of $\mathbf{G}^*_L(k)$, we have $\mathbf{G}_L(X, \subsetneq) \cap \mathbf{G}^*_L(n+1) \neq \emptyset$, and thus,

$$\#(\mathbf{G}_L(X_\oplus, \subsetneq)) < \#(\mathbf{G}_L(X, \subsetneq)).$$

Also, by $(*)$, we have

$$\{\mathbf{ed}_L(n+1, \beta) \mid \beta \in W, \alpha R \beta, \beta \not R \alpha\} = \{\mathbf{ed}_L(n, \beta) \mid \beta \in W, \alpha R \beta, \beta \not R \alpha\} \subseteq W_L.$$

Using the induction hypothesis, there exists

$$k \leq \#(\mathbf{G}_L(X_\oplus, \subsetneq)) + (n+1) + 1 \leq \#(\mathbf{G}_L(X, \subsetneq)) + n + 1$$

such that $\mathbf{ed}_L(k, \alpha) \in W_L$. ∎

Definition 4.12 *We define the number $\mathbf{a}_L(n, k)$ as*

$$\mathbf{a}_L(n,k) = \begin{cases} n & \text{if } k = 0 \\ \mathbf{a}_L(n, k-1) + \#(\mathbf{G}_L(\mathbf{a}_L(n, k-1)) + 1) & \text{if } k > 0. \end{cases}$$

Lemma 4.13 *If $0 \leq k < \ell$, then $\mathbf{a}_L(\mathbf{a}_L(n, 1), k) \leq \mathbf{a}_L(n, \ell)$.*

Proof. By an induction on k. ∎

Lemma 4.14 *Let $M = \langle W, R, P \rangle$ be a finite transitive $L \cap \mathbf{F}$-model and let α be a world in W. Then for any n, there exists $k \leq \mathbf{a}_L(n, k_{n,\alpha})$ such that $\mathbf{ed}_L(k, \alpha) \in W_L$, where*

$$k_{n,\alpha} = \#(\{\mathbf{ed}_L(n, \beta) \mid \beta \in W, \alpha R \beta, \beta \not R \alpha\} - W_L) + 1.$$

Proof. We use an induction on $k_{n,\alpha}$. First, we note that

$$\begin{array}{rl} & \#(\mathbf{G}_L(\mathbf{ed}_L(n+1, \alpha), \subsetneq)) + n + 1 \\ \leq & \#(\mathbf{G}_L(n+1) - \{\mathbf{ed}_L(n+1, \alpha)\}) + n + 1 \\ = & \#(\mathbf{G}_L(n+1)) + n = \mathbf{a}_L(n, 1). \end{array} \quad (1)$$

Basis($k_{n,\alpha} = 1$) is clear from Lemma 4.11 and (1).

We show Induction step($k_{n,\alpha} > 1$). By $k_{n,\alpha} > 1$, we have

$$\{\mathbf{ed}_L(n, \beta) \mid \beta \in W, \alpha R \beta, \beta \not R \alpha\} - W_L \neq \emptyset.$$

Using the finiteness of M, there exists a world $\gamma \in W$ such that

$$\alpha R\gamma, \gamma \not R \alpha, \mathbf{ed}_L(n,\gamma) \notin W_L, \text{ and } \{\mathbf{ed}_L(n,\beta) \mid \beta \in W, \gamma R\beta, \beta \not R \gamma\} \subseteq W_L.$$

Using Lemma 4.11 and (1), there exists $k \leq \mathbf{a}_L(n,1)$ such that $\mathbf{ed}_L(k,\gamma) \in W_L$. Therefore, we have

$$\mathbf{ed}_L(n,\gamma) \notin W_L, \mathbf{ed}_L(\mathbf{a}_L(n,1),\gamma) \in W_L, \text{ and}$$

$$\{\mathbf{ed}_L(n,\beta) \mid \beta \in W, \alpha R\beta, \beta \not R \alpha\} \cap W_L \subseteq \{\mathbf{ed}_L(\mathbf{a}_L(n,1),\beta) \mid \beta \in W, \alpha R\beta, \beta \not R \alpha\} \cap W_L.$$

Hence, we have

$$k_{n,\alpha} > k_{\mathbf{a}_L(n,1),\alpha}.$$

Using the induction hypothesis, there exists $\ell \leq \mathbf{a}_L(\mathbf{a}_L(n,1), k_{\mathbf{a}_L(n,1),\alpha})$ such that $\mathbf{ed}_L(\ell,\alpha) \in W_L$. Moreover, using Lemma 4.13 we have $\ell \leq \mathbf{a}_L(n, k_{n,\alpha})$. ■

Lemma 4.15 *Let $M = \langle W, R, P\rangle$ be a Kripke model. If $\alpha \in W$ and $X \in \mathbf{G}^*_L(n)$ satisfy $(M,\alpha) \not\models \mathbf{for}(X)$, then for any $Y \in \mathbf{G}_L(k)$,*

$$(M,\alpha) \not\models \mathbf{for}(Y) \text{ if and only if } X \in Y{\Downarrow_L}.$$

Proof. By $(M,\alpha) \not\models \mathbf{for}(X)$, we note that

$$\mathbf{ant}(X) \subseteq \{A \mid (M,\alpha) \models A\} \text{ and } \mathbf{suc}(X) \subseteq \{B \mid (M,\alpha) \not\models B\}.$$

Also, we note that $(M,\alpha) \not\models \mathbf{for}(Y)$ is equivalent to

$$\mathbf{ant}(Y) \subseteq \{A \mid (M,\alpha) \models A\} \text{ and } \mathbf{suc}(Y) \subseteq \{B \mid (M,\alpha) \not\models B\}.$$

Therefore, if $n \geq k$, then we have

$$(M,\alpha) \not\models \mathbf{for}(Y) \text{ if and only if } \mathbf{ant}(Y) \subseteq \mathbf{ant}(X) \text{ and } \mathbf{suc}(Y) \subseteq \mathbf{suc}(X),$$

and using Lemma 3.5 we obtain the lemma. If $n < k$, then we have $X \notin Y{\Downarrow_L}$; and we have

$$(M,\alpha) \not\models \mathbf{for}(Y) \text{ implies } \mathbf{ant}(X) \subseteq \mathbf{ant}(Y) \text{ and } \mathbf{suc}(X) \subseteq \mathbf{suc}(Y),$$

and using Lemma 3.5 we obtain that $(M,\alpha) \not\models \mathbf{for}(Y)$ is in contradiction with $X \in \mathbf{G}^*_L(n)$. ■

Lemma 4.16 *Let $M = \langle W, R, P\rangle$ be an $L\cap\mathbf{F}$-model. If $\alpha \in W$ and $X \in \mathbf{G}^*_L(n)$ satisfy $(M,\alpha) \not\models \mathbf{for}(X)$, then for any $Y \in \mathbf{G}_L(k)$,*

$$(M,\alpha) \not\models \Box\mathbf{for}(Y) \text{ if and only if either } \Box\mathbf{for}(Y) \in \mathbf{suc}(X) \text{ or } Y \in \mathbf{clus}_L(X).$$

Proof. By Corollary 2.12(2), we obtain the "if" part.

We show the "only if" part. Suppose that $(M,\alpha) \not\models \Box\mathbf{for}(Y)$. By Theorem 2.6 we have

$$\Box\mathbf{for}(Y) \equiv_L \begin{cases} \bigwedge \mathbf{for}(S \cup \mathbf{pclus}_L(Y)) & \text{if } Y \in \mathbf{G}^\circ_L(k) \\ \bigwedge \mathbf{for}(S) & \text{if } Y \notin \mathbf{G}^\circ_L(k), \end{cases}$$

where $S = \{Z \in \mathbf{ED}_L(k+1) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(Z)\}$. Therefore, either one of the following two conditions holds:

(1) $(M, \alpha) \not\models \mathbf{for}(Z)$ for some $Z \in S$,

(2) $Y \in \mathbf{G}^{\circ}_{L}(k)$ and $(M, \alpha) \not\models \mathbf{for}(Z)$ for some $Z \in \mathbf{pclus}_L(Y)$.

If (1) holds, then by Lemma 4.15 we have $X \in Z{\Downarrow_L}$, and using $Z \in S$ and Lemma 3.5 we obtain

$$\Box\mathbf{for}(Y) \in \mathbf{suc}(Z) \subseteq \mathbf{suc}(X).$$

So, we assume that (2) holds. Then by Lemma 4.15 we have $X \in Z{\Downarrow_L}$. By $Y \in \mathbf{G}^{\circ}_{L}(k)$ and $Z \in \mathbf{pclus}_L(Y)$, we have $Z \in \mathbf{G}^{*}_{L}(k)$ and $Y \in \mathbf{clus}_L(Z)$. Using $X \in Z{\Downarrow_L}$, we have $X = Z$, and hence, we obtain $Y \in \mathbf{clus}_L(Z) = \mathbf{clus}_L(X)$. ■

Lemma 4.17 *If L has the finite model property for $\mathbf{F}$, then $\mathbf{EM}_L$ is the exact model for $\mathbf{F}$ in L.*

Proof. By the finite model property for $\mathbf{F}$ of L, for any $A \in \mathbf{F} - L$, there exists a finite transitive $L \cap \mathbf{F}$-model $M_A = \langle W_A, R_A, P_A \rangle$ such that $M_A \not\models A$. By Lemma 4.9 and Lemma 4.14 for any $\alpha \in W_A$, there exists the sequent $seq(\alpha) \in W_L$ such that $(M_A, \alpha) \not\models \mathbf{for}(seq(\alpha))$. We define the Kripke model $M = \langle W, R, P \rangle$ as

- $W = (\bigcup_{A \in \mathbf{F}-L} W_A)/\sim$,
- $[\alpha]R[\beta] \Leftrightarrow \alpha' R_A \beta'$ for some $A \in \mathbf{F} - L, \alpha' \in [\alpha] \cap W_A$, and $\beta' \in [\beta] \cap W_A$,
- $[\alpha] \in P(p_i) \Leftrightarrow \alpha' \in P_A(p_i)$ for some $A \in \mathbf{F}$ and $\alpha' \in [\alpha] \cap W_A$,

where $\alpha \sim \beta \Leftrightarrow seq(\alpha) = seq(\beta)$.

We show that M is the exact model for $\mathbf{F}$ in L. By Lemma 4.15 for any $\alpha \in W_A$, $\beta \in W_B$, and $Y \in \mathbf{G}_L(k)$, we have

$$\alpha \sim \beta \text{ implies } (M_A, \alpha) \models \mathbf{for}(Y) \Leftrightarrow (M_B, \beta) \models \mathbf{for}(Y),$$

and using Theorem 2.7 we have

$$\alpha \sim \beta \text{ implies } \{C \mid (M_A, \alpha) \models C\} = \{C \mid (M_B, \beta) \models C\}.$$

Therefore, by an induction on C, we can show for any $C \in \mathbf{F}$ and $[\alpha] \in W$,

$$(M, [\alpha]) \models C \text{ if and only if for any } \alpha' \in [\alpha], (M_A, \alpha') \models C, \tag{1}$$

where M_A is the model satisfying $\alpha' \in W_A$. Thus, for any $C \in \mathbf{F}$, we have

$$M \models C \text{ if and only if for any } A \in \mathbf{F} - L,\ M_A \models C.$$

Also, by (1) and Lemma 4.15 we have

$$P(\neg\mathbf{for}(seq(\alpha))) = \{[\alpha]\} \text{ for any } [\alpha] \in W. \tag{2}$$

Hence, M is the exact model for $\mathbf{F}$ in L.

Therefore, we have only to show that M is isomorphic to $\mathbf{EM}_L$. Specifically, we show that the mapping $seq^* : W \to W_L$ defined as $seq^*([\alpha]) = seq(\alpha)$ is an isomorphism from M to $\mathbf{ED}_L$. We can easily observe that seq^* is one-to-one and onto. Also, by (2), we have

$$[\alpha]R[\beta] \text{ if and only if } (M, [\alpha]) \not\models \Box\mathbf{for}(seq^*([\beta])),$$

and using Lemma 4.16, we obtain

$$[\alpha]R[\beta] \text{ if and only if } seq^*([\alpha])R_L seq^*([\beta]).$$

We can also observe

$$[\alpha] \in P(p_i) \Leftrightarrow p_i \in \mathbf{ant}(seq(\alpha)) \Leftrightarrow seq^*([\alpha]) \in P_L(p_i).$$

■

Lemma 4.18 *For any $X \in \mathbf{G}^*_L(n)$,*

$$\mathbf{suc}(X) - \mathbf{suc}(X_\ominus) \subseteq \Box\mathbf{for}(\mathbf{clus}_L(X_\ominus) \cup \mathbf{G}^*_L(n-1)).$$

Proof. First, we note that $X \notin L$ from $X \in \mathbf{G}^*_L(n)$. Suppose that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X) - \mathbf{suc}(X_\ominus) - \Box\mathbf{for}(\mathbf{G}^*_L(n-1))$. Then by $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ and $X \notin L$, we have

$$(\mathbf{ant}(X))^\Box \to \Box\mathbf{for}(Y) \notin L,$$

and using ($\Box$), we have

$$(\mathbf{ant}(X))^\Box, \mathbf{ant}(Y) \to \mathbf{suc}(Y) \notin L.$$

Using Lemma 2.8, there exist Φ and Ψ such that

$$\Phi \cup \Psi = \mathbf{for}(\mathbf{G}_L(n-1) - \mathbf{ant}(X)), \quad \Phi \cap \Psi = \emptyset,$$

and

$$\Box\Phi, (\mathbf{ant}(X))^\Box, \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Box\Psi \notin L.$$

We refer to the above sequent as $Y_\oplus$. By $Y \notin \mathbf{G}_L(n-1) - \mathbf{G}^*_L(n-1)$, we have

$$Y_\oplus \in \mathbf{next}_L(Y) \subseteq \mathbf{G}_L(n),$$

Also, we note that

$$(\mathbf{ant}(X))^\Box \subseteq \mathbf{ant}(Y_\oplus).$$

Using $X \in \mathbf{G}^*_L(n)$, we have

$$(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y_\oplus))^\Box \text{ and } Y_\oplus \in \mathbf{G}^*_L(n).$$

By $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ and $X \notin L$, we have

$$\Box\mathbf{for}(Y_\ominus) \notin (\mathbf{ant}(X_\ominus))^\Box = (\mathbf{ant}(Y))^\Box.$$

Hence, we obtain

$$Y \in \mathbf{clus}_L(X_\ominus).$$

■

Lemma 4.19 *Let X be a sequent in $\mathbf{G}_L(n+1)$ and let Y be a sequent satisfying $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$. Then*

$$\Box\mathbf{for}(\{Y_\oplus \in \mathbf{next}_L(Y) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y_\oplus))^\Box\}) \to \mathbf{for}(X) \in L.$$

Proof. By $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ and Corollary 2.9, we have

$$\Box\mathbf{for}(\mathbf{next}_L(Y)) \to \mathbf{for}(X) \in L,$$

and using Lemma 2.10, we obtain the lemma. ■

Lemma 4.20 *Let X be a sequent in $\mathbf{G}_L(n+1)$ and let Y be a sequent in $\mathbf{G}_L(k)$ satisfying $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$. Then there exists a sequent $Y^* \in \mathbf{ED}_L(n)$ such that $\Box\mathbf{for}(Y^*) \in \mathbf{suc}(X)$ and $Y^* \in Y{\Downarrow_L}$.*

Proof. We use an induction on $n \geq k$. Basis ($n = k$) is clear since Y satisfies the conditions.

Induction step ($n > k$). By $n > k$, we have $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_\ominus)$. Using the induction hypothesis, there exists a sequent $Y' \in \mathbf{ED}_L(n-1)$ such that $\Box\mathbf{for}(Y') \in \mathbf{suc}(X_\ominus)$ and $Y' \in Y{\Downarrow_L}$. If $Y' \in \bigcup_{i=0}^{n-1} \mathbf{G}^*_L(i)$, then Y' satisfies the conditions. We assume that $Y' \in \mathbf{G}_L(n-1) - \mathbf{G}^*_L(n-1)$. Then by $\Box\mathbf{for}(Y') \in \mathbf{suc}(X_\ominus)$, $X \notin L$, and Lemma 4.19, we have

$$\Box\mathbf{for}(\{Y_\oplus \in \mathbf{next}_L(Y') \mid (\mathbf{ant}(X_\ominus))^\Box \subseteq (\mathbf{ant}(Y_\oplus))^\Box\}) \not\subseteq \mathbf{ant}(X).$$

Therefore, there exists a sequent $Y_\oplus \in \mathbf{next}_L(Y')$ such that $\Box\mathbf{for}(Y_\oplus) \in \mathbf{suc}(X)$. Also, by $Y' \in Y{\Downarrow_L}$ and $Y_\oplus \in \mathbf{next}_L(Y')$, we have $Y_\oplus \in Y{\Downarrow_L}$. ■

Lemma 4.21 *For any $X \in \mathbf{G}^*_L(n)$ and for any $Y \in \mathbf{G}_L(k)$,*

(1) $(\mathbf{EM}_L, X) \not\models p_i$ *if and only if* $p_i \in \mathbf{suc}(X)$,

(2) $(\mathbf{EM}_L, X) \not\models \Box\mathbf{for}(Y)$ *if and only if either one of the following two conditions holds:*

  (2.1) $k < n$ *and* $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$,

  (2.2) $k = n$ *and* $Y \in \mathbf{clus}_L(X)$.

Proof. From the definition of P_L, we obtain (1). We show (2) by an induction on $n + k$. A proof of Basis is included in Induction step.

Induction step. We first note that for any $X' \in \mathbf{G}^*_L(n')$ and for any $Y' \in \mathbf{G}_L(k')$, if $n' \leq n$ and $k' \leq \min\{n', k\}$, then the following four conditions are equivalent:

- $(\mathbf{EM}_L, X') \not\models \mathbf{for}(Y')$,
- $\mathbf{ant}(Y') \subseteq \{A \mid (\mathbf{EM}_L, X') \models A\}$ and $\mathbf{suc}(Y') \subseteq \{B \mid (\mathbf{EM}_L, X') \not\models B\}$,

- $\mathbf{ant}(Y') \subseteq \mathbf{ant}(X')$ and $\mathbf{suc}(Y') \subseteq \mathbf{suc}(X')$,
- $X' \in Y'\Downarrow_L$.

The equivalence between the second one and the third one is from (1) and the induction hypothesis. The equivalence between the third and the fourth is from Lemma 3.5

We show the "only if" part. Suppose that $(\mathbf{EM}_L, X) \not\models \Box\mathbf{for}(Y)$. Then there exist a number ℓ and a sequent $Z \in \mathbf{G}_L^*(\ell)$ such that XR_LZ and $(\mathbf{EM}_L, Z) \not\models \mathbf{for}(Y)$. By XR_LZ, we have either one of the following two conditions:

$$\ell < n \text{ and } \Box\mathbf{for}(Z) \in \mathbf{suc}(X), \tag{3.1}$$

$$\ell = n \text{ and } Z \in \mathbf{clus}_L(X). \tag{3.2}$$

Therefore, we have $\ell \leq n$. Also, by $(\mathbf{EM}_L, Z) \not\models \mathbf{for}(Y)$, we have $(\mathbf{EM}_L, Z) \not\models \mathbf{for}(Y(\ell))$. Using $\ell \leq n$ and the equivalence we noted first, we have $Z \in Y(\ell)\Downarrow_L$. Using Lemma 3.5 we have either one of the following two conditions:

$$k < \ell \text{ and } Y = Z(k), \tag{4.1}$$

$$k \geq \ell \text{ and } Y(\ell) = Z. \tag{4.2}$$

We divide the cases.

The case that (4.1) hold. Clearly, we have $k < n$. Also, we have the following two conditions:

$$\Box\mathbf{for}(Y) \to \Box\mathbf{for}(Z) \in L, \tag{4.1.1}$$

$$\Box\mathbf{for}(Y) \to \Box\mathbf{for}(Z(\ell - 1)) \in L. \tag{4.1.2}$$

If (3.1) holds, then by (4.1.1), we obtain (2.1). If (3.2) holds, then we have

$$\Box\mathbf{for}(Z(\ell - 1)) \in (\mathbf{suc}(Z))^{\Box} = (\mathbf{suc}(X))^{\Box},$$

and using (4.1.2), we obtain (2.1).

The case that (4.2) holds. We have $Y(\ell) = Z \in \mathbf{G}_L^*(\ell)$, and thus,

$$k = \ell \text{ and } Y = Y(\ell) = Z.$$

Hence, we have that (3.1) implies (2.1) and that (3.2) implies (2.2).

We show the "if" part. If there exists a sequent Y' satisfying

$$Y' \in Y\Downarrow_L \cap W_L \cap \mathbf{ED}_L(n) \text{ and } XR_LY', \tag{5}$$

then by the equivalence we noted first, we have $(\mathbf{EM}_L, Y') \not\models \mathbf{for}(Y)$, and thus, we have $(\mathbf{EM}_L, X) \not\models \Box\mathbf{for}(Y)$. Therefore, we have only to show the existence of Y' satisfying (5).

Suppose that (2.1) holds. Then by Lemma 4.20 there exists a sequent $Y^* \in \mathbf{ED}_L(n-1)$ such that $\Box\mathbf{for}(Y^*) \in \mathbf{suc}(X)$ and $Y^* \in Y\Downarrow_L$.

If $Y^* \in W_L \cap \mathbf{ED}_L(n-1) \subseteq W_L \cap \mathbf{ED}_L(n)$, then by $\Box\mathbf{for}(Y^*) \in \mathbf{suc}(X)$, we have XR_LY^*, and hence, we obtain that Y^* satisfies (5).

So, we assume that that $Y^* \in \mathbf{G}_L(n-1) - \mathbf{G}^*_L(n-1)$. By Lemma 4.18, we have $Y^* \in \mathbf{clus}_L(X_\ominus)$. We define $Y^*_\oplus$ as

$$Y^*_\oplus = (\mathbf{ant}(X) - \mathbf{ant}(X_\ominus), \mathbf{ant}(Y^*) \to \mathbf{suc}(Y^*), \mathbf{suc}(X) - \mathbf{suc}(X_\ominus)).$$

Then by $Y^* \in \mathbf{clus}_L(X_\ominus)$ and $\Box\mathbf{for}(Y^*) \in \mathbf{suc}(X)$, we have

$$Y^*_\oplus \in \mathbf{next}^+_L(Y^*) \subseteq \mathbf{G}^+_L(n),\ (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y^*_\oplus))^\Box, \text{ and } \Box\mathbf{for}(Y^*) \in \mathbf{suc}(Y^*_\oplus).$$

Using Lemma 2.11, we have $Y^*_\oplus \in \mathbf{G}^\circ_L(n)$, and thus, we have

$$Y^*_\oplus \in \mathbf{clus}_L(X) \text{ and } XR_LY^*_\oplus.$$

Also, we have

$$Y^*_\oplus \in \mathbf{next}_L(Y^*) \cap \mathbf{G}^\circ_L(n) \subseteq Y{\Downarrow_L} \cap W_L \cap \mathbf{ED}_L(n).$$

Hence, $Y^*_\oplus$ satisfies (5).

Suppose that (2.2) holds. Then we have $Y \in \mathbf{G}^*_L(n)$ and XR_LY. Hence, Y satisfies (5). ∎

Lemma 4.22 *For any $X \in \mathbf{G}^*_L(n)$ and for any $Y \in \mathbf{G}_L(k)$,*

$$(\mathbf{EM}_L, X) \not\models \mathbf{for}(Y) \textit{ if and only if } X \in Y{\Downarrow_L}\ .$$

Proof. By Lemma 4.21, the equivalence noted first in the proof of Lemma 4.21 holds. Therefore, we have $(\mathbf{EM}_L, X) \not\models \mathbf{for}(X)$, and using Lemma 4.15, we obtain the lemma. ∎

Corollary 4.23 *For any $Y \in \mathbf{G}_L(n)$,*

$$P_L(\mathbf{for}(Y)) = W_L - Y{\Downarrow_L} \textit{ and } P_L(\neg\mathbf{for}(Y)) = W_L \cap Y{\Downarrow_L}.$$

5 Exact models for $\mathbf{F}(n)$

In the present section, we only treat the normal modal logics with the finite model property for $\mathbf{F}$. By Theorem 4.3, for a logic L we treat here, $\mathbf{EM}_L$ is the exact L-model for $\mathbf{F}$. The purpose of the present section is to extend (VI) and (VII) in subsection 1.4 to these logics.

We let L satisfy the finite model property for $\mathbf{F}$.

First, we introduce an exact set $\mathcal{E}$ for $\mathbf{F}(n)$. The Kripke model $\mathbf{EM}_L|_\mathcal{E}$ is shown to be exact for $\mathbf{F}(n)$.

Definition 5.1 *A set $\mathcal{E}$ is said to be exact for $\mathbf{F}(n)$ in L if the following three conditions hold:*

(1) $\mathbf{ED}_L(n) \cap W_L \subseteq \mathcal{E} \subseteq W_L$,

(2) *for any* $X \in \mathbf{ED}_L(n)$, $\#(X\Downarrow_L \cap \mathcal{E}) = 1$,

(3) *for any* $X \in \mathcal{E}$ *and for any* $Y \in W_L$, XR_LY *implies* $Y \in \mathcal{E}$.

By the following theorem, we can extend (VI) and (VII) in subsection 1.4. Specifically, by (1.1), (2), and (3) of the theorem, we can extend (VI); in other words, we obtain a finite method to list all exact L-models for $\mathbf{F}(n)$ in L. Also, by (1.2), we can extend (VII); in other words, we obtain a finite method to find $S \in 2^{\mathbf{ED}_L(n)}$ satisfying $A \equiv_L \bigwedge \mathbf{for}(S)$ for each $A \in \mathbf{F}(n)$. The method is different from the method provided by Theorem 2.6 in section 2.

Theorem 5.2 *We refer to* κ *as* $\mathbf{a}_L(n, \#(\mathbf{ED}_L(n)))$.

(1) *For any exact set* $\mathcal{E}$ *for* $\mathbf{F}(n)$ *in* L,

  (1.1) $\mathbf{EM}_L|_{\mathcal{E}}$ *is an exact* L*-model for* $\mathbf{F}(n)$ *in* L,

  (1.2) *for any* $A \in \mathbf{F}(n)$,

$$A \equiv_L \bigwedge\{\mathbf{for}(X(n)) \mid X \in \mathcal{E}, (\mathbf{EM}_L|_{\mathcal{E}}, X) \not\models A\}.$$

(2) *For any exact* L*-model* $M = \langle W, R, P\rangle$ *for* $\mathbf{F}(n)$ *in* L, *there exists an exact set* $\mathcal{E}$ *for* $\mathbf{F}(n)$ *in* L *satisfying the following two conditions:*

- $\mathcal{E} \subseteq \bigcup_{i=0}^{\kappa} \mathbf{G}_L^*(i)$,
- M *is isomorphic to* $\mathbf{EM}_L|_{\mathcal{E}}$.

(3) *Every exact set for* $\mathbf{F}(n)$ *in* L *is a subset of* $\bigcup_{i=0}^{\kappa} \mathbf{G}_L^*(i)$.

Proof. For (1). Since $\mathbf{EM}_L$ is an L-model, we have

$$(\mathbf{EM}_L|_{\mathcal{E}}, X) \models L. \tag{1.3}$$

By Lemma 4.22, we have

$$(\mathbf{EM}_L|_{\mathcal{E}}, X) \not\models \mathbf{for}(Y) \text{ if and only if } X \in Y\Downarrow_L,$$

for any $X \in \mathcal{E}$ and for any $Y \in \mathbf{G}_L(k)$. Using (1.3) and Theorem 2.7, we obtain (1.1). Also, we have

$$(P_L|_{\mathcal{E}})(X) = \mathcal{E} - \{X\},$$

and using (1.1), we obtain (1.2).

For (2). Since M is exact $L \cap \mathbf{F}$-model for $\mathbf{F}(n)$ in L, we have

$$\text{for any } A \in \mathbf{F}, (M, \alpha) \models \Box A \supset \Box\Box A, \tag{2.1}$$

$$\mathbf{ED}_L(n) = \{\mathbf{ed}_L(n, \alpha) \mid \alpha \in W\}, \tag{2.2}$$

and

$$P(\mathbf{ed}_L(n, \alpha)) = W - \{\alpha\}. \tag{2.3}$$

Since $\mathbf{ED}_L(n)$ is finite, we can observe that M is finite. Also, by (2.3) and (2.1), we can observe that M is transitive.

We define $\mathcal{E}$ as

$$\mathcal{E} = \{\mathbf{ed}_L(\kappa, \alpha) \mid \alpha \in W\}.$$

By (2.2) and (2.3), we can define a one-to-one mapping f from W onto $\mathbf{ED}_L(n)$ and an one-to-one mapping g from $\mathbf{ED}_L(n)$ onto $\mathcal{E}$ as

$$f(\alpha) = \mathbf{ed}_L(n, \alpha), \quad g(\mathbf{ed}(n, \alpha)) = \mathbf{ed}_L(\kappa, \alpha).$$

In order to prove (2), we have only to show

(2a) $\mathcal{E}$ is an exact set for $\mathbf{F}(n)$ in L and $\mathcal{E} \subseteq \bigcup_{i=0}^{\kappa} \mathbf{G}^*_L(i)$,

(2b) $g \circ f$ is an isomorphism from M to $\mathbf{EM}_L|_{\mathcal{E}}$.

We show (2a). By the definition of $\mathbf{ed}_L(n, \alpha)$,

$$\mathbf{ED}_L(n) \cap W_L = \{\mathbf{ed}_L(\kappa, \alpha) \mid \mathbf{ed}_L(n, \alpha) \in W_L, \alpha \in W\} \subseteq \mathcal{E}.$$

By the finiteness and transitivity of M and Lemma 4.14, we have

$$\mathcal{E} \subseteq \bigcup_{i=0}^{\kappa} \mathbf{G}^*_L(i) \subseteq W_L.$$

Also, we have

$$\text{for any } \mathbf{ed}_L(n, \alpha) \in \mathbf{ED}_L(n), \mathbf{ed}_L(n, \alpha){\Downarrow_L} \cap \mathcal{E} = \{\mathbf{ed}_L(\kappa, \alpha)\}.$$

Therefore, it is sufficient to show

$$\text{for any } \mathbf{ed}_L(\kappa, \alpha) \in \mathcal{E} \text{ and for any } Y \in W_L, \mathbf{ed}_L(\kappa, \alpha) R_L Y \text{ implies } Y \in \mathcal{E}.$$

Suppose that $\mathbf{ed}_L(\kappa, \alpha) \in \mathcal{E}$, $Y \in W_L$, and $\mathbf{ed}_L(\kappa, \alpha) R_L Y$. By $\mathbf{ed}_L(\kappa, \alpha) R_L Y$ and Corollary 2.12(2), we have

$$\Box\mathbf{for}(Y) \to \mathbf{for}(\mathbf{ed}_L(\kappa, \alpha)) \in L.$$

Using (2.3) and $M \models L \cap \mathbf{F}$, we have $(M, \alpha) \not\models \Box\mathbf{for}(Y)$, and thus,

$$(M, \beta) \not\models \mathbf{for}(Y) \text{ for some } \beta \in \{\beta' \in W \mid \alpha R \beta'\}.$$

Using (2.3) and Lemma 4.15, we have

$$\mathbf{ed}_L(\kappa, \beta) \in Y{\Downarrow_L},$$

and using $Y \in W_L$, we have

$$Y = \mathbf{ed}_L(\kappa, \beta) \in \mathcal{E}.$$

We show (2b). We note that $g \circ f$ is one-to-one and onto. We can easily observe that $\{\mathbf{ed}_L(\kappa, \alpha) \mid \alpha \in P(p_i)\} = P_L(p_i)$. By (2.3), we have

$$\alpha R \beta \text{ if and only if } (M, \alpha) \not\models \Box\mathbf{for}(\mathbf{ed}_L(\kappa, \beta)).$$

Using (2.3), $M \models L \cap \mathbf{F}$, and Lemma 4.16 we obtain

$$\alpha R \beta \text{ if and only if } \mathbf{ed}_L(\kappa, \alpha) R_L \mathbf{ed}_L(\kappa, \beta).$$

For (3). Let $\mathcal{E}$ be an exact set for $\mathbf{F}(n)$ in L. Then by (1), $\mathbf{EM}_L|_{\mathcal{E}}$ is an exact $L \cap \mathbf{F}$-model for $\mathbf{F}(n)$ in L. We define g and f as in the proof of (2). Then $g \circ f$ is identity. Using (2), we obtain (3). ■

References

Bellissima, F. (1985). An effective representation for finitely generated free algebras. *Algebra Universalis*, **20**, 302–317.

Chagrov, A., & Zakharyaschev, M. (1997). *Modal logic.* New York: Oxford University Press.

de Bruijn, N. G. (1975). *Exact finite models for minimal propositional calculus over a finite alphabet.* Report 75-WSK-02, Eindhoven, The Netherlands: Technological University Eindhoven.

Diego, A. (1966). *Sur les algèbres de Hilbert.* Paris: Gauthier-Villars.

Esakia, L., & Grigolia, R. (1975). Christmas trees. On free cyclic algebras in some varieties of closure algebras. *Bulletin of the Section of Logic*, **4**, 95–102.

Esakia, L., & Grigolia, R. (1977). The criterion of Brouwerian and closure algebras to be finitely generated. *Bulletin of the Section of Logic*, **6**, 46–52.

Fine, K. (1975). Normal forms in modal logic. *Notre Dame Journal of Formal Logic*, **16**, 229–237.

Gentzen, G. (1934–35). Untersuchungen über das logisch Schliessen. *Mathematische Zeitschrift*, **39**, 176–210, 405–431.

Ghilardi, G. (1995). An algebraic theory of normal forms. *Annals of Pure and Applied Logic*, **71**, 189–245.

Hendriks, L. (1996). *Computations in propositional logic.* ILLC Dissertation Series DS-1996-01, Amsterdam, The Netherland: Institute for Logic, Language and Computation, University of Amsterdam.

Moss, L. S. (2007). Finite models constructed from canonical formulas. *Journal of Philosophical Logic*, **36**, 605–640.

Nishimura, I. (1960). On formulas of one variable in intuitionistic propositional calculus. *Journal of Symbolic Logic*, **25**, 327–331.

Rieger, L. (1949). On lattice of Brouwerian propositional logics. *Acta Universitatis Carolinae. Mathematica et Physica*, **189**, 1–40.

Sasaki, K. (2001) *Logics and Provability.* ILLC Dissertation Series DS-2001-07, Amsterdam, The Netherland: Institute for Logic, Language and Computation, University of Amsterdam.

Sasaki, K. (2010a). Formulas in modal logic **S4**. *The Review of Symbolic Logic*, **3**, 600–627.

Sasaki, K. (2010b). *Constructions of normal forms in modal logic* **K4**. Technical Report of the Nanzan Academic Society Information Sciences and Engineering, NANZAN-TR-2010-01, Seto, Japan: Nanzan University.

Shehtman, V. B. (1978). Rieger-Nishimura ladders. *Doklady Akademii Nauk SSSR*, **241**, 1288–1291.

Urquhart, A. (1973). Free Heyting algebras. *Algebras Universalis*, **3**, 94–97.

Urquhart, A. (1974). Implicational formulas in intuitionistic logic. *The Journal of Symbolic Logic*, **39**, 661-664.

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