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Abstract

We propose a new neighborhood, the (c_1, c_2, γ) - neighborhood, to describe the departure of data from an assumed model distribution. The neighborhood is generated from a special capacity determined by the three parameters, and as special cases it includes not only the commonly used neighborhoods defined in terms of ε - contamination and total variation but also various interesting new neighborhoods. We give the characterization of the (c_1, c_2, γ) - neighborhood in three forms, which reveals that the neighborhood is intuitively understandable and useful. As an important application to robust inference, when the data distribution is unknown and the data may be contaminated, under the (c_1, c_2, γ) - neighborhood we construct robust nonparametric confidence intervals and tests for the median based on the sign test statistics. These constructed tests and confidence intervals are the effectively robustified versions of the sign test and its associated confidence interval. We investigate their robustness and efficiency by deriving the maximum asymptotic lengths and consistency distances.

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Key words: Breakdown point, (c_1, c_2, γ) - neighborhood, consistency distance, maximum asymptotic length, median, robust nonparametric confidence interval, robust nonparametric test, sign test, special capacity.

1 Introduction

The theory of robust statistical inference aims at deriving reliable parameter estimates and their associated tests and confidence intervals when the data not only exactly but also approximately follow an assumed model distribution. Since Huber treated robust statistical inference mathematically well in his pioneer works (1964, 1965, 1968), a large number of contributions have been made to this field. Their results are found in the influential books such as Huber (1981),

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Hampel, Ronchetti, Rousseeuw and Stahel (1986), Rieder (1994), Maronna, Martin and Yohai (2006), Huber and Ronchetti (2009) and others. In this robust theory the departure of data from an assumed model is usually expressed by some suitably chosen neighborhood of the model distribution. The various types of neighborhoods have been used to describe the departure to date. Among them, the neighborhoods defined in terms of ε - contamination and total variation distance have been most frequently adopted in the literatures. As a combination of such two neighborhoods, Rieder (1977) introduced a neighborhood generated from a special capacity, which we call Rieder's neighborhood, and he used it in his many works on robust inference (see Rieder, 1978, 1980, 1981a, 1981b). Ando and Kimura (2003) proposed the (c, γ) - neighborhood which is a generalization of Rieder's neighborhood and gave its applications to robust estimation (see Ando and Kimura, 2004). Although the (c, γ) - neighborhood includes many important special neighborhoods, the class of (c, γ) - neighborhoods seems to be a little large in practical use.

In this paper, from the practical point of view, as a natural and useful neighborhood we introduce a new neighborhood, the (c_1, c_2, γ) - neighborhood, generated from a special capacity determined by the three parameters. By changing the values of the three parameters, the (c_1, c_2, γ) - neighborhood yields not only the (c, γ) - neighborhood but also various important neighborhoods. We present a list of such representative neighborhoods. In particular, we are interested in two neighborhoods, that is, the total variation neighborhood partially narrowed by ε - contamination ($TN\varepsilon$ - neighborhood, for short) and the neighborhood indicating a gap from the model distribution (G - neighborhood, for short). The special capacity, which was comprehensively studied by Bednarski (1981), satisfies all the conditions of Choquet's 2-alternating capacity except condition (4) in Huber and Strassen (1973). Since the (c_1, c_2, γ) - neighborhood is generated from a special capacity, it has nice properties for developing minimax theory in robust inference. We give three characterizations (Theorems 3.1 and 3.2, Corollary 3.1) of the (c_1, c_2, γ) - neighborhood, which show that the neighborhood consists of all γ - contamination of distributions in a certain neighborhood indicating a gap from the model distributions determined by c_1 and c_2. The characterization (Theorem 3.2) reveals that the (c_1, c_2, γ) - neighborhood is useful and intuitively understandable. We also find the stochastically smallest and largest (improper) distributions and the upper bound of Kolmogorov distance for the (c_1, c_2, γ) - neighborhood. These results play important roles in construction of robust procedures.

As an important application of the (c_1, c_2, γ) - neighborhood to robust inference, when the data may be contaminated, we consider the problem of constructing robust confidence intervals and tests based on the sign test statistics for the median of an unknown model distribution F°. This problem was first treated under the ε - contamination neighborhood by Yohai and Zamar (2004), and then under the (c, γ) - neighborhood by Ando, Kakiuchi and Kimura (2009), which showed that the standard sign test and its associated confidence interval for the median are not robust. Hettmansperger and Mckean (2011) also discussed robust nonparametric statistical methods derived by the weighted L_1 norm. We construct robust confidence intervals and tests for the median of an unknown model distribution F° under the (c_1, c_2, γ) - neighborhood (Theorems 4.1 and 5.1). In order to investigate their robustness and efficiency, we consider the asymptotic behaviors of intervals and tests under the the $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ - neighborhood, where $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ denotes the real size. We should understand the difference between the real size and the design size (c_1, c_2, γ). The latter design size is used to construct the robust confidence intervals and tests. We derive the maximum asymptotic length and the breakdown point of the confidence intervals

(Theorems 4.2 and 4.3). We also show that when the data is uncontaminated, the constructed confidence interval is efficient in the sense that it has the smallest asymptotic length among all nonparametric robust confidence intervals for the median under the (c_1, c_2, γ) - neighborhood (Theorem 4.4). As for the constructed tests, using the notions (power robustness, power distance, power breakdown point) introduced by Yohai and Zamar (2004) we study their robustness and efficiency (Theorems 5.2 and 5.3). Our results refine those of Yohai and Zamar (2004) and Ando, Kakiuchi and Kimura (2009), and include all their results as special cases.

The papers is organized as follows. Section 2 presents the definition of the (c_1, c_2, γ) - neighborhood and a list of representative special cases. Section 3 gives three forms of characterizations of the (c_1, c_2, γ) - neighborhood. Section 4 constructs robust nonparametric confidence intervals under the (c_1, c_2, γ) - neighborhood, and study their robustness and efficiency. Section 5 considers robust nonparametric tests and treats their construction and robustness. Section 6 collects all the proofs of lemmas and theorems.

2 The $(\mathbf{c_1}, \mathbf{c_2}, \boldsymbol{\gamma})$ - neighborhood

Let $\mathbb{X}$ be a Polish space (a complete, separable and metrizable space), $\mathscr{B}$ the Borel σ - algebra of subsets of $\mathbb{X}$ and $\mathscr{M}$ the set of all probability measures on $(\mathbb{X}, \mathscr{B})$. For some specified $F^\circ \in \mathscr{M}$ we propose the following new class of neighborhoods $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ of F° with three parameters c_1, c_2 and γ, which we call the (c_1, c_2, γ) - neighborhood.

Definition 2.1 *The (c_1, c_2, γ) - neighborhood is defined by*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G \in \mathscr{M} \mid c_1 F^\circ\{A\} \le G\{A\} \le c_2 F^\circ\{A\} + \gamma, {}^\forall A \in \mathscr{B}\}, \tag{2.1}$$

where c_1, c_2 and γ are the parameters such that

$$0 \le c_1 \le 1 - \gamma \le c_2 < \infty, \;\; c_1 \ne c_2 \;\; \textit{and} \;\; 0 \le \gamma < 1. \tag{2.2}$$

We note that $c_1 F^\circ\{A\} \le G\{A\}$ for all $A \in \mathscr{B}$ is equivalent to $G\{A\} \le c_1 F^\circ\{A\} + 1 - c_1$ for all $A \in \mathscr{B}$. Therefore the (c_1, c_2, γ) - neighborhood is expressed in the form of

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G \in \mathscr{M} \mid G\{A\} \le \min(c_2 F^\circ\{A\} + \gamma,\; c_1 F^\circ\{A\} + 1 - c_1), {}^\forall A \in \mathscr{B}\}. \tag{2.3}$$

This shows that the (c_1, c_2, γ) - neighborhood is generated from the following special capacity v_h: Let

$$h(t) = \min(c_2 t + \gamma,\; c_1 t + 1 - c_1), \quad 0 \le t \le 1, \tag{2.4}$$

and let

$$v_h\{A\} = \begin{cases} h(F^\circ\{A\}) & \text{if} \quad \phi \ne {}^\forall A \in \mathscr{B}, \\ 0 & \text{if} \quad A = \phi. \end{cases} \tag{2.5}$$

Then, from (2.3) we obtain

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G \in \mathscr{M} \mid G\{A\} \leq v_h\{A\} \text{ for } {}^\forall A \in \mathscr{B}\}. \tag{2.6}$$

By Lemma 3.1 of Bednarski (1981) v_h is a special capacity, which satisfies all the conditions of Choquet's 2-alternating capacity except condition (4) in Huber and Strassen (1973). Thus the (c_1, c_2, γ)-neighborhood is generated from the special capacity v_h with (2.4) and it has nice properties for developing minimax theory in robust inference.

Remark 2.1 We removed the case of $c_1 = c_2$ in the condition (2.2). The reason is that when $c_1 = c_2$ $(= 1-\gamma)$, we have $h(t) = (1-\gamma)t + \gamma$, which is the γ-contamination case. We also note that either $c_1 = 1-\gamma$ or $c_2 = 1-\gamma$ leads to the γ-contamination case.

Changing the values of c_1, c_2 and γ, we can get various neighborhoods, for example, the (c, γ)-neighborhood introduced by Ando and Kimura (2003), and hence as its special cases Rieder's neighborhood as well as the neighborhoods defined in terms of ε-contamination and total variation distance. We now present a list of representative and important neighborhoods and the corresponding h functions given by (2.4).

(i) ε-contamination neighborhood $\mathscr{P}_{c_1,1-\varepsilon,\varepsilon}(F^\circ)$ or $\mathscr{P}_{1-\varepsilon,c_2,\varepsilon}(F^\circ)$: With $(c_1, c_2, \gamma) = (1-\varepsilon, c_2, \varepsilon)$ or $(c_1, 1-\varepsilon, \varepsilon)$, and $h(t) = (1-\varepsilon)t + \varepsilon$, where $0 \leq c_1 < 1-\varepsilon < c_2$ and $0 \leq \varepsilon < 1$.

(ii) Total variation neighborhood $\mathscr{P}_{0,1,\delta}(F^\circ)$: With $(c_1, c_2, \gamma) = (0, 1, \delta)$ and $h(t) = \min(t+\delta,\ 1)$, where $0 \leq \delta < 1$.

(iii) Rieder's neighborhood $\mathscr{P}_{0,1-\varepsilon,\varepsilon+\delta}(F^\circ)$: With $(c_1,\ c_2, \gamma) = (0, 1-\varepsilon, \varepsilon+\delta)$ and $h(t) = \min\{(1-\varepsilon)t + \varepsilon + \delta, 1\}$, where $0 \leq \varepsilon,\ 0 \leq \delta$ and $\varepsilon + \delta < 1$.

(iv) (c, γ)-neighborhood $\mathscr{P}_{0,c,\gamma}(F^\circ)$: With $(c_1, c_2, \gamma) = (0, c, \gamma)$ and $h(t) = \min(ct + \gamma, 1)$, where $0 \leq \gamma < 1$ and $1-\gamma \leq c < \infty$.

(v) $TN\varepsilon$-neighborhood $\mathscr{P}_{1-\varepsilon,1,\delta}(F^\circ)$: With $(c_1, c_2, \gamma) = (1-\varepsilon, 1, \delta)$ and $h(t) = \min\{t+\delta, (1-\varepsilon)t + \varepsilon\}$, where $0 < \delta < \varepsilon < 1$.

(vi) G-neighborhood $\mathscr{P}_{c_1,c_2,0}(F^\circ)$: With $(c_1, c_2, \gamma) = (c_1, c_2, 0)$ and $h(t) = \min(c_2 t,\ c_1 t + 1 - c_1)$, where $0 < c_1 < 1 < c_2 < \infty$.

We should notice that if $c_1 \neq 0$, $c_1 \neq 1-\gamma$ and $c_2 \neq 1-\gamma$, then the graph of h corresponding to the (c_1, c_2, γ)-neighborhood is a broken line intersecting at the inside of the square and hence so do the graphs of h corresponding to the $TN\varepsilon$-neighborhood (the total variation neighborhood partially narrowed by the ε-contamination neighborhood) and the G-neighborhood (the neighborhood without contamination indicating a gap from the model distribution). These two neighborhoods are new, and we have a special interest.

3 Characterization of $(\mathbf{c_1}, \mathbf{c_2}, \boldsymbol{\gamma})$-neighborhood

We give three characterizations of the (c_1, c_2, γ)-neighborhood. The first two are for the general case and the third one is for the case of $\mathbb{X} = \mathbb{R}$, the real line.

Theorem 3.1 *It holds that*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G = c_2(F^\circ - W) + \gamma K \in \mathscr{M} \mid \ W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ),\ K \in \mathscr{M}\}, \tag{3.1}$$

where $\mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$ *is the set of all measures* W *on* $(\mathbb{X},\mathscr{B})$ *such that* $0 \le W\{A\} \le \{(c_2 - c_1)/c_2\}F^\circ\{A\}$ *for any* $A \in \mathscr{B}$ *and* $W\{\mathbb{X}\} = (c_2 - 1 + \gamma)/c_2$.

Corollary 3.1 *Suppose that* $c_1 \neq 0$. *Then it holds that*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G = c_1(F^\circ + V) + \gamma K \in \mathscr{M} \mid V \in \mathscr{V}_{c_1,c_2,\gamma}(F^\circ),\ K \in \mathscr{M}\}, \tag{3.2}$$

where $\mathscr{V}_{c_1,c_2,\gamma}(F^\circ)$ *is the set of all measures* V *on* $(\mathbb{X},\mathscr{B})$ *such that* $0 \le V\{A\} \le \{(c_2 - c_1)/c_1\}F^\circ\{A\}$ *for all* $A \in \mathscr{B}$ *and* $V\{\mathbb{X}\} = (1-\gamma-c_1)/c_1$.

From Theorem 3.1 and Corollary 3.1 it can be seen that the (c_1, c_2, γ) - neighborhood consists of all the γ contamination of elements in a certain class of continuous distributions. In order to express the (c_1, c_2, γ) - neighborhood in the intuitively more understandable form by using density functions, we hereafter consider the case of $\mathbb{X} = \mathbb{R}$, the real line. Let F° be an absolutely continuous distribution function on $\mathbb{R}$ and let f° be a density function of F° (with respect to the Lebesgue measure). Also, let $\mathscr{M}_c(\subset \mathscr{M})$ be the set of all absolutely continuous distributions on $(\mathbb{R}, \mathscr{B})$.

Theorem 3.2 *It holds that*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G = (1-\gamma)F + \gamma K \in \mathscr{M} \mid F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ),\ K \in \mathscr{M}\}, \tag{3.3}$$

where

$$\mathscr{F}_{c_1,c_2,\gamma}(F^\circ) = \left\{F \in \mathscr{M}_c \;\Big|\; \frac{c_1}{1-\gamma}f^\circ \le f \le \frac{c_2}{1-\gamma}f^\circ\right\} \tag{3.4}$$

and f *is a density function of* F.

Remark 3.1 The concrete forms of (3.3) and (3.4) for the neighborhoods (i) $-$ (vi) in Section 2 are obtained by substituting the respective values of the parameters (c_1, c_2, γ). For example, the $TN\varepsilon$ - neighborhood is represented as

$$\mathscr{P}_{1-\varepsilon,1,\delta}(F^\circ) \ = \{G = (1-\delta)F + \delta K \in \mathscr{M} \mid F \in \mathscr{F}_{1-\varepsilon,1,\delta}(F^\circ),\ K \in \mathscr{M}\}$$

and

$$\mathscr{F}_{1-\varepsilon,1,\delta}(F^\circ) \ = \left\{F \in \mathscr{M}_c \;\Big|\; \frac{1-\varepsilon}{1-\delta}f^\circ \le f \le \frac{1}{1-\delta}f^\circ\right\}.$$

Next, we give the stochastically smallest and largest distribution functions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$. Let F_L° and F_R° be the distributions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ defined as

$$F_L^\circ(x) = \begin{cases} \dfrac{c_2}{1-\gamma}F^\circ(x) & \text{if } \ x \le x_L, \\[2ex] \dfrac{c_1}{1-\gamma}F^\circ(x) + \left(1 - \dfrac{c_1}{1-\gamma}\right) & \text{if } \ x > x_L, \end{cases} \tag{3.5}$$

and

$$F_R^\circ(x) = \begin{cases} \dfrac{c_1}{1-\gamma} F^\circ(x) & \text{if } \ x \le x_R, \\ \dfrac{c_2}{1-\gamma} F^\circ(x) + \left(1 - \dfrac{c_2}{1-\gamma}\right) & \text{if } \ x > x_R, \end{cases} \tag{3.6}$$

where

$$x_L = (F^\circ)^{-1}\left(\frac{1-\gamma-c_1}{c_2-c_1}\right) \tag{3.7}$$

and

$$x_R = (F^\circ)^{-1}\left(\frac{c_2-1+\gamma}{c_2-c_1}\right). \tag{3.8}$$

Theorem 3.3 *Let F_L° and F_R° be given in* (3.5) *and* (3.6)*. Then the following results hold:*

(i) *For any $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$*

$$F_R^\circ(x) \le F(x) \le F_L^\circ(x) \quad \textit{for all } x \in \mathbb{R}. \tag{3.9}$$

(ii) *For any $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$*

$$(1-\gamma)F_R^\circ(x) \le G(x) \le (1-\gamma)F_L^\circ(x) + \gamma \quad \textit{for all } x \in \mathbb{R}. \tag{3.10}$$

Theorem 3.4 *Let $d_K(G,H) = \sup_x |G(x) - H(x)|$ be the Kolmogorov distance between G and H in $\mathscr{M}$. Then it holds that*

$$\sup_{G,H \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)} d_K(G,H) = \min\{(1-\gamma) - c_1, c_2 - (1-\gamma)\} + \gamma. \tag{3.11}$$

We define the *size* of the (c_1, c_2, γ) - neighborhood as

$$\lambda = \min\{(1-\gamma) - c_1, c_2 - (1-\gamma)\} + \gamma. \tag{3.12}$$

The λ is decomposed into $g = \min\{(1-\gamma) - c_1,\ c_2 - (1-\gamma)\}$ and γ. Here we note

$$(1-\gamma) \sup_{F,F' \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)} d_K(F, F') = g.$$

Since any element G in $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ is written as $G = (1-\gamma)F + \gamma K,\ F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ),\ K \in \mathscr{M}$, we can regard the (c_1, c_2, γ) - neighborhood as a mixture of a gap from the model distribution F° and its γ-contamination. We call g and γ the *gap size* and the *contamination size* of the (c_1, c_2, γ) - neighborhood, respectively. Table 1 shows the values λ, g and γ for the representative neighborhoods (i) – (vi) in Section 2. In particular, we have that $\lambda = \varepsilon\,(0 \le \varepsilon < 1)$ for the G-neighborhood (vi) when either $c_1 = 1 - \varepsilon$, $c_2 \ge 1 + \varepsilon$ or $c_1 \le 1 - \varepsilon$, $c_2 = 1 + \varepsilon$. It should be noted that although the ε - contamination neighborhood (i) and the G - neighborhood are

Table 1: The values of λ, g and γ for the neighborhoods (i) – (vi)

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
λ	ε	2δ	$\varepsilon + 2\delta$	$c + 2\gamma - 1$	$\min(\varepsilon, 2\delta)$	$\min(1 - c_1, c_2 - 1)$
g	0	δ	δ	$c + \gamma - 1$	$\min(\varepsilon - \delta, \delta)$	$\min(1 - c_1, c_2 - 1)$
γ	ε	δ	$\varepsilon + \delta$	γ	δ	0

heterogeneous, their λ are the same value ε. We can also see that λ of the $TN\varepsilon$ - neighborhood (v) is smaller than those of the ε - contamination neighborhood and the total variation neighborhood (ii) and that γ of the G - neighborhood is 0.

It should be emphasized that in modeling of the data distribution the decomposition of the neighborhood into the gap and contamination is important for practical and conceptual analysis of robust inference.

4 Robust nonparametric confidence intervals

The purpose of this section is to construct robust nonparametric confidence intervals for the median of the unknown model (target) distribution F° when the data distribution is in $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$. Note that our interest is in the median of the model distribution but not in that of the data distribution. We now assume the following two conditions:

(C1) F° is absolutely continuous with a unique median $\theta = (F^\circ)^{-1}(1/2)$.

(C2) $0 \leq \gamma < 1/2$ and $c_2 < 2(1 - \gamma)$.

The second condition of (C2) guarantees that $0 < F(\theta) < 1$ holds for all $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$, that is, the median θ always lies inside the support of F.

Let $\boldsymbol{X}_n = (X_1, \cdots, X_n)$ be an independent and identically distributed random sample with a common distribution $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$, where F° is unknown, and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics of $\boldsymbol{X}_n$. The sign test statistic can be used to construct the robust nonparametric confidence interval, which is given by

$$T_{n,\theta}(\boldsymbol{X}_n) = \sum_{i=1}^{n} I_{(0,\infty)}(X_i - \theta), \tag{4.1}$$

where $I_{(0,\infty)}$ denotes the indicator function on $(0, \infty)$. Then, the associated confidence interval for θ is

$$I_n = [X_{(k_n+1)}, X_{(n-k_n)}), \tag{4.2}$$

which is obtained by inverting the acceptance region of the sign test, $k_n < T_{n,\theta}(\boldsymbol{X}_n) < n - k_n$, where k_n is the integer determined by given n, $\alpha\,(0 < \alpha < 1)$ and $\lambda\,(0 \leq \lambda < 1)$ as follows:

$$k_n = k_n(n, \alpha, \lambda) = \arg\min_{k} |\alpha^*(n, k, \lambda) - \alpha|, \tag{4.3}$$

where

$$\alpha^* = \alpha^*(n, k, \lambda) = 1 - P(k < Z_n < n - k) \tag{4.4}$$

and Z_n is distributed with the Binomial distribution $B\,(n, (1-\lambda)/2)$.

The problem is to determine the value of λ for which I_n is a nonparametric and robust confidence interval under $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$, and now we let λ be the size of the (c_1, c_2, γ) - neighborhood given by (3.12).

4.1 The $(\mathbf{c_1}, \mathbf{c_2}, \boldsymbol{\gamma})$ - robust confidence interval

Definition 4.1 A confidence interval $I_n = [a_n(\boldsymbol{X}_n), b_n(\boldsymbol{X}_n))$ is said to have *nonparametric* (c_1, c_2, γ) *- robust coverage* $1 - \alpha$ if for all F°

$$\inf_{G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)} P_G\{a_n(\boldsymbol{X}_n) \leq \theta < b_n(\boldsymbol{X}_n)\} = 1 - \alpha, \tag{4.5}$$

where P_G means the probability under G.

Theorem 4.1 *Let* $I_n = [X_{(k_n+1)}, X_{(n-k_n)})$ *be the confidence interval given by* (4.3). *Then the following results hold:*

(i) I_n *has nonparametric* (c_1, c_2, γ)*-robust coverage* $1 - \alpha^*(n, k_n, \lambda)$, *that is, for all* F°

$$\inf_{G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)} P_G\{X_{(k_n+1)} \leq \theta < X_{(n-k_n)}\} = 1 - \alpha^*(n, k_n, \lambda). \tag{4.6}$$

(ii) *The infimum in* (4.6) *is achieved for any* γ *- contaminating distribution of* F_L° (*or* F_R°) *which places all its mass to the left* (*or the right*) *of* θ, *where* F_L° *and* F_R° *are the stochastically smallest and largest distributions in* $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ *given by* (3.5) *and* (3.6), *respectively.*

Remark 4.1 (i) We note that $\lim_{n\to\infty} \alpha^*(n, k_n, \lambda) = \alpha$, that is, the exact (i.e., real) coverage probability $1 - \alpha^*$ of the confidence interval (4.2) asymptotically converges to the nominal coverage probability $1 - \alpha$.

(ii) Since $0 \leq \gamma \leq \lambda < 1$, it follows that $\lambda = 0$ implies $\gamma = 0$ and hence $c_1 = 1$ or $c_2 = 1$. Thus, if $\lambda = 0$, then we have $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{F^\circ\}$ and I_n becomes the usual confidence interval derived from the sign test.

Theorem 4.1 states that the nonparametric (c_1, c_2, γ) - robust confidence interval I_n with coverage probability $1-\alpha^*(n, k_n, \lambda)$ is determined only through λ. The confidence intervals I_n with coverage probability $1-\alpha^*(n, k_n, \lambda)$ for $\lambda = \varepsilon$ and $\lambda = c+2\gamma-1$ are the same as the nonparametric ε - robust confidence interval in Yohai and Zamar (2004) and the nonparametric (c, γ) - robust confidence interval in Ando, Kakiuchi and Kimura (2009), respectively. This fact implies that their confidence intervals have also nonparametric (c_1, c_2, γ) - robust coverage $1-\alpha^*(n, k_n, \lambda)$ for all c_1, c_2 and γ such that $\lambda = \varepsilon$ or $\lambda = c + 2\gamma - 1$, respectively.

4.2 Robustness of the $(\mathbf{c_1}, \mathbf{c_2}, \gamma)$ - robust confidence interval

As shown in Theorem 4.1, for some given n, α and $\lambda(= \lambda(c_1, c_2, \gamma))$ we can construct the (c_1, c_2, γ) - robust coverage $1 - \alpha^*$ confidence interval I_n. We call the (c_1, c_2, γ) the design size. In what follows, we investigate the robustness and efficiency of I_n under the real size $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$, which is assumed to satisfy the conditions (2.2) and (C2) with (c_1, c_2, γ) replaced by $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$, that is,

$$0 \leq \tilde{c}_1 \leq 1 - \tilde{\gamma} \leq \tilde{c}_2 < 2(1 - \tilde{\gamma}), \ \tilde{c}_1 \neq \tilde{c}_2 \quad \text{and} \quad 0 \leq \tilde{\gamma} < 1/2. \tag{4.7}$$

Thus we have to understand the difference between (c_1, c_2, γ) and $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ clearly.

First, as a measure of the efficiency of $\{I_n\}$ we consider its maximum asymptotic length $L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ under the real size $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ at F°, which is defined as

$$L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = \sup_{G \in \mathscr{P}_{\tilde{c}_1, \tilde{c}_2, \tilde{\gamma}}(F^\circ)} \operatorname{essup} \limsup_{n \to \infty} (X_{(n-k_n)} - X_{(k_n+1)}),$$

where essup stands for essential supremum.

Theorem 4.2 *Suppose that F° has a symmetric (around θ) and unimodal density. Let $0 < \alpha < 1$ and let λ be fixed and given by (3.12). Let the sequence $\{I_n\}$ of confidence intervals $I_n = [X_{(k_n+1)}, X_{(n-k_n)})$ with k_n given by (4.3). Then, for $0 \leq \lambda < 1 - 2\tilde{\gamma}$ the following results hold:*

(i) *If* $0 \leq \lambda < \max\left\{0, 1 - 2\tilde{\gamma} - \dfrac{2\tilde{c}_2(1 - \tilde{\gamma} - \tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1}\right\}$, *then*

$$L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = (F^\circ)^{-1}\left(1 - \frac{1 - \lambda}{2\tilde{c}_1}\right) - (F^\circ)^{-1}\left(1 - \frac{1 + \lambda}{2\tilde{c}_1}\right). \tag{4.8}$$

(ii) *If* $\max\left\{0, 1 - 2\tilde{\gamma} - \dfrac{2\tilde{c}_2(1 - \tilde{\gamma} - \tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1}\right\} \leq \lambda < \min\left\{\dfrac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1}, 1 - 2\tilde{\gamma}\right\}$, *then*

$$L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = (F^\circ)^{-1}\left(\frac{\lambda}{\tilde{c}_1} + \frac{1 - 2\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\frac{1 - 2\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right). \tag{4.9}$$

(iii) *If* $\min\left\{\dfrac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1}, 1 - 2\tilde{\gamma}\right\} \leq \lambda < 1 - 2\tilde{\gamma}$, *then*

$$L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = (F^\circ)^{-1}\left(1 - \frac{1 - \lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\frac{1 - 2\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right). \tag{4.10}$$

Remark 4.2 If $\tilde{c}_1 \neq 0$, then $L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ is classified into the three cases of (i)-(iii) of Theorem 4.2. If $\tilde{c}_1 = 0$, then only the case (iii) is applied and it is the same with that in Theorem 2.2 of Ando, Kakiuchi and Kimura (2009). If either $\tilde{c}_1 = 1 - \tilde{\gamma}$ or $\tilde{c}_2 = 1 - \tilde{\gamma}$, then the case (i) or the case (iii) is applied, respectively, and they are the same with that in Theorem 2 of Yohai and Zamar (2004).

The following Lemma gives the supplementary results of Theorem 4.2 with respect to the conditions of λ. As a consequence, we obtain the conditions of $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ for which the maximum asymptotic lengths (4.8), (4.9) and (4.10) in Theorem 4.2 holds.

Lemma 4.1 *Assume that* $\tilde{c}_1 \neq 0$, $\tilde{c}_1 \neq 1-\tilde{\gamma}$ *and* $\tilde{c}_2 \neq 1-\tilde{\gamma}$. *Let* $A = \max(0, a)$ *and* $B = \min(b, 1-2\tilde{\gamma})$, *where* $a = 1-2\tilde{\gamma}-2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)/(\tilde{c}_2-\tilde{c}_1)$ *and* $b = \tilde{c}_1(\tilde{c}_2-1+\tilde{\gamma})/(\tilde{c}_2-\tilde{c}_1)$, *respectively. Then the following results hold:*

(i) *If* $0 < \tilde{c}_1 \leq 1/2$, *then*

$$(A, B) = \begin{cases} (0, b) & \text{for} \quad 0 \leq \tilde{\gamma} < \tilde{c}_2(1-\tilde{c}_1)/(2\tilde{c}_2-\tilde{c}_1), \\ (0, 1-2\tilde{\gamma}) & \text{for} \quad \tilde{c}_2(1-\tilde{c}_1)/(2\tilde{c}_2-\tilde{c}_1) \leq \tilde{\gamma} < 1/2. \end{cases}$$

(ii) *If* $1/2 < \tilde{c}_1 \leq 2\tilde{c}_2/(4\tilde{c}_2-1)$, *then*

$$(A, B) = \begin{cases} (0, b) & \text{for} \quad 0 \leq \tilde{\gamma} < \tilde{c}_2(1-\tilde{c}_1)/(2\tilde{c}_2-\tilde{c}_1), \\ (0, 1-2\tilde{\gamma}) & \text{for} \quad \tilde{c}_2(1-\tilde{c}_1)/(2\tilde{c}_2-\tilde{c}_1) \leq \tilde{\gamma} < 1/2+\tilde{c}_2(1-2\tilde{c}_1)/(2\tilde{c}_1), \\ (a, 1-2\tilde{\gamma}) & \text{for} \quad 1/2+\tilde{c}_2(1-2\tilde{c}_1)/(2\tilde{c}_1) \leq \tilde{\gamma} < 1/2. \end{cases}$$

(iii) *If* $2\tilde{c}_2/(4\tilde{c}_2-1) < \tilde{c}_1 \leq \tilde{c}_2/(2\tilde{c}_2-1)$, *then*

$$(A, B) = \begin{cases} (0, b) & \text{for} \quad 0 \leq \tilde{\gamma} < 1/2+\tilde{c}_2(1-2\tilde{c}_1)/(2\tilde{c}_1), \\ (a, b) & \text{for} \quad 1/2+\tilde{c}_2(1-2\tilde{c}_1)/(2\tilde{c}_1) \leq \tilde{\gamma} < \tilde{c}_2(1-\tilde{c}_1)/(2\tilde{c}_2-\tilde{c}_1), \\ (a, 1-2\tilde{\gamma}) & \text{for} \quad \tilde{c}_2(1-\tilde{c}_1)/(2\tilde{c}_2-\tilde{c}_1) \leq \tilde{\gamma} < 1/2. \end{cases}$$

(iv) *If* $\tilde{c}_2/(2\tilde{c}_2-1) < \tilde{c}_1$, *then*

$$(A, B) = \begin{cases} (a, b) & \text{for} \quad 0 \leq \tilde{\gamma} < \tilde{c}_2(1-\tilde{c}_1)/(2\tilde{c}_2-\tilde{c}_1), \\ (a, 1-2\tilde{\gamma}) & \text{for} \quad \tilde{c}_2(1-\tilde{c}_1)/(2\tilde{c}_2-\tilde{c}_1) \leq \tilde{\gamma} < 1/2. \end{cases}$$

Remark 4.3 If $\tilde{c}_2 \leq 1$, then $\tilde{c}_2/(2\tilde{c}_2-1) > 1$, and hence the part (iv) of Lemma 4.1 is not applied.

Example 4.1 We consider the maximum asymptotic length $L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ under the $TN\varepsilon$- neighborhood, where $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma}) = (1-\tilde{\varepsilon}, 1, \tilde{\delta})$. The assumptions of Lemma 4.1 are satisfied with $0 < \tilde{\delta} < \tilde{\varepsilon} < 1$ *and* $\tilde{\delta} < 1/2$, and the conditions of (i), (ii) and (iii) in Lemma 4.1 correspond to (i) $1/2 \leq \tilde{\varepsilon} < 1$, (ii) $1/3 \leq \tilde{\varepsilon} < 1/2$ and (iii) $0 < \tilde{\varepsilon} < 1/3$, respectively. Here, from the viewpoint of similarity of the calculation we only show $L\{I_n, F^\circ, (\tilde{\varepsilon}, \tilde{\delta})\}$ for the case (iii). From Theorem 4.2 we obtain the following results:

(a) Let $0 < \tilde{\delta} < \tilde{\varepsilon}/\{2(1-\tilde{\varepsilon})\}$. If $0 \leq \lambda < (1-\tilde{\varepsilon})\tilde{\delta}/\tilde{\varepsilon}$, then

$$L\{I_n, F^\circ, (\tilde{\varepsilon}, \tilde{\delta})\} = (F^\circ)^{-1}\left(\frac{1-2\tilde{\delta}+\lambda}{2(1-\tilde{\varepsilon})}\right) - (F^\circ)^{-1}\left(\frac{1-2\tilde{\delta}-\lambda}{2}\right), \tag{4.11}$$

and if $(1-\tilde{\varepsilon})\tilde{\delta}/\tilde{\varepsilon} \leq \lambda < 1-2\tilde{\delta}$, then

$$L\{I_n, F^\circ, (\tilde{\varepsilon}, \tilde{\delta})\} = (F^\circ)^{-1}\left(\frac{1+\lambda}{2}\right) - (F^\circ)^{-1}\left(\frac{1-2\tilde{\delta}-\lambda}{2}\right). \tag{4.12}$$

(b) Let $\tilde{\varepsilon}/\{2(1-\tilde{\varepsilon})\} \le \tilde{\delta} < \tilde{\varepsilon}/(1+\tilde{\varepsilon})$. If $0 \le \lambda < -1-2\tilde{\delta}+2\tilde{\delta}/\tilde{\varepsilon}$, then

$$L\{I_n, F^\circ, (\tilde{\varepsilon}, \tilde{\delta})\} = (F^\circ)^{-1}\left(\frac{1-2\tilde{\varepsilon}+\lambda}{2(1-\tilde{\varepsilon})}\right) - (F^\circ)^{-1}\left(\frac{1-2\tilde{\varepsilon}-\lambda}{2(1-\tilde{\varepsilon})}\right), \tag{4.13}$$

if $-1-2\tilde{\delta}+2\tilde{\delta}/\tilde{\varepsilon} \le \lambda < (1-\tilde{\varepsilon})\tilde{\delta}/\tilde{\varepsilon}$, then it is (4.11), and if $(1-\tilde{\varepsilon})\tilde{\delta}/\tilde{\varepsilon} \le \lambda < 1-2\tilde{\delta}$, then it is (4.12).

(c) Let $\tilde{\varepsilon}/(1+\tilde{\varepsilon}) \le \tilde{\delta} < \tilde{\varepsilon}$. If $0 \le \lambda < -1-2\tilde{\delta}+2\tilde{\delta}/\tilde{\varepsilon}$, then it is (4.13), and if $-1-2\tilde{\delta}+2\tilde{\delta}/\tilde{\varepsilon} \le \lambda < 1-2\tilde{\delta}$, then it is (4.11). □

Example 4.2 The maximum asymptotic length under the G-neighborhood (i.e., $\tilde{\gamma}=0$, $\lambda = \min(c_2-1, 1-c_1)$) can be easily obtained from Lemma 4.1 and Theorem 4.2. By Lemma 4.1 we have $(A,B)=(0,b)$ for $0<\tilde{c}_1 \le \tilde{c}_2/(2\tilde{c}_2-1)$ and $(A,B)=(a,b)$ for $\tilde{c}_1 > \tilde{c}_2/(2\tilde{c}_2-1)$, where $a = 1-2\tilde{c}_2(1-\tilde{c}_1)/(\tilde{c}_2-\tilde{c}_1)$ and $b = \tilde{c}_1(\tilde{c}_2-1)/(\tilde{c}_2-\tilde{c}_1)$. We first assume $0<\tilde{c}_1 \le \tilde{c}_2/(2\tilde{c}_2-1)$. It directly follows from Theorem 4.2 that if $0 \le \lambda < b$, then (4.9) holds with $\tilde{\gamma}=0$ and $\lambda = \min(c_2-1, 1-c_1)$, and if $b \le \lambda < 1$, then (4.10) holds. Next, assume $\tilde{c}_2/(2\tilde{c}_2-1) < \tilde{c}_1 < 1$. Then we have (4.8), (4.9) and (4.10) according as $0 \le \lambda < a$, $a \le \lambda < b$ and $b \le \lambda < 1$. □

We now turn to stability or robustness properties. The next definition of the asymptotic length robustness of the proposed confidence interval determined by λ under the real size $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ is the counterpart of Hampel's breakdown point .

Definition 4.2 The *length breakdown size* $\lambda^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ of the sequence $\{I_n\}$ under the real size $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ at F° is defined by

$$\lambda^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = \sup\{\lambda \mid L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} < \infty\}.$$

Theorem 4.3 *Under the assumptions of Theorem 4.2, the following results hold for* $0 \le \lambda < 1-2\tilde{\gamma}$*:*

(i) $\lambda^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = 1-2\tilde{\gamma}$.

(ii) *If* $\tilde{\gamma}=\gamma$*, then the sequence* $\{I_n\}$ *has the finite maximum asymptotic length under the real size* $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ *at* F° *if and only if*

$$\min\{(1-\gamma)-c_1,\, c_2-(1-\gamma)\} + 3\gamma < 1.$$

Since the length breakdown size λ^* dose not depend on $\tilde{c}_1$ and $\tilde{c}_2$, in the same way as Ando, Kakiuchi and Kimura (2009) we can define the *length breakdown point* by

$$\tilde{\gamma}^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = \sup\{\tilde{\gamma} \mid L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} < \infty\}.$$

Then, it follows from (i) of Theorem 4.3 that

$$\tilde{\gamma}^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = (1-\lambda)/2.$$

In the cases of the ε - contamination neighborhood and (c,γ) - neighborhood, their length breakdown points $\tilde{\gamma}^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ are equal to those of Yohai and Zamar (2004), and Ando, Kakiuchi and Kimura (2009), respectively. As easily seen, for the ε - contamination neighborhood and total variation neighborhood we have $\lambda^* = \gamma^* = 1/3$, and $\lambda^* = 1/2$ and $\gamma^* = 1/4$, respectively. Also, as for the $TN\varepsilon$ - neighborhood, we have (a) $1/3 \le \lambda^* \le 1/2$ and $1/4 \le \gamma^* \le 1/3$ for all $\delta < \varepsilon$, (b) if ε approaches to δ, then both λ^* and γ^* approach to 1/3, and (c) if $2\delta < \varepsilon$, then λ^* and γ^* are equal to 1/2 and 1/4, respectively.

Theorem 4.4 *Let $\{I_n\}$ be a sequence of confidence intervals $I_n = [A_n(\boldsymbol{X}_n), B_n(\boldsymbol{X}_n))$ such that*

$$\inf_{G \in \mathscr{P}_{c_1,c_2,\gamma}(G^\circ)} P_G\{A_n(\boldsymbol{X}_n) \le (G^\circ)^{-1}(1/2) < B_n(\boldsymbol{X}_n)\} = 1-\alpha$$

for any absolutely continuous distribution G°. Suppose that $\lim_{n\to\infty} A_n(\boldsymbol{X}_n) = A_0$ and $\lim_{n\to\infty} B_n(\boldsymbol{X}_n) = B_0$ almost surely when the sample comes from F°. Then it holds that

$$A_0 \le (F^\circ)^{-1}((1-\lambda)/2) \quad \text{and} \quad B_0 \ge (F^\circ)^{-1}((1+\lambda)/2).$$

Theorem 4.4 states that in the case of uncontaminated data (i.e., $\tilde{c}_1 = 1-\tilde{\gamma}$ and $\tilde{\gamma} = 0$, or $\tilde{c}_2 = 1-\tilde{\gamma}$ and $\tilde{\gamma} = 0$), the proposed interval I_n in Theorem 4.1 is efficient in the sense that it has the smallest asymptotic length among all nonparametric (c_1, c_2, γ) - robust confidence intervals for the median whose upper and lower confidence bounds converge.

5 The $(\mathbf{c_1}, \mathbf{c_2}, \boldsymbol{\gamma})$ - robust nonparametric test

In the framework of Section 4 we consider robust nonparametric tests for the problem of testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \ne \theta_0$, where θ_0 is fixed.

Definition 5.1 A non-randomized test $\varphi_{n,\theta_0}(\boldsymbol{X}_n)$ for H_0 versus H_1 is said to have *nonparametric (c_1, c_2, γ) - robust level α* if for all F°

$$\sup_{G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)} P_G\{\varphi_{n,\theta_0}(\boldsymbol{X}_n) = 1\} = \alpha.$$

This definition implies that the probability of rejecting H_0 is less than or equal to α, not only at F° but also at any $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$.

Theorem 5.1 *A non-randomized modified sign test $\varphi_{n,\theta_0}(\boldsymbol{X}_n)$ given by*

$$\varphi_{n,\theta_0}(\boldsymbol{X}_n) = \begin{cases} 1 & \text{if } \; T_{n,\theta_0}(\boldsymbol{X}_n) \le k_n \; \text{ or } \; T_{n,\theta_0}(\boldsymbol{X}_n) \ge n - k_n, \\ 0 & \text{if } \; k_n < T_{n,\theta_0}(\boldsymbol{X}_n) < n - k_n, \end{cases} \tag{5.1}$$

has nonparametric (c_1, c_2, γ)*-robust level* $\alpha^*(n, k_n, \lambda)$*, where* $T_{n,\theta}(\boldsymbol{X}_n)$ *is defined by* (4.1) *and* k_n *and* λ *are given in* Theorem 4.1.

As in the case of confidence intervals, we clearly distinguish the design size (c_1, c_2, γ) and the real size $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$, and establish the asymptotic behavior of the power of the test (5.1) under the $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ which satisfies the condition of (4.7). The following definitions are closely related to those of power robustness, power distance and power breakdown point which were introduced by Yohai and Zamar (2004) as measures of efficiency and robustness of tests.

Definition 5.2 Let $F^\circ_\eta(x) = F^\circ(x - \eta)$, where F° is fixed. A sequence (φ_{n,θ_0}), $n \geq n_0$, of non-randomized tests is said to have $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ - *robust power at* F° if there exists a positive real number M such that

$$\inf_{G \in \mathscr{P}_{\tilde{c}_1, \tilde{c}_2, \tilde{\gamma}}(F^\circ_\eta)} \lim_{n\to\infty} P_G\{\varphi_{n,\theta_0}(\boldsymbol{X}_n) = 1\} = 1 \quad \text{for all } |\eta| > M. \tag{5.2}$$

Definition 5.3 The $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ - *consistency distance* $M^*\{(\varphi_{n,\theta_0}), F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ of a sequence (φ_{n,θ_0}), $n \geq n_0$, of tests at F° is defined as the infimum of the set of values M for which (5.2) holds.

Definition 5.4 The *power breakdown size* $\lambda^*\{(\varphi_{n,\theta_0}), F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ of a sequence (φ_{n,θ_0}), $n \geq n_0$, of tests at F° is defined as the supremum of the set of values λ for which the sequence of tests is $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ - robust power at F°.

Theorem 5.2 *Suppose that* F° *has a symmetric* (*around* θ) *and unimodal density. Let* $0 < \alpha < 1$ *and consider the sequence of tests* (φ_{n,θ_0}), $n \geq n_0$, *for* $H_0 : \theta = \theta_0$ *versus* $H_1 : \theta \neq \theta_0$ *given by* (5.1)*. Then the following results hold:*

(i) *Let* $0 \leq \tilde{c}_1 \leq 1/2$*. If* $0 \leq \tilde{\gamma} < 1/2$*, then*

$$M^*\{(\varphi_{n,\theta_0}), F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = (F^\circ)^{-1}\left(1 - \frac{1 - 2\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right). \tag{5.3}$$

holds for $0 \leq \lambda < 1 - 2\tilde{\gamma}$.

(ii) *Let* $1/2 < \tilde{c}_1 < 1 - \tilde{\gamma}$ *and* $\tilde{c}_1 \leq \tilde{c}_2/(2\tilde{c}_2 - 1)$*. If* $0 \leq \tilde{\gamma} < 1/2 + \{\tilde{c}_2(1 - 2\tilde{c}_1)\}/\{2\tilde{c}_1\}$*, then* (5.3) *holds for* $0 \leq \lambda < 1 - 2\tilde{\gamma}$*, and if* $1/2 + \{\tilde{c}_2(1 - 2\tilde{c}_1)\}/\{2\tilde{c}_1\} \leq \tilde{\gamma} < 1/2$*, then*

$$M^*\{(\varphi_{n,\theta_0}), F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = (F^\circ)^{-1}\left(\frac{1 + \lambda}{2\tilde{c}_1}\right) \tag{5.4}$$

holds for $0 \leq \lambda < 1 - 2\tilde{\gamma} - 2\tilde{c}_2(1 - \tilde{\gamma} - \tilde{c}_1)/\{\tilde{c}_2 - \tilde{c}_1\}$ *, and* (5.3) *holds for* $1 - 2\tilde{\gamma} - 2\tilde{c}_2(1 - \tilde{\gamma} - \tilde{c}_1)/\{\tilde{c}_2 - \tilde{c}_1\} \leq \lambda < 1 - 2\tilde{\gamma}$.

(iii) *Let* $\tilde{c}_2/(2\tilde{c}_2 - 1) < \tilde{c}_1 < 1 - \tilde{\gamma}$*. If* $0 \leq \tilde{\gamma} < 1/2$*, then* (5.3) *holds for* $0 \leq \lambda < 1 - 2\tilde{\gamma} - 2\tilde{c}_2(1 - \tilde{\gamma} - \tilde{c}_1)/\{\tilde{c}_2 - \tilde{c}_1\}$*, and* (5.4) *holds for* $1 - 2\tilde{\gamma} - 2\tilde{c}_2(1 - \tilde{\gamma} - \tilde{c}_1)/\{\tilde{c}_2 - \tilde{c}_1\} \leq \lambda < 1 - 2\tilde{\gamma}$.

Theorem 5.3 *Under the assumptions of Theorem* 5.2, *the following results hold:*

(i) $\lambda^*\{(\varphi_{n,\theta_0}), F^\circ\} = 1 - 2\tilde{\gamma}$.

(ii) *If* $\tilde{\gamma} = \gamma$, *then the sequence* $(\varphi_{n,\theta_0}), n \geq n_0$, *of tests has* $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$*-robust power at* F° *if and only if*

$$\min\{(1-\gamma) - c_1,\, c_2 - (1-\gamma)\} + 3\gamma < 1.$$

Remark 5.1 (i) If $c_1 = \tilde{c}_1 = 0$, then (5.3) holds, which is the same as (i) in Theorem 3.2 of Ando, Kakiuchi and Kimura (2009) with $c_2 = c$ and $\tilde{c}_2 = \tilde{c}$.
(ii) As in the length breakdown point for confidence interval, the *power breakdown point* γ^* of a sequence (φ_{n,θ_0}), $n \geq n_0$, of tests at F° is defined as the supremum of the set of values $\tilde{\gamma}$ for which the sequence of tests is $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$-robust power at F°. In this case we can obtain $\tilde{\gamma}^* = (1-\lambda)/2$ from (i) in Theorem 5.3. This power breakdown point is the same as that of the confidence interval.

6 Proofs

Proof of Theorem 3.1

First, in the case of $\gamma \neq 0$, we show that for any element G of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (2.1) G can be expressed by the form of (3.1). Let f° and g be the density functions of F° and G with respect to a σ-finite measure μ (e.g. $\mu = F^\circ + G$), respectively, and let

$$A = \{x \in \mathbb{X} \mid c_1 f^\circ(x) \leq g(x) \leq c_2 f^\circ(x)\} \quad (A^c = \{x \in \mathbb{X} \mid g(x) > c_2 f^\circ(x)\}).$$

Noting that $G\{B\} \geq \max(c_1 F^\circ\{B\}, 1 - c_2 - \gamma + c_2 F^\circ\{B\})$ for any $B \in \mathscr{B}$, we have

$$F^\circ\{B\} - \frac{1}{c_2}G\{B\} \leq \min\left(\frac{c_2 - c_1}{c_2}F^\circ\{B\},\ \frac{c_2 - 1 + \gamma}{c_2}\right). \tag{6.1}$$

From (6.1) it follows that if $F^\circ\{A\} \geq (c_2 - 1 + \gamma)/(c_2 - c_1)$, then

$$0 \leq F^\circ\{A\} - \frac{1}{c_2}G\{A\} \leq \frac{c_2 - 1 + \gamma}{c_2} \leq \frac{c_2 - c_1}{c_2}F^\circ\{A\},$$

and if $F^\circ\{A\} < (c_2 - 1 + \gamma)/(c_2 - c_1)$, then

$$0 \leq \frac{c_2 - 1 + \gamma}{c_2} - \frac{c_2 - c_1}{c_2}F^\circ\{A\} \leq \frac{c_2 - c_1}{c_2}F^\circ\{A^c\}.$$

Therefore, there exist two functions $\phi_1(x)$ and $\phi_2(x)$ defined on A and A^c, respectively, such that

$$0 \leq f^\circ(x) - \frac{1}{c_2}g(x) \leq \phi_1(x) \leq \frac{c_2 - c_1}{c_2}f^\circ(x), \quad 0 \leq \phi_2(x) \leq \frac{c_2 - c_1}{c_2}f^\circ(x)$$

and that if $F^\circ\{A\} \geq (c_2 - 1 + \gamma)/(c_2 - c_1)$EEEEEEECthen

$$\int_A \phi_1(x)d\mu(x) = \frac{c_2 - 1 + \gamma}{c_2}, \quad \phi_2(x) \equiv 0,$$

and if $F^\circ\{A\} < (c_2 - 1 + \gamma)/(c_2 - c_1)$, then

$$\phi_1(x) = \frac{c_2 - c_1}{c_2} f^\circ(x), \qquad \int_{A^c} \phi_2(x) d\mu(x) = \frac{c_2 - 1 + \gamma}{c_2} - \frac{c_2 - c_1}{c_2} F^\circ\{A\}.$$

Define ϕ on $\mathbb{X}$ as

$$\phi(x) = \phi_1(x) I_A(x) + \phi_2(x) I_{A^c}(x),$$

where $I_B(x)$ is the indicator function on B. It is clear that

$$0 \le \phi(x) \le \frac{c_2 - c_1}{c_2} f^\circ(x), \qquad \int_{\mathbb{X}} \phi(x)\, d\mu(x) = \frac{c_2 - 1 + \gamma}{c_2}.$$

Let

$$W\{B\} = \int_B \phi(x)\, d\mu(x) \quad \text{for any } B \in \mathscr{B}. \tag{6.2}$$

Then W is a measure with the density ϕ such that $W\{\mathbb{X}\} = (c_2 - 1 + \gamma)/c_2$, which implies $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$. Let

$$K\{B\} = \frac{1}{\gamma} \left\{ G\{B\} - c_2(F^\circ\{B\} - W\{B\}) \right\}, \quad {}^\forall B \in \mathscr{B}.$$

Then we see that K is a probability measure on $(\mathbb{X}, \mathscr{B})$ and

$$G\{B\} = c_2(F^\circ\{B\} - W\{B\}) + \gamma K\{B\}, \quad {}^\forall B \in \mathscr{B}. \tag{6.3}$$

Secondly, we consider the case of $\gamma = 0$. Any $G \in \mathscr{P}_{c_1,c_2,0}(F^\circ)$ by (2.1) has a density g and $G\{A\} = G\{\mathbb{X}\} = 1$. Let

$$\phi(x) = f^\circ(x) - \frac{1}{c_2} g(x), \quad {}^\forall x \in \mathbb{X}.$$

Then, using W in (6.2) with this ϕ, we have

$$G\{B\} = c_2(F^\circ\{B\} - W\{B\}), \quad {}^\forall B \in \mathscr{B}. \tag{6.4}$$

The equations (6.3) and (6.4) imply that G belongs to $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ by (3.1).

Conversely, let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ by (3.1). Then, for any $B \in \mathscr{B}$ there exist $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$ and $K \in \mathscr{M}$ such that $G\{B\} = c_2(F^\circ\{B\} - W\{B\}) + \gamma K\{B\}$. Since $0 \le W\{B\} \le \{(c_2 - c_1)/c_2\} F^\circ\{B\}$, we have

$$c_2 F^\circ\{B\} + \gamma \ge G\{B\} \ge c_2 \left(F^\circ\{B\} - \frac{c_2 - c_1}{c_2} F^\circ\{B\} \right) = c_1 F^\circ\{B\}.$$

This implies that G belongs to $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ by (2.1). $\square$

Proof of Corollary 3.1

It is sufficient to show that $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.2) is equal to that by (3.1) whenever $c_1 \neq 0$. Let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.1), which implies $G = c_2(F^\circ - W) + \gamma K$. Then, it is rewritten as

$$\begin{aligned} G &= c_1 \left\{ F^\circ + \left(\frac{c_2 - c_1}{c_1} F^\circ - \frac{c_2}{c_1} W \right) \right\} + \gamma K \\ &= c_1 \{ F^\circ + V \} + \gamma K, \end{aligned}$$

where $V = \{(c_2 - c_1)/c_1\}F^\circ - (c_2/c_1)W$. Then, from $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$ it follows that V is a measure on $(\mathbb{X}, \mathscr{B})$ such that $0 \le V(A) \le \{(c_2 - c_1)/c_1\}F^\circ\{A\}$ for all $A \in \mathscr{B}$ and $V\{\mathbb{X}\} = (1 - \gamma - c_1)/c_1$. Thus $V \in \mathscr{V}_{c_1,c_2,\gamma}(F^\circ)$, and hence G is in $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.2).

Conversely, let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.2). Then, it is also written as

$$\begin{aligned}
G &= c_1(F^\circ + V) + \gamma K \\
&= c_2 \left\{ F^\circ - \left(\frac{c_2 - c_1}{c_2} F^\circ - \frac{c_1}{c_2} V \right) \right\} + \gamma K \\
&= c_2(F^\circ - W) + \gamma K,
\end{aligned}$$

where $W = \{(c_2 - c_1)/c_2\}F^\circ - (c_1/c_2)V$. From $V \in \mathscr{V}_{c_1,c_2,\gamma}(F^\circ)$, it is easy to see $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$, which implies $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.2). $\square$

Proof of Theorem 3.2

Let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.1). Then there exist $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$ and $K \in \mathscr{M}$ such that $G = c_2(F^\circ - W) + \gamma K$. Hence we have

$$\begin{aligned}
G &= (1 - \gamma) \left\{ \frac{c_2}{1 - \gamma}(F^\circ - W) \right\} + \gamma K \\
&= (1 - \gamma)F + \gamma K,
\end{aligned}$$

where $F = \{c_2/(1 - \gamma)\}(F^\circ - W)$. We note that

$$F\{\mathbb{R}\} = \frac{c_2}{1 - \gamma} \left(F^\circ\{\mathbb{R}\} - W\{\mathbb{R}\} \right) = \frac{c_2}{1 - \gamma} \left(1 - \frac{c_2 - 1 + \gamma}{c_2} \right) = 1.$$

Since F° and W are absolutely continuous, F is an absolutely continuous distribution. Also, since $0 \le W\{B\} \le \{(c_2 - c_1)/c_2\}F^\circ\{B\}$ for any $B \in \mathscr{B}$, we have

$$\frac{c_2}{1 - \gamma} F^\circ\{B\} \ge F\{B\} \ge \frac{c_2}{1 - \gamma} \left(F^\circ\{B\} - \frac{c_2 - c_1}{c_2} F^\circ\{B\} \right) = \frac{c_1}{1 - \gamma} F^\circ\{B\}.$$

Therefore we obtain $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$, which implies $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.3).

Conversely, let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.3). Then there exists $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ such that $G = (1 - \gamma)F + \gamma K$. Letting

$$W = F^\circ - \frac{1 - \gamma}{c_2} F,$$

we can easily see $G = c_2(F^\circ - W) + \gamma K$ and $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$. This implies $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.1). $\square$

Proof of Theorem 3.3

To show the assertion (i), let f_L° and f_R° be the density functions of F_L° and F_R°, respectively. Then, from the form of density in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ it can be easily seen that f_L° and f_R° are stochastically smallest and largest density functions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$, respectively, and x_L and x_R are the constants such that they are density functions. This implies that (3.9) holds. Next, to show the assertion (ii), let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$. Then, by Theorem 3.2 there exist $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ and $K \in \mathscr{M}$ such that $G = (1 - \gamma)F + \gamma K$. Noting that K is a distribution function in $\mathscr{M}$, we obtain (3.10) from the assertion (i). $\square$

Proof of Theorem 3.4

Let $\delta(m)$ denote the point mass distribution at m. We note that $(1-\gamma)F_R^\circ(x)+\gamma\delta(m)$ is in $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ for all m and as $m\to\infty$ its limit approaches to $(1-\gamma)F_R^\circ(x)$, and similarly as $m\to\infty$, $(1-\gamma)F_L^\circ(x)+\gamma\delta(-m)$ approaches to $(1-\gamma)F_L^\circ(x)+\gamma$. Therefore, from (3.10) we have

$$\begin{aligned}\sup_{G,H\in\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)} d_K(G,H) &= \sup_x \{(1-\gamma)F_L^\circ(x)+\gamma-(1-\gamma)F_R^\circ(x)\}\\ &= (1-\gamma)\{F_L^\circ(x_L)-F_R^\circ(x_L)\}+\gamma.\end{aligned}$$

Also we have

$$\begin{aligned}(1-\gamma)\{F_L^\circ(x_L)-F_R^\circ(x_L)\} &= \begin{cases}1-\gamma-c_1 & \text{if} \quad x_L\le x_R\\ c_2-1+\gamma & \text{if} \quad x_R<x_L\end{cases}\\ &= \min(1-\gamma-c_1, c_2-1+\gamma)\end{aligned}$$

because of $x_L \le, > x_R$ according as $1-\gamma-c_1 \le, > c_2-1+\gamma$, which completes the proof of the theorem. $\square$

Proof of Theorem 4.1

Let $p_\theta(G)=P_G\{X_i>\theta\}=1-G(\theta)$. Since $T_{n,\theta}(\boldsymbol{X}_n)$ is distributed with Binomial $B_N(n,p_\theta(G))$, we have

$$P_G\{k_n<T_{n,\theta}(\boldsymbol{X}_n)<n-k_n\}=\sum_{i=k_n+1}^{n-k_n-1}\binom{n}{i}\{p_\theta(G)\}^i\{1-p_\theta(G)\}^{n-i}. \tag{6.5}$$

From Lemma 1 in Yohai and Zamar (2004), the right-hand side of (6.5) is symmetric at $p_\theta(G)=1/2$ and nondecreasing about $p_\theta(G)$ on $[0,1/2]$. Here we note that

$$F^\circ(\theta)-F^\circ(x_L)=\frac{1}{2}-\frac{1-\gamma-c_1}{c_2-c_1}=\frac{(c_2-1+\gamma)-(1-\gamma-c_1)}{2(c_2-c_1)}$$

and

$$F^\circ(\theta)-F^\circ(x_R)=F^\circ(\theta)-(1-F^\circ(x_L))=F^\circ(x_L)-F^\circ(\theta),$$

because of $F^\circ(x_L)+F^\circ(x_R)=1$, where x_L and x_R are given by (3.7) and (3.8), respectively. This implies that

$$\begin{cases}x_R\le\theta\le x_L & \text{if} \quad c_2-1+\gamma\le 1-\gamma-c_1,\\ x_L<\theta<x_R & \text{if} \quad c_2-1+\gamma>1-\gamma-c_1.\end{cases} \tag{6.6}$$

By Theorem 3.3 we have

$$(1-\gamma)F_R^\circ(\theta)\le G(\theta)\le(1-\gamma)F_L^\circ(\theta)+\gamma, \tag{6.7}$$

where F_L° and F_R° are given by (3.5) and (3.6), respectively. Then, from (6.6) we obtain

$$(1-\gamma)F_R^\circ(\theta)=\begin{cases}1-\gamma-\dfrac{c_2}{2} & \text{if} \quad c_2-1+\gamma\le 1-\gamma-c_1,\\ \dfrac{c_1}{2} & \text{if} \quad c_2-1+\gamma>1-\gamma-c_1\end{cases}$$

and

$$(1-\gamma)F_L^{\circ}(\theta)+\gamma=\begin{cases}\dfrac{c_2}{2}+\gamma & \text{if} \quad c_2-1+\gamma\leq 1-\gamma-c_1,\\ 1-\dfrac{c_1}{2} & \text{if} \quad c_2-1+\gamma>1-\gamma-c_1\end{cases}$$

From the definition of λ in (3.12) it follows that

$$(1-\gamma)F_R^{\circ}(\theta)=\frac{1-\lambda}{2} \quad \text{and} \quad (1-\gamma)F_L^{\circ}(\theta)+\gamma=\frac{1+\lambda}{2}.$$

Therefore, by (6.7) we have

$$\frac{1-\lambda}{2}\leq p_\theta(G)\leq\frac{1+\lambda}{2} \quad \text{for all} \quad G\in\mathscr{P}_{c_1,c_2,\gamma}(F^{\circ}),$$

and the infimum (the supremum) is attained for any γ-contaminating distribution of F_L° (F_R°) which places all its mass to the left (the right) of θ. Noting that

$$P_G(k_n<T_{n,\theta}(\boldsymbol{X}_n)<n-k_n)=P_G(X_{(k_n+1)}\leq\theta<X_{(n-k_n)}), \tag{6.8}$$

the infimum of the left-hand side of (6.8) is attained by $p_\theta(G)=(1-\lambda)/2$ or $p_\theta(G)=(1+\lambda)/2$, which completes the proof. $\square$

Proof of Theorem 4.2

Let $l_n(\boldsymbol{X}_n)=X_{(n-k_n)}-X_{(k_n+1)}$. Then, by Lemma 2 in Yohai and Zamar (2004), we have

$$\lim_{n\to\infty} l_n(\boldsymbol{X}_n)=G^{-1}\left(\frac{1+\lambda}{2}\right)-G^{-1}\left(\frac{1-\lambda}{2}\right), \tag{6.9}$$

where $G=(1-\tilde{\gamma})F+\tilde{\gamma}K\in\mathscr{P}_{c_1,c_2,\gamma}(F^{\circ})$. Let $x_0=\sup\{x \mid G(x)<1/2\}$. Then there exists $\xi\in[0,1]$ such that

$$(1-\tilde{\gamma})F(x_0)+\tilde{\gamma}\{K(x_0-0)+\xi(K(x_0)-K(x_0-0))\}=\frac{1}{2}.$$

Putting $\eta=(1-\xi)K(x_0-0)+\xi K(x_0)$, η takes the value on $[0,1]$ for all $K\in\mathscr{M}$. Let

$$H_\eta(x)=(1-\tilde{\gamma})F(x)+\tilde{\gamma}\eta, \tag{6.10}$$

and then $H_\eta(x)$ takes the value on $[\tilde{\gamma}\eta,1-\tilde{\gamma}(1-\eta)]$. Here we note that $H_\eta^{-1}(u)$ is defined on $(\tilde{\gamma},1-\tilde{\gamma})$ for all $\eta\in[0,1]$. Since $H_\eta(x)$ is continuous and strictly increasing on $(-\infty,\infty)$, we have that $H_\eta(x)\geq G(x)$ if $x<x_0$ and $H_\eta(x)\leq G(x)$ if $x\geq x_0$. This implies that for any $0\leq\lambda<1-2\tilde{\gamma}$

$$H_\eta^{-1}\left(\frac{1+\lambda}{2}\right)-H_\eta^{-1}\left(\frac{1-\lambda}{2}\right)\geq G^{-1}\left(\frac{1+\lambda}{2}\right)-G^{-1}\left(\frac{1-\lambda}{2}\right). \tag{6.11}$$

The left-hand side of (6.11) is written as

$$F^{-1}\left(\frac{1-2\eta\tilde{\gamma}+\lambda}{2(1-\tilde{\gamma})}\right)-F^{-1}\left(\frac{1-2\eta\tilde{\gamma}-\lambda}{2(1-\tilde{\gamma})}\right). \tag{6.12}$$

Denoting

$$d(\eta) = \frac{(2\eta - 1)\tilde{\gamma} + \lambda}{2(1 - \tilde{\gamma})}, \tag{6.13}$$

we have

$$\frac{1 - 2\eta\tilde{\gamma} - \lambda}{2(1 - \tilde{\gamma})} = \frac{1}{2} - d(\eta) \quad \text{and} \quad \frac{1 - 2\eta\tilde{\gamma} + \lambda}{2(1 - \tilde{\gamma})} = \frac{1}{2} + d(1 - \eta).$$

Also, since $d(\eta) + d(1 - \eta) = \lambda/(1 - \tilde{\gamma}) \geq 0$ and $d(\eta) - d(1 - \eta) = (2\eta - 1)\tilde{\gamma}/(1 - \tilde{\gamma})$, it follows that

$$\begin{cases} d(\eta) \geq |d(1 - \eta)| & \text{for} \quad \dfrac{1}{2} \leq \eta \leq 1, \\ d(1 - \eta) > |d(\eta)| & \text{for} \quad 0 \leq \eta < \dfrac{1}{2}. \end{cases}$$

In order to obtain the upper bound of (6.12) for all $F \in \mathscr{F}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)$, we first consider the case of $1/2 \leq \eta \leq 1$, and assume $\tilde{c}_1 \neq 0$, $\tilde{c}_1 \neq 1 - \tilde{\gamma}$ and $\tilde{c}_2 \neq 1 - \tilde{\gamma}$.

Let

$$x_2 = F^{-1}\left(\frac{1 - 2\eta\tilde{\gamma} - \lambda}{2(1 - \tilde{\gamma})}\right),$$

and let $\tilde{F} \in \mathscr{F}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)$ be defined by

$$\tilde{F}(x) = \begin{cases} \dfrac{\tilde{c}_2}{1 - \tilde{\gamma}} F^\circ(x) & \text{if} \quad x < x_1, \\ \dfrac{\tilde{c}_2 - \tilde{c}_1}{1 - \tilde{\gamma}} F^\circ(x_1) + \dfrac{\tilde{c}_1}{1 - \tilde{\gamma}} F^\circ(x) & \text{if} \quad x_1 \leq x \leq x_3, \\ \left(1 - \dfrac{\tilde{c}_2}{1 - \tilde{\gamma}}\right) + \dfrac{\tilde{c}_2}{1 - \tilde{\gamma}} F^\circ(x) & \text{if} \quad x_3 < x, \end{cases}$$

where x_1 ($-\infty \leq x_1 \leq x_2$) and x_3 ($x_2 \leq x_3 \leq \infty$) are determined so as to satisfy $\tilde{F}(x_2) = F(x_2)$ and $F^\circ(x_3) = F^\circ(x_1) + (\tilde{c}_2 - 1 + \tilde{\gamma})/(\tilde{c}_2 - \tilde{c}_1)$, respectively. Then we can easily see that $F(x) \leq, =, \geq \tilde{F}(x)$ according as $x <, =, > x_2$. Moreover, denoting

$$x_4 = (\tilde{F})^{-1}\left(\frac{1 - 2\eta\tilde{\gamma} + \lambda}{2(1 - \tilde{\gamma})}\right),$$

we obtain that (6.12) is bounded above by $x_4 - x_2$.

Next, let $\tilde{F}_L^\circ \in \mathscr{F}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)$ be the stochastically smallest distribution given by (3.5) with (c_1, c_2, γ) replaced by $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$. Also, let $\tilde{x}_L = (F^\circ)^{-1}\{(1 - \tilde{\gamma} - \tilde{c}_1)/(\tilde{c}_2 - \tilde{c}_1)\}$ and

$$\tilde{x}_{2L} = (\tilde{F}_L^\circ)^{-1}\left(\frac{1 - 2\eta\tilde{\gamma} - \lambda}{2(1 - \tilde{\gamma})}\right). \tag{6.14}$$

Now, we consider Case 1: $\tilde{x}_{2L} > \tilde{x}_L$ and Case 2: $\tilde{x}_{2L} \leq \tilde{x}_L$, separately. We first treat Case 1. From (6.14) it follows that

$$\tilde{x}_{2L} = (F^\circ)^{-1}\left(1 - \frac{1 - 2(1 - \eta)\tilde{\gamma} + \lambda}{2\tilde{c}_1}\right).$$

Therefore, expressing the condition $\tilde{x}_{2L} > \tilde{x}_L$ as that in λ we have

$$\text{Case 1:} \quad \lambda < 1 - 2\eta\tilde{\gamma} - \frac{2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1}.$$

Let

$$\tilde{x}_{4L} = (\tilde{F}_L^\circ)^{-1}\left(\frac{1-2\eta\tilde{\gamma}+\lambda}{2(1-\tilde{\gamma})}\right) = (F^\circ)^{-1}\left(1 - \frac{1-2(1-\eta)\tilde{\gamma}-\lambda}{2\tilde{c}_1}\right),$$

and we show that $\tilde{x}_{4L} - \tilde{x}_{2L} \geq x_4 - x_2$. It follows from the definition of $\tilde{F}_L^\circ$ that $\tilde{F}(x) - \tilde{F}(\tilde{x}_{2L}) \geq \tilde{F}_L^\circ(x) - \tilde{F}_L^\circ(\tilde{x}_{2L})$ for all $x \geq \tilde{x}_{2L}$. Since $\tilde{F}_L^\circ(\tilde{x}_{2L}) = \tilde{F}(x_2)$ and $\tilde{F}_L^\circ(\tilde{x}_{4L}) = \tilde{F}(x_4)$, we have

$$\tilde{F}(x_2) - \tilde{F}(\tilde{x}_{2L}) \geq \tilde{F}(x_4) - \tilde{F}(\tilde{x}_{4L}). \tag{6.15}$$

Here we assume $\theta = 0$ without loss of generality. If $\tilde{x}_{4L} < 0$, it is clear from the unimodality and the symmetry of f° that $\tilde{x}_{4L} - \tilde{x}_{2L} \geq x_4 - x_2$. Let $\tilde{x}_{4L} \geq 0$. Since $1 - \tilde{F}(x_4) = 1/2 - d(1-\eta) \geq 1/2 - d(\eta) = \tilde{F}_L^\circ(\tilde{x}_{2L})$, it follows from the density forms of $\tilde{F}_L^\circ$ and $\tilde{F}$ that $\tilde{x}_{2L} \leq -x_4$. Therefore, by (6.15) we obtain

$$x_2 - \tilde{x}_{2L} \geq x_4 - \tilde{x}_{4L}.$$

This means that (6.12) is bounded above by $\tilde{x}_{4L} - \tilde{x}_{2L}$, that is,

$$(F^\circ)^{-1}\left(1 - \frac{1-2(1-\eta)\tilde{\gamma}-\lambda}{2\tilde{c}_1}\right) - (F^\circ)^{-1}\left(1 - \frac{1-2(1-\eta)\tilde{\gamma}+\lambda}{2\tilde{c}_1}\right). \tag{6.16}$$

Next, we consider

$$\text{Case 2:} \quad \lambda \geq 1 - 2\eta\tilde{\gamma} - \frac{2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1}.$$

Let $\hat{F}_L^\circ \in \mathscr{F}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)$ be defined by

$$\hat{F}_L^\circ(x) = \begin{cases} \dfrac{\tilde{c}_2}{1-\tilde{\gamma}} F^\circ(x) & \text{if} \quad x < \hat{x}_{2L}, \\ \left(1 - \dfrac{\tilde{c}_1}{\tilde{c}_2}\right)\dfrac{1-2\eta\tilde{\gamma}-\lambda}{2(1-\tilde{\gamma})} + \dfrac{\tilde{c}_1}{1-\tilde{\gamma}} F^\circ(x) & \text{if} \quad \hat{x}_{2L} \leq x \leq \hat{x}_{3L}, \\ \left(1 - \dfrac{\tilde{c}_2}{1-\tilde{\gamma}}\right) + \dfrac{\tilde{c}_2}{1-\tilde{\gamma}} F^\circ(x) & \text{if} \quad \hat{x}_{3L} < x, \end{cases} \tag{6.17}$$

where

$$\hat{x}_{2L} = (F^\circ)^{-1}\left(\frac{1-2\eta\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right)$$

and $\hat{x}_{3L} = (F^\circ)^{-1}\{(1-2\eta\tilde{\gamma}-\lambda)/(2\tilde{c}_2) + (\tilde{c}_2 - 1 + \tilde{\gamma})/(\tilde{c}_2 - \tilde{c}_1)\}$. Let

$$\hat{x}_{4L} = (\hat{F}_L^\circ)^{-1}\left(\frac{1-2\eta\tilde{\gamma}+\lambda}{2(1-\tilde{\gamma})}\right).$$

We consider the case (a): $\hat{x}_{4L} \in (\hat{x}_{2L}, \hat{x}_{3L})$ and the case (b): $\hat{x}_{4L} \in [\hat{x}_{3L}, \infty)$, separately. We note that the conditions $\hat{x}_{4L} \in (\hat{x}_{2L}, \hat{x}_{3L})$ and $\hat{x}_{4L} \in [\hat{x}_{3L}, \infty)$ are expressed in the forms of

$\lambda < \tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})/(\tilde{c}_2 - \tilde{c}_1)$ and $\lambda \geq \tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})/(\tilde{c}_2 - \tilde{c}_1)$, respectively. Hence, from (6.17) it follows that

$$\hat{x}_{4L} = \begin{cases} (F^\circ)^{-1}\left(\dfrac{\lambda}{\tilde{c}_1} + \dfrac{1 - 2\eta\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right) & \text{if } \lambda < \dfrac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1} \quad \text{(the case (a))}, \\ (F^\circ)^{-1}\left(1 - \dfrac{1 - 2(1-\eta)\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right) & \text{if } \lambda \geq \dfrac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1} \quad \text{(the case (b))}. \end{cases} \tag{6.18}$$

As easily seen, if $\tilde{F}_L^\circ$ is replaced by $\hat{F}_L^\circ$, then the same argument as Case 1 yields the inequality $\hat{x}_{4L} - \hat{x}_{2L} \geq x_4 - x_2$ in the case (a). Noting $\hat{x}_{4L} = x_4$, we can also see that the same result holds for the case (b). Thus, (6.12) is bounded above by

$$(F^\circ)^{-1}\left(\frac{\lambda}{\tilde{c}_1} + \frac{1 - 2\eta\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\frac{1 - 2\eta\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right) \quad \text{if } \lambda < \frac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1} \tag{6.19}$$

and

$$(F^\circ)^{-1}\left(1 - \frac{1 - 2(1-\eta)\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\frac{1 - 2\eta\tilde{\gamma} - \lambda}{2\tilde{c}_2}\right) \quad \text{if } \lambda \geq \frac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1}. \tag{6.20}$$

Next we show that all the upper bounds of (6.16), (6.19) and (6.20) are attained at $\eta = 1$. Let $g_i(\eta), i = 1, 2, 3$ be the upper bounds given in (6.16), (6.19) and (6.20) and $g_i'(\eta), i = 1, 2, 3$ be the derivatives of $g_i(\eta)$ with respect to η, respectively. For Case 1, we have

$$g_1'(\eta) = -\frac{\tilde{\gamma}}{\tilde{c}_1}\left\{\frac{1}{f^\circ\left((F^\circ)^{-1}\left(\frac{1}{2} + d_1(\eta)\right)\right)} - \frac{1}{f^\circ\left((F^\circ)^{-1}\left(\frac{1}{2} - d_2(\eta)\right)\right)}\right\},$$

where

$$d_1(\eta) = \frac{1}{2} - \frac{1 - 2(1-\eta)\tilde{\gamma} - \lambda}{2\tilde{c}_1} \quad \text{and} \quad d_2(\eta) = \frac{1 - 2(1-\eta)\tilde{\gamma} + \lambda}{2\tilde{c}_1} - \frac{1}{2}.$$

Since $d_1(\eta) + d_2(\eta) = \lambda/\tilde{c}_1 \geq 0$, $d_2(\eta) - d_1(\eta) = \{(1 - \tilde{\gamma} - \tilde{c}_1) + (2\eta - 1)\tilde{\gamma}\}/\tilde{c}_1 \geq 0$ for $1/2 \leq \eta \leq 1$ and $d_2(\eta) \geq 0$, we have $d_2(\eta) \geq |d_1(\eta)|$ for $1/2 \leq \eta \leq 1$, and hence we obtain $g_1'(\eta) \geq 0$ for $1/2 \leq \eta \leq 1$ from the unimodality and the symmetry of $f^\circ(x)$. Thus the maximum value of $g_1(\eta)$ on $1/2 \leq \eta \leq 1$ is attained at $\eta = 1$, which is given by

$$g_1(1) = (F^\circ)^{-1}\left(1 - \frac{1-\lambda}{2\tilde{c}_1}\right) - (F^\circ)^{-1}\left(1 - \frac{1+\lambda}{2\tilde{c}_1}\right).$$

For the case (a) of Case 2, $g_2'(\eta)$ is given by

$$g_2'(\eta) = -\frac{\tilde{\gamma}}{\tilde{c}_2}\left\{\frac{1}{f^\circ\left((F^\circ)^{-1}\left(\frac{1}{2} + d_3(\eta)\right)\right)} - \frac{1}{f^\circ\left((F^\circ)^{-1}\left(\frac{1}{2} - d_4(\eta)\right)\right)}\right\}, \tag{6.21}$$

where

$$d_3(\eta) = \frac{\lambda}{\tilde{c}_1} - \frac{1}{2} + \frac{1 - 2\eta\tilde{\gamma} - \lambda}{2\tilde{c}_2} \quad \text{and} \quad d_4(\eta) = \frac{1}{2} - \frac{1 - 2\eta\tilde{\gamma} - \lambda}{2\tilde{c}_2}. \tag{6.22}$$

Then we have $d_3(\eta) + d_4(\eta) = \lambda/\tilde{c}_1 \geq 0$, $d_4(\eta) - d_3(\eta) = -(\tilde{c}_2 - \tilde{c}_1)\lambda/(\tilde{c}_1\tilde{c}_2) + (\tilde{c}_2 - 1 + 2\eta\tilde{\gamma})/\tilde{c}_2 \geq 0$ because of the condition for the case (a) of Case 2 given in (6.18) and $d_4(\eta) = (\tilde{c}_2 - 1 + 2\eta\tilde{\gamma} +$

$\lambda)/\tilde{c}_2 \geq ((1-\tilde{\gamma})-1+2\eta\tilde{\gamma}+\lambda)/\tilde{c}_2 = ((2\eta-1)\tilde{\gamma}+\lambda)/\tilde{c}_2 \geq 0$ for $1/2 \leq \eta \leq 1$. By the same argument as in Case 1, the maximum value of $g_2(\eta)$ on $1/2 \leq \eta \leq 1$ is given by

$$g_2(1) = (F^\circ)^{-1}\left(\frac{\lambda}{\tilde{c}_1} + \frac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\frac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right).$$

For the case (b) of Case 2, the derivative $g_3'(\eta)$ is given by (6.21) with $d_3(\eta)$ replaced by

$$d_5(\eta) = \frac{1}{2} - \frac{1-2\eta\tilde{\gamma}-\lambda}{2\tilde{c}_2}.$$

Therefore, similarly, the maximum value of $g_3(\eta)$ on $1/2 \leq \eta \leq 1$ is given by

$$g_3(1) = (F^\circ)^{-1}\left(1 - \frac{1-\lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\frac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right).$$

The upper bound of (6.12) for the case of $0 \leq \eta < 1/2$ is given as follows: Let $F^*(x) = 1 - F(2\theta - x)$. Then, we have that (6.12) is expressed by

$$(F^*)^{-1}\left(\frac{1}{2} + d(1-\eta^*)\right) - (F^*)^{-1}\left(\frac{1}{2} - d(\eta^*)\right), \tag{6.23}$$

where $d(\eta^*)$ is given by (6.13) with $\eta^* = 1 - \eta$. Therefore, it can be easily seen that the upper bounds of (6.23) for all $F^* \in \mathscr{F}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)$ and $\eta^* \in (1/2, 1]$ are the same as those for the case of $1/2 \leq \eta \leq 1$.

Here, we check the conditions about λ for Case 1, (1) and (2) of Case 2 with $\eta = 1$, respectively. Since

$$\frac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1} - \left\{1 - 2\tilde{\gamma} - \frac{2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1}\right\} = \frac{\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1) + (\tilde{c}_2 - \tilde{c}_1)\tilde{\gamma}}{\tilde{c}_2 - \tilde{c}_1} \geq 0,$$

we have three cases for $\lambda\,(0 \leq \lambda < 1 - 2\tilde{\gamma})$ when $\tilde{c}_1 \neq 0, \tilde{c}_1 \neq 1 - \tilde{\gamma}$ and $\tilde{c}_2 \neq 1 - \tilde{\gamma}$: Let $A = 1 - 2\tilde{\gamma} - 2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)/(\tilde{c}_2 - \tilde{c}_1)$ and $B = \tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})/(\tilde{c}_2 - \tilde{c}_1)$. Then it holds that (i) if $0 \leq \lambda < \max(0, A)$, then the upper bound is $g_1(1)$, (ii) if $\max(0, A) \leq \lambda < \min(B, 1 - 2\tilde{\gamma})$, then $g_2(1)$ and (iii) if $\min(B, 1-2\tilde{\gamma}) \leq \lambda < 1 - 2\tilde{\gamma}$, then $g_3(1)$.

We note that if either $\tilde{c}_1$ or $\tilde{c}_2$ equals to $1 - \tilde{\gamma}$, then $\mathscr{F}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ) = \{F^\circ\}$. In this case, the maximum asymptotic length $L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ is

$$(F^\circ)^{-1}\left(\frac{1-2\tilde{\gamma}+\lambda}{2(1-\tilde{\gamma})}\right) - (F^\circ)^{-1}\left(\frac{1-2\tilde{\gamma}-\lambda}{2(1-\tilde{\gamma})}\right) \quad \text{for} \;\; 0 \leq \lambda < 1 - 2\tilde{\gamma},\; 0 \leq \tilde{\gamma} < \frac{1}{2},$$

which is included in (i) or (iii) of this theorem according as $\tilde{c}_1 = 1 - \tilde{\gamma}$ or $\tilde{c}_2 = 1 - \tilde{\gamma}$, respectively.

Finally, let $\tilde{H}_{\delta(-m)}$ be defined by (6.10) with F and η replaced by $\tilde{F}_L^\circ$ and $\delta(-m)$, respectively, where $\tilde{F}_L^\circ$ is the smallest distribution in $\mathscr{F}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)$ and $\delta(-m)$ denotes the point mass distribution at $-m$. Then it is easy to see that $\tilde{H}_{\delta(-m)} \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)$ and the limit of

$$\tilde{H}_{\delta(-m)}^{-1}\left(\frac{1+\lambda}{2}\right) - \tilde{H}_{\delta(-m)}^{-1}\left(\frac{1-\lambda}{2}\right) \quad \text{as} \;\; m \to \infty$$

is equal to $g_1(1)$. Similarly, let $\hat{H}_{\delta(-m)} \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)$ be defined by (6.10) with F and η replaced by $\hat{F}_L^\circ$ and $\delta(-m)$, respectively, where $\hat{F}_L^\circ$ is given by (6.17). Then, the limit of $\hat{H}_{\delta(-m)}^{-1}((1+\lambda)/2) - \hat{H}_{\delta(-m)}^{-1}((1-\lambda)/2)$ as $m \to \infty$ is equal to $g_2(1)$ or $g_3(1)$ according as the condition of λ in (6.19). Thus, the theorem is proved. $\square$

Proof of Lemma 4.1

We first calculate A. Since

$$a = \frac{2\tilde{c}_1}{(\tilde{c}_2 - \tilde{c}_1)}\left(\tilde{\gamma} - \frac{1}{2} - \frac{\tilde{c}_2(1-2\tilde{c}_1)}{2\tilde{c}_1}\right),$$

we have

$$A = \begin{cases} 0 & \text{if} \quad \tilde{\gamma} < \dfrac{1}{2} + \dfrac{\tilde{c}_2(1-2\tilde{c}_1)}{2\tilde{c}_1}, \\ a & \text{if} \quad \tilde{\gamma} \geq \dfrac{1}{2} + \dfrac{\tilde{c}_2(1-2\tilde{c}_1)}{2\tilde{c}_1}. \end{cases}$$

If $0 < \tilde{c}_1 \leq 1/2$, then $1/2 + \tilde{c}_2(1-2\tilde{c}_1)/(2\tilde{c}_1) > 1/2$, and hence we have $A = 0$ for $0 \leq \tilde{\gamma} < 1/2$. If $1/2 < \tilde{c}_1 \leq 1 - \tilde{\gamma}$, then we have that

$$\frac{1}{2} + \frac{\tilde{c}_2(1-2\tilde{c}_1)}{2\tilde{c}_1} \geq, < 0 \text{ according as } \tilde{c}_1 \leq, > \frac{\tilde{c}_2}{2\tilde{c}_2 - 1}.$$

Therefore, if $1/2 < \tilde{c}_1 \leq 1 - \tilde{\gamma}$ and $\tilde{c}_1 \leq \tilde{c}_2/(2\tilde{c}_2 - 1)$, then it follows that

$$A = \begin{cases} 0 & \text{if} \quad \tilde{\gamma} < \dfrac{1}{2} + \dfrac{\tilde{c}_2(1-2\tilde{c}_1)}{2\tilde{c}_1}, \\ a & \text{if} \quad \tilde{\gamma} \geq \dfrac{1}{2} + \dfrac{\tilde{c}_2(1-2\tilde{c}_1)}{2\tilde{c}_1}, \end{cases} \tag{6.24}$$

and also if $1/2 < \tilde{c}_1 \leq 1 - \tilde{\gamma}$ and $\tilde{c}_1 > \tilde{c}_2/(2\tilde{c}_2 - 1)$, then $A = a$ for $0 \leq \tilde{\gamma} < 1/2$.

Next, we consider B. From

$$1 - 2\tilde{\gamma} - b = \frac{1}{(\tilde{c}_2 - \tilde{c}_1)(2\tilde{c}_2 - \tilde{c}_1)}\left(\frac{\tilde{c}_2(1-\tilde{c}_1)}{2\tilde{c}_2 - \tilde{c}_1} - \tilde{\gamma}\right)$$

and

$$0 < \frac{\tilde{c}_2(1-\tilde{c}_1)}{2\tilde{c}_2 - \tilde{c}_1} = \frac{1}{2} - \frac{\tilde{c}_1(\tilde{c}_2 - 1/2)}{2\tilde{c}_2 - \tilde{c}_1} < \frac{1}{2},$$

we have

$$B = \begin{cases} b & \text{if} \quad 0 < \tilde{\gamma} < \dfrac{\tilde{c}_2(1-\tilde{c}_1)}{2\tilde{c}_2 - \tilde{c}_1}, \\ 1 - 2\tilde{\gamma} & \text{if} \quad \dfrac{\tilde{c}_2(1-\tilde{c}_1)}{2\tilde{c}_2 - \tilde{c}_1} \leq \tilde{\gamma} < \dfrac{1}{2}. \end{cases} \tag{6.25}$$

Now we have

$$\frac{\tilde{c}_2(1-\tilde{c}_1)}{2\tilde{c}_2 - \tilde{c}_1} - \left\{\frac{1}{2} + \frac{\tilde{c}_2(1-2\tilde{c}_1)}{2\tilde{c}_1}\right\} = \frac{1}{2\tilde{c}_1(2\tilde{c}_2 - \tilde{c}_1)}(\tilde{c}_2 - \tilde{c}_1)\{(4\tilde{c}_2 - 1)\tilde{c}_1 - 2\tilde{c}_2\} \tag{6.26}$$

and so the left-hand side of (6.26) is non-positive or positive according as $\tilde{c}_1 \leq$ or $> 2\tilde{c}_2/(4\tilde{c}_2 - 1)$. Thus, from (6.24) and (6.25) the lemma is obtained. □

Proof of Theorem 4.3

The assertion (i) is the direct consequence of Theorem 4.2 and the assertion (ii) is the special case of the assertion (i). □

Proof of Theorem 4.4

First, we show $(F^\circ)^{-1}\{(1+\lambda)/2\} \le B_0$. Let

$$F_R^*(x) = \begin{cases} 0 & \text{if} \quad x < (F^\circ)^{-1}(\gamma), \\ \dfrac{F^\circ(x) - \gamma}{1-\gamma} & \text{if} \quad x \ge (F^\circ)^{-1}(\gamma), \end{cases}$$

and

$$H_L^*(x) = \begin{cases} \dfrac{F^\circ(x)}{\gamma} & \text{if} \quad x < (F^\circ)^{-1}(\gamma), \\ 1 & \text{if} \quad x \ge (F^\circ)^{-1}(\gamma). \end{cases}$$

Then we have $F^\circ = (1-\gamma)F_R^* + \gamma H_L^*$. Also, for $M = (F^\circ)^{-1}\{(1+\lambda)/2\}$ let

$$V_M(x) = \begin{cases} 0 & \text{if} \quad x < M, \\ \dfrac{1}{1 - F_R^*(M)} \left(F_R^*(x) - F_R^*(M)\right) & \text{if} \quad x \ge M, \end{cases}$$

and

$$G_R^\circ(x) = \frac{1-\gamma}{1+\lambda-2\gamma} F_R^*(x) + \left(1 - \frac{1-\gamma}{1+\lambda-2\gamma}\right) V_M(x).$$

Then we have $G_R^\circ(M) = 1/2$, that is, M is the median of G_R°. Letting $g_R^\circ(x)$ be the density function of $G_R^\circ(x)$, it is easy to see that

$$g_R^\circ(x) = \begin{cases} \dfrac{1-\gamma}{1+\lambda-2\gamma} f_R^*(x) & \text{if} \quad x < M, \\ \dfrac{1-\gamma}{1-\lambda} f_R^*(x) & \text{if} \quad x \ge M. \end{cases}$$

From $1-\lambda \le 1+\lambda-2\gamma$, it follows that for every $x \in \mathbb{R}$

$$\frac{c_1}{1-\gamma} g_R^\circ(x) \le \frac{c_1}{1-\lambda} f_R^*(x) \tag{6.27}$$

and

$$\frac{c_2}{1-\gamma} g_R^\circ(x) \ge \frac{c_2}{1+\lambda-2\gamma} f_R^*(x). \tag{6.28}$$

Noting $1-\lambda = \max\left(c_1, 2(1-\gamma) - c_2\right) \ge c_1$ and $1+\lambda-2\gamma = \min\left(2(1-\gamma) - c_1, c_2\right) \le c_2$, we obtain from (6.27) and (6.28) that $\{c_1/(1-\gamma)\}\, g_R^\circ(x) \le f_R^*(x) \le \{c_2/(1-\gamma)\}\, g_R^\circ(x)$, and hence $F_R^* \in \mathscr{F}_{c_1,c_2,\gamma}(G_R^\circ)$. Since $F^\circ = (1-\gamma)F_R^* + \gamma H_L^*$, we have $F^\circ \in \mathscr{P}_{c_1,c_2,\gamma}(G_R^\circ)$. By the assumptions of the theorem, this implies $M = (F^\circ)^{-1}\{(1+\lambda)/2\} = (G_R^\circ)^{-1}(1/2) \in [A_0, B_0]$, and therfore we obtain $(F^\circ)^{-1}\{(1+\lambda)/2\} \le B_0$.

Next, in order to show $(F^\circ)^{-1}\{(1-\lambda)/2\} \ge A_0$, let

$$F_L^*(x) = \begin{cases} \dfrac{F^\circ(x)}{1-\gamma} & \text{if} \quad x < (F^\circ)^{-1}(1-\gamma), \\ 1 & \text{if} \quad x \ge (F^\circ)^{-1}(1-\gamma), \end{cases}$$

and

$$H_R^*(x) = \begin{cases} 0 & \text{if} \quad x < (F^\circ)^{-1}(1-\gamma), \\ \dfrac{F^\circ(x) - (1-\gamma)}{\gamma} & \text{if} \quad x \geq (F^\circ)^{-1}(1-\gamma). \end{cases}$$

Then we have $F^\circ = (1-\gamma)F_L^* + \gamma H_R^*$. Also, for $N = (F^\circ)^{-1}((1-\lambda)/2)$ let

$$W_N(x) = \begin{cases} \dfrac{1}{F_L^*(N)} F_L^*(x) & \text{if} \quad x < N, \\ 1 & \text{if} \quad x \geq N, \end{cases}$$

and

$$G_L^\circ(x) = \frac{1-\gamma}{1+\lambda-2\gamma} F_L^*(x) + \left(1 - \frac{1-\gamma}{1+\lambda-2\gamma}\right) W_N(x).$$

Then we have $G_L^\circ(N) = 1/2$, that is, N is the median of G_L°. Letting $g_L^\circ(x)$ be the density function of $G_L^\circ(x)$, we can see

$$g_L^\circ(x) = \begin{cases} \dfrac{1-\gamma}{1-\lambda} f_L^*(x) & \text{if} \quad x < N, \\ \dfrac{1-\gamma}{1+\lambda-2\gamma} f_L^*(x) & \text{if} \quad x \geq N. \end{cases}$$

By the same argument as in g_R° it is clear that $F_L^* \in \mathscr{F}_{c_1,c_2,\gamma}(G_L^\circ)$. Since $F^\circ = (1-\gamma)F_L^* + \gamma H_R^*$, we have $F^\circ \in \mathscr{P}_{c_1,c_2,\gamma}(G_0)$. Therefore, from the assumptions of the theorem it follows that $N = (F^\circ)^{-1}\{(1-\lambda)/2\} = (G_L^\circ)^{-1}(1/2) \in [A_0, B_0]$ and hence $(F^\circ)^{-1}\{(1-\lambda)/2\} \geq A_0$. $\square$

Proof of Theorem 5.1

Let $I_n(\boldsymbol{X}_n)$ be given in Theorem 4.1. Then we have

$$P_G\{\varphi_{n,\theta}(\boldsymbol{X}_n) = 1\} = P_G\{\theta \notin I_n(\boldsymbol{X}_n)\}. \tag{6.29}$$

This implies that the theorem immediately follows from Theorem 4.1. $\square$

Proof of Theorem 5.2

We assume $\theta = 0$ without loss of generality. Since, by (6.9) we have $X_{(k_n)} \to G^{-1}\{(1-\lambda)/2\}$ and $X_{(n-k_n)} \to G^{-1}\{(1+\lambda)/2\}$, it follows from (6.29) that

$$\lim_{n\to\infty} P_G\{\varphi_{n,\theta}(\boldsymbol{X}_n) = 1\} = \begin{cases} 1 & \text{if} \quad G^{-1}\{(1-\lambda)/2\} > 0 \text{ or } G^{-1}\{(1+\lambda)/2\} < 0, \\ 0 & \text{if} \quad G^{-1}\{(1-\lambda)/2\} < 0 < G^{-1}\{(1+\lambda)/2\}. \end{cases}$$

Then

$$\inf_{G \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F_\eta^\circ)} \lim_{n\to\infty} P_G\{\varphi_{n,\theta}(\boldsymbol{X}_n) = 1\} = 1 \quad \text{for all } |\eta| > M$$

holds either if

$$\sup_{G \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F_\eta^\circ)} G^{-1}\left(\frac{1+\lambda}{2}\right) = \eta + \sup_{G \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)} G^{-1}\left(\frac{1+\lambda}{2}\right) < 0 \tag{6.30}$$

or

$$\inf_{G \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ_\eta)} G^{-1}\left(\frac{1-\lambda}{2}\right) = \eta + \inf_{G \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)} G^{-1}\left(\frac{1-\lambda}{2}\right) > 0. \tag{6.31}$$

Also, it is easy to see that

$$\sup_{G \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)} G^{-1}\left(\frac{1+\lambda}{2}\right) = \{(1-\tilde{\gamma})\tilde{F}_R\}^{-1}\left(\frac{1+\lambda}{2}\right)$$

and

$$\inf_{G \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)} G^{-1}\left(\frac{1-\lambda}{2}\right) = \{(1-\tilde{\gamma})\tilde{F}_L + \tilde{\gamma}\}^{-1}\left(\frac{1-\lambda}{2}\right),$$

where $\tilde{F}_L(x)$ and $\tilde{F}_R(x)$ are given by (3.5) and (3.6) with respect to $\tilde{x}_L, \tilde{x}_R, \tilde{c}_1, \tilde{c}_2$ and $\tilde{\gamma}$. Here we assume $c_1 \neq 0$. Note that the solution of the equation $(1-\tilde{\gamma})\tilde{F}_R(\tilde{x}_R) = (1+\lambda)/2$ is $\lambda = 1 - 2\tilde{\gamma} - 2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)/(\tilde{c}_2-\tilde{c}_1)$, and that of the equation $(1-\tilde{\gamma})\tilde{F}_L(\tilde{x}_L) + \tilde{\gamma} = (1-\lambda)/2$ is the same. Therefore, we easily obtain that

$$G^{-1}\left(\frac{1+\lambda}{2}\right) \leq \begin{cases} (F^\circ)^{-1}\left(\dfrac{1+\lambda}{2\tilde{c}_1}\right) & \text{if} \quad 0 \leq \lambda \leq \max(0,a), \\ (F^\circ)^{-1}\left(1 - \dfrac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right) & \text{if} \quad \max(0,a) < \lambda < 1-2\tilde{\gamma}, \end{cases}$$

and

$$G^{-1}\left(\frac{1-\lambda}{2}\right) \geq \begin{cases} (F^\circ)^{-1}\left(1 - \dfrac{1+\lambda}{2\tilde{c}_1}\right) & \text{if} \quad 0 \leq \lambda \leq \max(0,a), \\ (F^\circ)^{-1}\left(\dfrac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right) & \text{if} \quad \max(0,a) < \lambda < 1-2\tilde{\gamma}, \end{cases}$$

where $a = 1 - 2\tilde{\gamma} - \{2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)\}/(\tilde{c}_2-\tilde{c}_1)$. Therefore, since F° is symmetric about 0, (6.30) and (6.31) hold if

$$|\eta| > \begin{cases} (F^\circ)^{-1}\left(\dfrac{1+\lambda}{2\tilde{c}_1}\right) & \text{if} \quad 0 \leq \lambda \leq \max(0,a), \\ (F^\circ)^{-1}\left(1 - \dfrac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right) & \text{if} \quad \max(0,a) < \lambda < 1-2\tilde{\gamma}. \end{cases}$$

According to the criteria of $A = \max(0,a)$ in the proof of Lemma 4.1, this implies that the theorem holds for $\tilde{c}_1 \neq 0$. As for the case of $\tilde{c}_1 = 0$, the result is included. □

Proof of Theorem 5.3

The assertions (i) is the immediate consequences of Theorem 5.2 and the assertion (ii) readily follows from the assertion (i). □

References

Ando, M. and Kimura, M. (2003). A characterization of the neighborhoods defined by certain special capacities and their applications to bias-robustness of estimates, *J. Statist. Plann. Inference.*, **116**, 61-90.

Ando, M. and Kimura, M. (2004). The maximum asymptotic bias of S-estimates for regression over the neighborhoods defined by certain special capacities, *J. Multivariate Anal.*, **90**, 407-425.

Ando, M., Kakiuchi, I. and Kimura, M. (2009). Robust nonparametric confidence intervals and tests for the median in the presence of (c, γ) - contamination, *J. Statist. Plann. Inference*, **139**, 1836-1846.

Bednarski, T. (1981). On solutions of minimax tests problems for special capacities, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **58**, 397-405.

Hampel, F. R., Ronchetti, E. M., Rousseeuw, P. J. and Stahel, W. A. (1986). *Robust Statistics: The approach based on influence functions*, Wiley, New York.

Hettmansperger, T. P. and Mckean, J. W. (2011). *Robust Nonparametric Statistical Methods*, Second Edition, Chapman and Hall, London.

Huber, P. J. (1964). Robust estimation of a location parameter, *Ann. Math. Statist.*, **35**, 73-101.

Huber, P. J. (1965). Robust version of the probability ratio tests, *Ann. Math. Statist.*, **36**, 1753-1758.

Huber, P. J. (1968). Robust confidence limits, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete*, **10**, 269-278.

Huber, P. J. and Strassen, V.(1973). Minimax tests and the Neyman-Pearson lemma for capacities, *Ann Statist.*, **1**, 256-263.

Huber, P. J. (1981). *Robust Statistics*, Wiley, New York.

Huber, P. J. and Ronchetti, E. M. (2009). *Robust Statistics*, Second Edition, Wiley, New York.

Maronna, A. M, Martin R. D. and Yohai, V. J. (2006) *Robust Statistics: Theory and Methods*, Wiley, New York.

Rieder, H. (1977). Least favorable pairs for special capacities, *Ann. Statist.*, **6**, 1080-1094.

Rieder, H.(1978). A robust asymptotic testing model. *Ann. Statist.*, **6**, 1080-1094.

Rieder, H.(1980). Estimates derived from robust test. *Ann. Statist.*, **8**, 106-115.

Rieder, H.(1981a). Robustness of one and two sample rank tests against gross errors, *Ann. Statist.*, **9**, 245-265.

Rieder, H.(1981b). On local asymptotic minimaxity and admissibility in robust estimation. *Ann. Statist.*, **9**, 266-277.

Rieder, H.(1994). *Robust Asymptotic Statistics*, Springer-Verlag, Berlin.

Yohai, V. J. and Zamar, R. H. (2004) Robust nonparametric inference for the median, *Ann. Statist.*, **32**, 1841-1857.