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Passivity Degradation under the Discretization with the Zero-Order Hold and the Ideal Sampler*

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Quantitative study is made on how much passivity is lost when a continuous-time nonlinear system is discretized for digital implementation of a controller. The standard discretization with the zero-order hold and the ideal sampler is mainly considered though other types of discretization are also considered. The result can be used for the choice of the sampling period. Namely, the sampling period has to be small when the original continuous-time system is oscillatory. The general input-output framework is used for the discussion.

Keywords: passivity, discretization, sampling period, sampled-data systems.

1. Introduction

Passivity is an important notion in systems analysis and control especially in the case of nonlinear systems [1, 2, 3]. It is well-known however that this notion is not compatible with discretization, which is required for digital implementation of a controller. Namely, even if the original continuous-time system is passive, the discretized system may not be passive any more.

Various authors have analyzed this phenomenon. Laila–Nešić–Teel [4] and de la Sen [5] gave conditions for passivity to be preserved with a small enough sampling period. Although these authors used standard discretization with the zero-order hold and the ideal sampler, Costa–Castelló–Fossas [6] considered a different type of discretization with the average of the output of the system. They showed that this discretization preserves passivity for any sampling period. Stramigioli–Secchi–van der Schaft–Fantuzzi [7] considered the same discretization for port-Hamiltonian systems. A drawback of this discretization is that the average of the output requires the future value of the output, which is difficult to compute for a nonlinear system.

In this paper, we quantitatively investigate how much passivity is lost due to discretization. In particular, we express the passivity degradation as a function of the sampling period. Our main interest is on the standard discretization with the zero-order hold and the ideal sampler though other types of discretization are considered as well. The result gives an insight for the choice of the sampling period. Namely, a small sampling period is required for an oscillatory system. We use the input-output framework for the sake of generality.

This paper is organized as follows. In Section 2, the notion of passivity and two types of discretization are presented. Passivity loss under discretization is also discussed there. Section 3 is the main section and evaluates the passivity degradation under the standard discretization. Finally in Section 4, extension of the result is discussed.

Notation is standard. The symbol \mathbb{R}^m stands for the space of the m -dimensional real vectors. The symbol T expresses the transpose of a vector. For a vector $v \in \mathbb{R}^m$, its norm $\|v\|$ is defined as $\sqrt{v^T v}$.

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2. Passivity and Discretization

The continuous-time system S is an operator mapping a piecewise continuous signal $u(t)$ to a continuous signal $y(t)$, each of which takes a value in \mathbb{R}^m . It is assumed to map the zero input $u(t) \equiv 0$ to the zero output $y(t) \equiv 0$.

Suppose that the system S satisfies the passivity inequality

$$\int_0^T u(t)^T y(t) dt \geq \epsilon \int_0^T \|u(t)\|^2 dt + \delta \int_0^T \|y(t)\|^2 dt \quad (1)$$

for any $T \geq 0$ and any piecewise continuous input $u(t)$, where ϵ and δ are real numbers. Although ϵ and δ can be negative, we are interested in the case that they are nonnegative. In particular, S is called (i) *passive* when $\epsilon = \delta = 0$; (ii) *input strictly passive* when $\epsilon > 0$ and $\delta = 0$; (iii) *output strictly passive* when $\epsilon = 0$ and $\delta > 0$; (iv) *very strictly passive* when $\epsilon > 0$ and $\delta > 0$ [1, 2, 3].

We next consider discretization of S . With h being some positive number called the *sampling period*, we define the discrete-time input $u_d(k)$, $k = 1, 2, \dots$, and the discrete-time output $y_d(k)$, $k = 1, 2, \dots$, so as to be related to their continuous-time counterparts in a way described later. The values $u_d(k)$ and $y_d(k)$ are considered as the signals at the time $t = kh$. Now the operator from $\{u_d(k)\}$ to $\{y_d(k)\}$ is regarded as a discrete-time system. This procedure as well as the resulting discrete-time system is called the discretization of S .

Passivity can be considered also on a discrete-time system. Suppose that the discrete-time system considered above satisfies the discrete-time passivity inequality

$$h \sum_{k=0}^{K-1} u_d(k)^T y_d(k) \geq \epsilon_d h \sum_{k=0}^{K-1} \|u_d(k)\|^2 + \delta_d h \sum_{k=0}^{K-1} \|y_d(k)\|^2 \quad (2)$$

for any positive integer K and any input $u_d(k)$, where ϵ_d and δ_d are real numbers. Then this system is called (i) *passive* when $\epsilon_d = \delta_d = 0$; (ii) *input strictly passive* when $\epsilon_d > 0$ and $\delta_d = 0$; (iii) *output strictly passive* when $\epsilon_d = 0$ and $\delta_d > 0$; (iv) *very strictly passive* when $\epsilon_d > 0$ and $\delta_d > 0$. The factor h appears in (2) for making its dimension in time consistent with the continuous-time inequality (1).

The most commonly used discretization is the following. The discrete-time input $u_d(k)$ is related to the continuous-time input $u(t)$ by the zero-order hold, that is,

$$u(t) = u_d(k), \quad kh \leq t < (k+1)h, \quad k = 0, 1, \dots; \quad (3)$$

the discrete-time output $y_d(k)$ is corresponded to the continuous-time output $y(t)$ by the ideal sampler, that is,

$$y_d(k) = y(kh), \quad k = 0, 1, \dots \quad (4)$$

We call this procedure *standard discretization* and write the resulting discrete-time system as S_d^{sta} .

Important observation here is that, even if S is passive, its standard discretization S_d^{sta} may not be so. See Example 2 below. The approach of [7, 6] to avoid this problem is to define the discrete-time output by

$$y_d(k) = \frac{1}{h} \int_{kh}^{(k+1)h} y(t) dt, \quad k = 0, 1, \dots, \quad (5)$$

in place of (4). We call this discretization *average discretization* and write the resulting discrete-time system as S_d^{ave} . This discretization preserves passivity. We state the result in a stronger form than

that in [6]. Namely, we consider preservation not only of (nonstrict) passivity but also of the three types of strict passivity. Moreover, the result is given in a quantitative way. The real number δ is assumed to be nonnegative here.

Theorem 1. *Suppose that the original continuous-time system S satisfies the continuous-time passivity inequality (1) with some real numbers ϵ and $\delta \geq 0$. Then its average discretization S_d^{ave} satisfies the discrete-time passivity inequality (2) with $\epsilon_d = \epsilon$ and $\delta_d = \delta$. In particular, if S is passive, input strictly passive, output strictly passive, and very strictly passive, so is its average discretization S_d^{ave} , respectively.*

Proof. Assuming the relationships (3) and (5) and $T = Kh$, we compare the three integrals in the continuous-time passivity inequality (1) with the corresponding summations in (2), respectively.

The first integral in (1) is equal to the corresponding summation in (2) because

$$\int_0^{Kh} u(t)^T y(t) dt = \sum_{k=0}^{K-1} u_d(k)^T \int_{kh}^{(k+1)h} y(t) dt = h \sum_{k=0}^{K-1} u_d(k)^T y_d(k). \quad (6)$$

As easily seen, the second integral is again equal to the corresponding summation, *i.e.*,

$$\int_0^{Kh} \|u(t)\|^2 dt = h \sum_{k=0}^{K-1} \|u_d(k)\|^2. \quad (7)$$

Finally, the third integral is rewritten as

$$\int_0^{Kh} \|y(t)\|^2 dt = \sum_{k=0}^{K-1} \sum_{i=1}^m \int_{kh}^{(k+1)h} y_i(t)^2 dt$$

with $y(t) = (y_1(t) \ y_2(t) \ \cdots \ y_m(t))^T$. Here, we have

$$\sqrt{h} \sqrt{\int_{kh}^{(k+1)h} y_i(t)^2 dt} \geq \left| \int_{kh}^{(k+1)h} y_i(t) dt \right|$$

by the Cauchy–Schwarz inequality. It hence follows that

$$\int_0^{Kh} \|y(t)\|^2 dt \geq \sum_{k=0}^{K-1} \sum_{i=1}^m \frac{1}{h} \left(\int_{kh}^{(k+1)h} y_i(t) dt \right)^2 = h \sum_{k=0}^{K-1} \|y_d(k)\|^2. \quad (8)$$

Using (6)–(8) with the passivity inequality (1), we have the desired result. Note that δ has to be nonnegative because (8) is an inequality. \square

Example 2. Let S be a linear system

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t), & x(0) &= 0, \\ y(t) &= x(t). \end{aligned}$$

This system satisfies $\dot{y}(t)y(t) = -y(t)^2 + u(t)y(t)$ and then

$$0 \leq \frac{1}{2}y(T)^2 = \int_0^T \dot{y}(t)y(t) dt = - \int_0^T y(t)^2 dt + \int_0^T u(t)y(t) dt,$$

which implies that S is output strictly passive with $\epsilon = 0$ and $\delta = 1$.

This system is not input strictly passive. To see this, let a be a positive integer and consider the input $u(t)$ equal to $\sin at$ for $0 \leq t < 2\pi$ and to 0 otherwise. The integral $\int_0^T u(t)^2 dt = \pi$ for any $T \geq 2\pi$ irrespective of a . On the other hand, $\int_0^T u(t)y(t) dt$ approaches zero as $a \rightarrow \infty$. This means that the continuous-time passivity inequality (1) does not hold with $\epsilon > 0$ and $\delta = 0$.

Its standard discretization S_d^{sta} can be written in the state-space form as

$$\begin{aligned} x((k+1)h) &= e^{-h}x(kh) + (1 - e^{-h})u_d(k), \quad x(0) = 0, \\ y_d(k) &= x(kh). \end{aligned}$$

This system is however not passive irrespectively of the sampling period h . To see this, let the input $u_d(k)$ be equal to 1 for $k = 0$, to -1 for $k = 1$, and to 0 otherwise. This input gives $y_d(0) = 0$ and $y_d(1) = 1 - e^{-h}$, resulting in $h \sum_{k=0}^{K-1} u_d(k)y_d(k) = -h(1 - e^{-h}) < 0$ for $K \geq 2$. Hence, the discrete-time passivity inequality (2) does not hold with $\epsilon_d = \delta_d = 0$.

On the other hand, the average discretization S_d^{ave} remains output strictly passive due to Theorem 1. \diamond

The average discretization is theoretically attractive because it makes the passivity-based methods available after the discretization. A drawback of this discretization is that computation of the average sample $y_d(k) = (1/h) \int_{kh}^{(k+1)h} y(t) dt$ may not be easy at the time $t = kh$ because it requires the future output value $y(t)$, $kh < t < (k+1)h$. When the original system S is linear, computation of $y_d(k)$ is possible at $t = kh$ because the future value is explicitly obtained as a function of the present state $x(kh)$ and input $u_d(k)$. When S is nonlinear, however, such explicit expression is difficult usually.

Motivated by this fact, we go back to the standard discretization and investigate how much passivity is lost with it.

3. Passivity Degradation under the Standard Discretization

This is the main section of the paper. We evaluate how much passivity is lost under the standard discretization. The result is summarized in the following theorem. Nonnegativity of δ is not required this time.

Theorem 3. *Suppose that the original continuous-time system S satisfies the continuous-time passivity inequality (1) with some real numbers ϵ and δ . Suppose also that the operator from $u(t)$ to $\dot{y}(t)$ has the finite L^2 -gain $\gamma > 0$, that is,*

$$\int_0^T \|\dot{y}(t)\|^2 dt \leq \gamma^2 \int_0^T \|u(t)\|^2 dt$$

for any $T \geq 0$ and admissible $u(t)$. Then its standard discretization S_d^{sta} satisfies the discrete-time passivity inequality (2) with

$$\begin{aligned} \epsilon_d &= \epsilon - h\gamma - h\gamma|\delta| - h^2\gamma^2|\delta|, \\ \delta_d &= \delta - h\gamma|\delta|. \end{aligned}$$

In particular, if S is input strictly passive and very strictly passive, so is its standard discretization S_d^{sta} , respectively, for a small enough sampling period h .

This theorem tells importance of input strict passivity for preservation of passivity. Indeed, if the original system S lacks input strict passivity, its passivity or output strict passivity may not be preserved however small the sampling period would be. This is what we observed in Example 2. The result is consistent with the results of [4, 5].

It is notable that Theorem 3 presents the degradation of passivity as a function of the sampling period h . In order to make the degradation small, we need to choose h small. It is especially so when γ is large, that is, when the operator from $u(t)$ to $\dot{y}(t)$ has a high gain. This is understandable because such a system should be oscillatory and require fast enough sampling for its behavior to be captured.

Proof. We again compare the three integrals in the continuous-time passivity inequality (1) with the corresponding summations but assuming (3) and (4) this time.

The discrepancy between the first integral and the corresponding summation is

$$D_1 := \left| \int_0^{Kh} u(t)^T y(t) dt - h \sum_{k=0}^{K-1} u_d(k)^T y_d(k) \right|.$$

Rewriting this with (3) and (4), we obtain

$$\begin{aligned} D_1 &= \left| \sum_{k=0}^{K-1} u_d(k)^T \int_{kh}^{(k+1)h} y(t) - y(kh) dt \right| = \left| \sum_{k=0}^{K-1} u_d(k)^T \int_{kh}^{(k+1)h} \int_{kh}^t \dot{y}(s) ds dt \right| \\ &\leq \sum_{k=0}^{K-1} \|u_d(k)\| \int_{kh}^{(k+1)h} \int_{kh}^t \|\dot{y}(s)\| ds dt. \end{aligned}$$

In the last expression, the integral with respect to s can be bounded from above as

$$\int_{kh}^t \|\dot{y}(s)\| ds \leq \int_{kh}^{(k+1)h} \|\dot{y}(s)\| ds \leq \sqrt{h} \sqrt{\int_{kh}^{(k+1)h} \|\dot{y}(s)\|^2 ds}.$$

We hence have

$$D_1 \leq h\sqrt{h} \sum_{k=0}^{K-1} \|u_d(k)\| \sqrt{\int_{kh}^{(k+1)h} \|\dot{y}(s)\|^2 ds} \leq h\sqrt{h} \sqrt{\sum_{k=0}^{K-1} \|u_d(k)\|^2} \sqrt{\sum_{k=0}^{K-1} \int_{kh}^{(k+1)h} \|\dot{y}(s)\|^2 ds}.$$

Since the L^2 -gain from $u(t)$ to $\dot{y}(t)$ is γ , the second summation in the last expression can be bounded from above by $\gamma^2 \int_0^{Kh} \|u(t)\|^2 dt$, which is equal to $\gamma^2 h \sum_{k=0}^{K-1} \|u_d(k)\|^2$. This gives

$$D_1 \leq h\gamma \cdot h \sum_{k=0}^{K-1} \|u_d(k)\|^2.$$

On the second integral in (1), the equation (7) remains to hold. On the third integral, we can make comparison in a similar way to the first one and obtain

$$\left| \int_0^{Kh} \|y(t)\|^2 dt - h \sum_{k=0}^{K-1} \|y_d(k)\|^2 \right| \leq (h\gamma + h^2\gamma^2)h \sum_{k=0}^{K-1} \|u_d(k)\|^2 + h\gamma \cdot h \sum_{k=0}^{K-1} \|y_d(k)\|^2.$$

Using the obtained relationships with (1) and arranging the terms, we have the theorem. \square

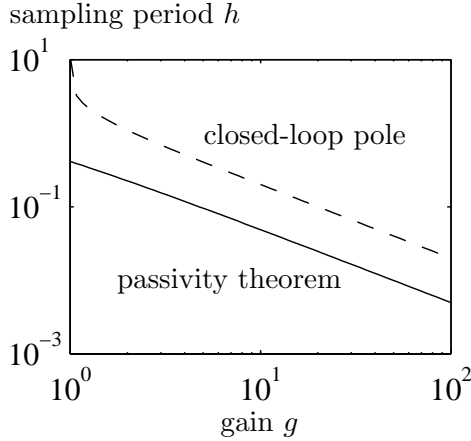


Figure 1. The stability bounds on the sampling period h obtained from the passivity theorem (solid line) and the closed-loop pole location (dashed line)

Example 4. We continue Example 2. Recall that the original system S is output strictly passive and its standard discretization S_d^{sta} is not passive for any sampling period $h > 0$. This phenomenon is explained by Theorem 3. Indeed, the lack of input strict passivity of S leads to the loss of passivity.

Quantitative discussion is possible. The operator from $u(t)$ to $\dot{y}(t)$ is in this case

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t), & x(0) &= 0, \\ \dot{y}(t) &= -x(t) + u(t), \end{aligned}$$

which has the L^2 -gain $\gamma = 1$. Theorem 3 states that the standard discretization S_d^{sta} satisfies the passivity inequality with $\epsilon_d = -2h - h^2$ and $\delta_d = 1 - h$.

This result can be used for stability analysis. Let us recall the passivity theorem [1, p. 350]. There, we consider a closed-loop system connecting two continuous-time systems by the negative feedback and assume it well-posed. Then, the closed-loop system is L^2 -stable if the two systems satisfy the passivity inequality (1) with ϵ, δ and ϵ', δ' , respectively, and they have $\epsilon + \delta' > 0$ and $\epsilon' + \delta > 0$. The same result holds on discrete-time systems, too.

We now consider the closed-loop system connecting S and a static linear gain $g > 0$ by the negative feedback. It is well-posed due to the strict properness of S . The system S has $\epsilon = 0$ and $\delta = 1$ as in Example 2 while the gain g has $\epsilon' = 0$ and $\delta' = 1/g$ as seen easily. The passivity theorem says that the closed-loop system is L^2 -stable for any $g > 0$. This can be confirmed also with the location of the closed-loop pole.

The situation is different for the discrete-time closed-loop system consisting of S_d^{sta} and g . Since $\epsilon_d = -2h - h^2$, $\delta_d = 1 - h$, $\epsilon'_d = 0$, and $\delta'_d = 1/g$, the stability condition is

$$0 < h < \min\left\{1, \sqrt{1 + \frac{1}{g}} - 1\right\}.$$

It claims that, when the gain g is large, the sampling period has to be chosen adequately small.

We can obtain a stability condition from the location of the closed-loop pole, which turns out

$$0 < h < \ln \frac{g+1}{g-1} \quad (g > 1).$$

The two stability conditions are compared in Figure 1. They show similar behavior for a large g . The condition from the passivity theorem looks more conservative than the other. This conservatism is considered to come from Theorem 3 and, possibly, from the passivity theorem as well. We need to notice, however, that the condition from the passivity theorem guarantees stability not only for the particular linear gain g but also all the static nonlinear gain belonging to the sector $[0, g]$. \diamond

4. Discussion

In this paper, we investigate how much passivity is lost under the standard discretization. Our investigation is quantitative and can be used, for example, for the choice of the sampling period as seen in the example. This is considered to be useful in practice.

An extension of the result is possible. The standard discretization can be improved with the information available at the time $t = kh$ as

$$y_d(k) = y(kh) + \frac{h}{2}\dot{y}(kh), \quad k = 0, 1, \dots$$

This is obtained by substitution of the first-order Taylor expansion $y(t) \approx y(kh) + (t - kh)\dot{y}(kh)$ into the definition of the average discretization (5). Let us call it the *first-order discretization* and write the resulting system as S_d^{fir} . Passivity degradation under this discretization can be evaluated similarly to the case of the standard discretization.

Theorem 5. *Suppose that the original continuous-time system S satisfies the continuous-time passivity inequality (1) with some real numbers ϵ and δ . Suppose also that the operator from $u(t)$ to $\dot{y}(t)$ has the finite L^2 -gain $\gamma > 0$ and the operator from $u(t)$ to $\ddot{y}(t)$ has the finite L^2 -gain $\beta > 0$. Then its first-order discretization S_d^{fir} satisfies the discrete-time passivity inequality (2) with*

$$\begin{aligned} \epsilon_d &= \epsilon - \frac{1}{2}h^2\beta - \left(h^2\gamma^2 + \frac{1}{2}h^2\beta + \frac{1}{4}h^4\beta^2\right)|\delta|, \\ \delta_d &= \delta - \frac{1}{2}h^2\beta|\delta|. \end{aligned}$$

In particular, if S is input strictly passive and very strictly passive, so is its first-order discretization S_d^{fir} , respectively, for a small enough sampling period h .

Proof. See Appendix. \square

Passivity degradation under the first-order discretization is of order h^2 while it is of order h under the standard discretization. This reduction is a consequence of the improvement of discretization. On the other hand, it is still not possible to preserve passivity without input strict passivity. Indeed, when the original system S is not input strictly passive, its discretization S_d^{fir} may not be passive however small the sampling period would be.

In order to quantify the passivity degradation, we need γ and β , which are the L^2 -gain from $u(t)$ to $\dot{y}(t)$ and that from $u(t)$ to $\ddot{y}(t)$, respectively. Their evaluation is easy for a linear S as seen in the example. It is however difficult for a nonlinear S in general. It is even possible that S does not have any finite γ or β . This difficulty arises because the present framework allows the input $u(t)$ to

be infinitely large. In this sense, a local framework of passivity might be useful for extending the discussion here.

Kawakami–Fujioka [8] analyzed stability of a sampled-data Hamiltonian system by extending the scaled small-gain theorem, which is more general than the passivity theorem. This direction of the research should be fruitful for tight stability analysis.

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Appendix: Proof of Theorem 5

The idea of the proof is the same as in Theorems 1 and 3. We compare the three integrals in (1) with the corresponding summations in (2). To do this, the following relationship is useful:

$$\begin{aligned}
y(t) - y_d(k) &= y(t) - y(kh) - \frac{h}{2}\dot{y}(kh) \\
&= \frac{1}{h} \int_{kh}^{(k+1)h} \left[y(t) - y(kh) - (s - kh)\dot{y}(kh) \right] ds \\
&= \frac{1}{h} \int_{kh}^{(k+1)h} \left[\int_{kh}^t \dot{y}(r) dr - \int_{kh}^s \dot{y}(kh) dr \right] ds \\
&= \frac{1}{h} \int_{kh}^{(k+1)h} \left[\int_s^t \dot{y}(r) dr + \int_{kh}^s \dot{y}(r) - \dot{y}(kh) dr \right] ds \\
&= \frac{1}{h} \int_{kh}^{(k+1)h} \int_s^t \dot{y}(r) dr ds + \frac{1}{h} \int_{kh}^{(k+1)h} \int_{kh}^s \int_{kh}^r \ddot{y}(q) dq dr ds. \tag{9}
\end{aligned}$$

We compare the first integral in (1) with the corresponding summation in (2). Their discrepancy is written as

$$D'_1 := \left| \int_0^{Kh} u(t)^T y(t) dt - h \sum_{k=0}^{K-1} u_d(k)^T y_d(k) \right| = \left| \sum_{k=0}^{K-1} u_d(k)^T \int_{kh}^{(k+1)h} y(t) - y_d(k) dt \right|.$$

We substitute (9) into the last expression. Noting that the first term of (9) vanishes after the integration with t , we have

$$\begin{aligned}
D'_1 &= \left| \sum_{k=0}^{K-1} u_d(k)^T \int_{kh}^{(k+1)h} \frac{1}{h} \int_{kh}^{(k+1)h} \int_{kh}^s \int_{kh}^r \ddot{y}(q) dq dr ds dt \right| \\
&\leq \sum_{k=0}^{K-1} \|u_d(k)\| \int_{kh}^{(k+1)h} \int_{kh}^s \int_{kh}^r \|\ddot{y}(q)\| dq dr ds \\
&\leq \sum_{k=0}^{K-1} \|u_d(k)\| \frac{h^2}{2} \int_{kh}^{(k+1)h} \|\ddot{y}(q)\| dq
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{K-1} \|u_d(k)\| \frac{h^2}{2} \sqrt{h} \sqrt{\int_{kh}^{(k+1)h} \|\ddot{y}(q)\|^2 dq} \\
&\leq \frac{h^2}{2} \sqrt{h} \sqrt{\sum_{k=0}^{K-1} \|u_d(k)\|^2} \sqrt{\sum_{k=0}^{K-1} \int_{kh}^{(k+1)h} \|\ddot{y}(q)\|^2 dq}.
\end{aligned}$$

The second summation in the last expression is bounded by $\beta^2 \int_0^{Kh} \|u(t)\|^2 dt = \beta^2 h \sum_{k=0}^{K-1} \|u_d(k)\|^2$. We hence have

$$D'_1 \leq \frac{h^2 \beta}{2} \cdot h \sum_{k=0}^{K-1} \|u_d(k)\|^2.$$

The second integral in (1) is equal to the corresponding summation in (2). On the third integral, its discrepancy with the corresponding summation can be evaluated similarly to the first one. Specifically we have

$$\left| \int_0^{Kh} \|y(t)\|^2 dt - h \sum_{k=0}^{K-1} \|y_d(k)\|^2 \right| \leq \left(h^2 \gamma^2 + \frac{1}{2} h^2 \beta + \frac{1}{4} h^4 \beta^2 \right) h \sum_{k=0}^{K-1} \|u_d(k)\|^2 + \frac{1}{2} h^2 \beta \cdot h \sum_{k=0}^{K-1} \|y_d(k)\|^2.$$

Using the obtained relationships with (1) and arranging the terms, we have the desired inequality.

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