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Itsuro Kakiuchi and Miyoshi Kimura

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Characterization of a new neighborhood determined by three parameters

Itsuro Kakiuchi [1] and Miyoshi Kimura [2]
Kobe University and Nanzan University

Abstract

A new neighborhood of distributions to describe the departure of data from a model is introduced. The neighborhood is generated from a special capacity determined by three parameters and as special cases it includes the commonly used neighborhoods defined in terms of ε - contamination and total variation distance. Characterization theorems of the neighborhood are given and it is shown that the neighborhood is a certain combination of contamination and gap from the model. Various new neighborhoods are obtained from changing the values of the three parameters. One of the parameters expresses the size of contamination and the others determine the size of gap from the model. It turns out that the introduced neighborhood is intuitively understandable and useful for developing minimax theory in robust inference.

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1 Introduction

The theory of modern robust statistical inference was initiated by Huber (1964, 1965, 1968). In this theory the departure of data from an assumed model is usually expressed by some suitably chosen neighborhood of the model distribution. The various types of neighborhoods have been used to describe the departure to date (see Huber and Ronchetti, 2009). Among them, the neighborhoods defined in terms of ε - contamination and total variation distance have been most frequently adopted in the literatures. As a combination form of such two neighborhoods, Rieder (1977) introduced a neighborhood defined by a special capacity, which we call Rieder's neighborhood, and he used it in his works (1978, 1981a, 1981b). Ando and Kimura (2003) proposed the (c, γ) - neighborhood which is a generalization of Rieder's neighborhood. The basic properties of the (c, γ) - neighborhood and its applications to robust inference are seen in Ando and Kimura (2003, 2004) and Ando, Kakiuchi and Kimura (2009).

[1] Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe, Hyougo, 657-8501
[2] Department of Information Systems and Mathematical Sciences, Nanzan University, 27 Seirei-cho, Seto, Aichi, 489-0863

In this paper we introduce a new neighborhood generated from a special capacity determined by three parameters, which we call (c_1, c_2, γ) - neighborhood. This (c_1, c_2, γ) - neighborhood includes not only (c, γ) - neighborhood but also various new neighborhoods produced by changing the values of the three parameters. We present a list of such representative neighborhoods. In particular, we are especially interested in two neighborhoods, called the extended ε - contamination neighborhood and the inlier contamination neighborhood, respectively. The special capacity, which was comprehensively studied by Bednarski (1981), satisfies all the conditions of Choquet's 2 - alternating capacity except condition (4) in Huber and Strassen (1973). Since (c_1, c_2, γ) - neighborhood is generated from a special capacity, it has nice properties for developing minimax theory in robust inference. We give three characterizations (Theorems 3.1, 3.2, 3.3) of (c_1, c_2, γ) - neighborhood, which show that the neighborhood consists of all γ contamination of distributions in a certain neighborhood of the model distribution determined by c_1 and c_2. The characterization (Theorem 3.3) from the density function point of view, our main theorem, reveals that (c_1, c_2, γ) - neighborhood is useful and intuitively understandable. We also find the stochastically smallest and largest (improper) distributions for (c_1, c_2, γ) - neighborhood. These distributions play important roles in construction of robust procedures. For example, we can effectively apply them to constructi robust confidence intervals and tests for the median of an unknown distribution under (c_1, c_2, γ) - neighborhood.

2 The $(\mathbf{c_1}, \mathbf{c_2}, \boldsymbol{\gamma})$ - neighborhood

Let $\mathbb{X}$ be a Polish space (a complete, separable and metrizable space), $\mathscr{B}$ the Borel σ - algebra of subsets of $\mathbb{X}$ and $\mathscr{M}$ the set of all probability measures on $(\mathbb{X}, \mathscr{B})$. For some specified $F^\circ \in \mathscr{M}$ we propose the following neighborhood of F° defined as

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G \in \mathscr{M} \mid c_1 F^\circ\{A\} \le G\{A\} \le c_2 F^\circ\{A\} + \gamma, {}^\forall A \in \mathscr{B}\}, \tag{2.1}$$

where $0 \le c_1 \le 1 - \gamma \le c_2 < \infty$, $c_1 \ne c_2$ and $0 \le \gamma < 1$.

This neighborhood, which we call (c_1, c_2, γ) - neighborhood, is restricted by upper and lower probabilities and includes (c, γ) - neighborhood introduced by Ando and Kimura (2003), and hence as special cases it includes Rieder's neighborhood as well as the neighborhoods defined in terms of ε - contamination and total variation distance. We can obtain various other neighborhoods by changing c_1, c_2 and γ. The (c_1, c_2, γ) - neighborhood is also expressed in the following form.

Lemma 2.1 *It holds that*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G \in \mathscr{M} \mid G\{A\} \le \min(c_2 F^\circ\{A\} + \gamma,\ c_1 F^\circ\{A\} + 1 - c_1), {}^\forall A \in \mathscr{B}\}. \tag{2.2}$$

Proof. The lemma easily follows from the fact that

$$c_1 F^\circ\{A\} \le G\{A\} \quad \text{for} \quad {}^\forall A \in \mathscr{B}$$

is equivalent to

$$G\{A\} \le c_1 F^\circ\{A\} + 1 - c_1 \quad \text{for} \quad {}^\forall A \in \mathscr{B}. \qquad \square$$

This lemma shows that the (c_1, c_2, γ) - neighborhood $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ is generated from the following special capacity. Let

$$h(t) = \min(c_2 t + \gamma,\ c_1 t + 1 - c_1), \quad 0 \le t \le 1, \tag{2.3}$$

where $0 \le c_1 \le 1-\gamma \le c_2 < \infty,\ c_1 \ne c_2$ and $0 \le \gamma < 1$, and let

$$v_h\{A\} = \begin{cases} h(F^\circ\{A\}), & \text{if } \ \phi \ne {}^\forall A \in \mathscr{B}, \\ 0, & \text{if } \ A = \phi. \end{cases} \tag{2.4}$$

Then, by Lemma 3.1 of Bednarski (1981) v_h is a special capacity, which satisfies all the conditions of Choquet's 2 - alternating capacity except condition (4) in Huber and Strassen (1973). By Lemma 2.1 we have

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G \in \mathscr{M} \mid G\{A\} \le v_h\{A\} \ \text{ for } \ {}^\forall A \in \mathscr{B}\}. \tag{2.5}$$

Thus the (c_1, c_2, γ) - neighborhood is generated from the special capacity v_h with (2.3). This fact means that it has nice properties for developing minimax theory in robust inference.

Remark 2.1. When $c_1 = c_2 = 1$ and $\gamma = 0$, we have $h(t) = t$ and $v_h = F^\circ$. This implies that a probability measure is a special capacity and $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = F^\circ$.

We present a list of representative neighborhoods and the corresponding h functions given by (2.3), which are obtained as special cases of the (c_1, c_2, γ) - neighborhood.

(i) ε - contamination neighborhood:

$$\begin{aligned} \mathscr{P}_{0,1-\varepsilon,\varepsilon}(F^\circ) &= \{G = (1-\varepsilon)F^\circ + \varepsilon K \in \mathscr{M} \mid K \in \mathscr{M}\} \\ &= \{G \in \mathscr{M} \mid G\{A\} \le (1-\varepsilon)F^\circ\{A\} + \varepsilon, {}^\forall A \in \mathscr{B}\}, \\ h(t) &= (1-\varepsilon)t + \varepsilon, \quad 0 \le \varepsilon < 1. \end{aligned}$$

We note that ε - contamination neighborhood is also obtained by $c_1 = 1-\varepsilon,\ \gamma = \varepsilon$ or $c_2 = 1-\varepsilon,\ \gamma = \varepsilon$.

(ii) Total variation neighborhood:

$$\begin{aligned} \mathscr{P}_{0,1,\delta}(F^\circ) &= \{G \in \mathscr{M} \mid \sup_A |G\{A\} - F^\circ\{A\}| \le \delta, {}^\forall A \in \mathscr{B}\} \\ &= \{G \in \mathscr{M} \mid G\{A\} \le F^\circ\{A\} + \delta, {}^\forall A \in \mathscr{B}\}, \\ h(t) &= \min(t + \delta,\ 1), \quad 0 \le \delta < 1. \end{aligned}$$

(iii) Rieder's neighborhood:

$$\begin{aligned} \mathscr{P}_{0,1-\varepsilon,\varepsilon+\delta}(F^\circ) &= \{G \in \mathscr{M} \mid G\{A\} \le (1-\varepsilon)F^\circ\{A\} + \varepsilon + \delta, {}^\forall A \in \mathscr{B}\}, \\ h(t) &= \min\{(1-\varepsilon)t + \varepsilon + \delta, 1\}, \quad 0 \le \varepsilon,\ 0 \le \delta,\ \varepsilon + \delta < 1. \end{aligned}$$

(iv) (c, γ) - neighborhood:

$$\begin{aligned} \mathscr{P}_{0,c,\gamma}(F^\circ) &= \{G \in \mathscr{M} \mid G\{A\} \le cF^\circ\{A\} + \gamma, {}^\forall A \in \mathscr{B}\}, \\ h(t) &= \min(ct + \gamma, 1), \quad 0 \le \gamma < 1,\ 1 - \gamma \le c < \infty. \end{aligned}$$

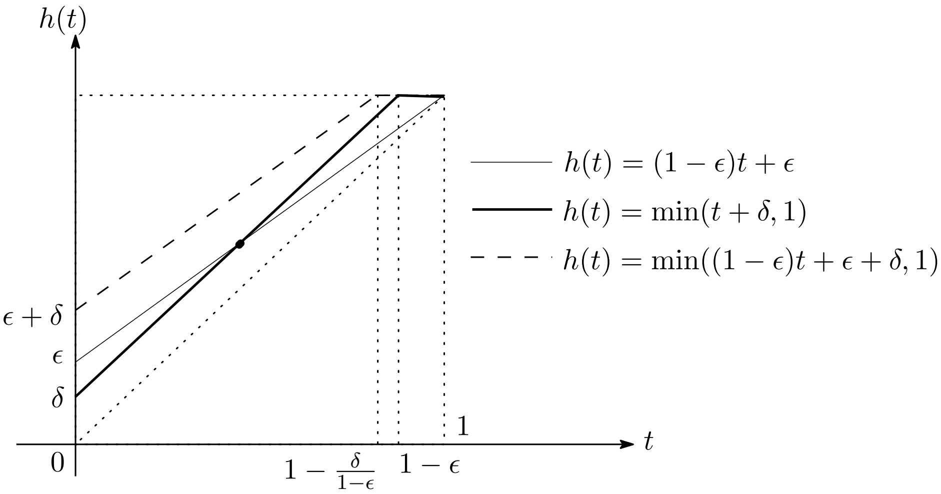

Figure 1: The graphs of h for the neighborhoods (i), (ii) and (iii) ($0 \leq \delta \leq \varepsilon$)

(v) Extended ε - contamination neighborhood:

$$\mathscr{P}_{1-\varepsilon,1,\delta}(F^\circ) = \{G \in \mathscr{M} \mid G\{A\} \leq \min\left(F^\circ\{A\} + \delta, (1-\varepsilon)F^\circ\{A\} + \varepsilon\right), {}^\forall A \in \mathscr{B}\},$$
$$h(t) = \min\{t+\delta, (1-\varepsilon)t + \varepsilon\}, \quad 0 < \delta < \varepsilon < 1.$$

(vi) Inlier contamination neighborhood:

$$\mathscr{P}_{c_1,c_2,0}(F^\circ) = \{G \in \mathscr{M} \mid G\{A\} \leq \min\left(c_2F^\circ\{A\}, c_1F^\circ\{A\} + 1 - c_1\right), {}^\forall A \in \mathscr{B}\},$$
$$h(t) = \min\left(c_2t, c_1t + 1 - c_1\right), \quad 0 \leq c_1 < 1 \leq c_2 < \infty.$$

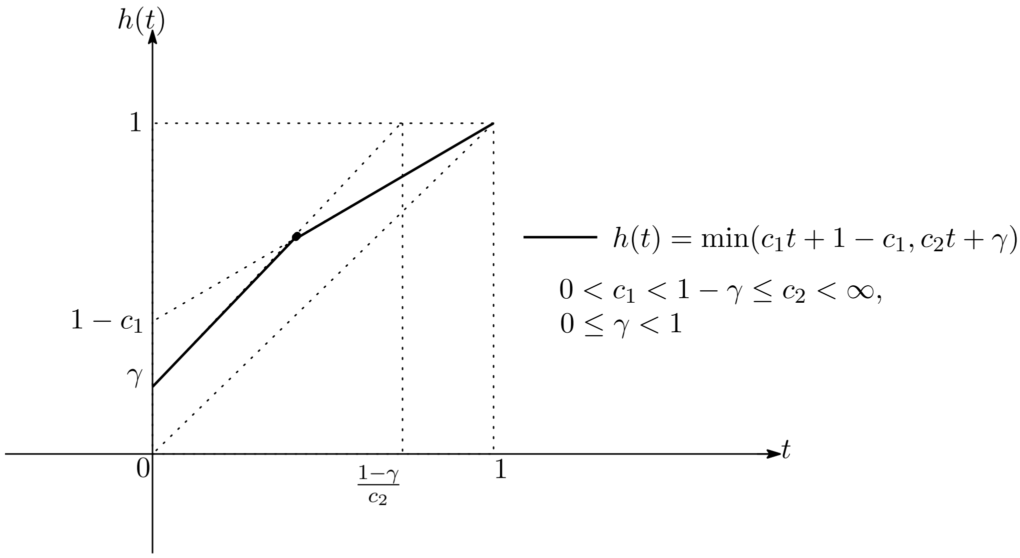

Figure 2: The graph of h for (c_1, c_2, γ) - neighborhood

Figures 1 and 2 are the graphs of the functions h corresponding to the neighborhoods (i), (ii), (iii) and (2.3). They show the features and diffferences of their neughborhoods. We should notice that the two line segments of h in Figure 2 connect at the inside of the square.

3 Characterization of $(\mathbf{c_1}, \mathbf{c_2}, \boldsymbol{\gamma})$ - neighborhood

First, from the measure theoretic point of view we give two characterizations of the (c_1, c_2, γ) - neighborhood.

Theorem 3.1 *For* $0 \le c_1 \le 1-\gamma \le c_2 < \infty,\ c_1 \ne c_2$ *and* $0 \le \gamma < 1$, *it holds that*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G = c_2(F^\circ - W) + \gamma K \in \mathscr{M} \mid\ W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ),\ K \in \mathscr{M}\}, \tag{3.1}$$

where $\mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$ *is the set of all measures* W *on* $(\mathbb{X}, \mathscr{B})$ *such that* $0 \le W\{A\} \le \{(c_2 - c_1)/c_2\}F^\circ\{A\}$ *for any* $A \in \mathscr{B}$ *and* $W\{\mathbb{X}\} = (c_2 - 1 + \gamma)/c_2$.

Proof. We first show that any element G of $\in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ in (2.1) is expressed in the form of (3.1). Since for any $A \in \mathscr{B}$

$$\max(c_1F^\circ\{A\}, 1-c_2-\gamma+c_2F^\circ\{A\}) \le G\{A\} \le \min(1-c_1+c_1F^\circ\{A\}, c_2F^\circ\{A\}+\gamma), \tag{3.2}$$

we obtain

$$\begin{aligned} F^\circ\{A\} - \frac{1}{c_2}G\{A\} &\le F^\circ\{A\} - \frac{1}{c_2}\max(c_1F^\circ\{A\}, 1-c_2-\gamma-c_2F^\circ\{A\}) \\ &= \min\left(\frac{c_2-c_1}{c_2}F^\circ\{A\},\ \frac{c_2+\gamma-1}{c_2}\right) \\ &= \begin{cases} \dfrac{c_2+\gamma-1}{c_2}, & \text{if } F^\circ\{A\} \ge \dfrac{c_2+\gamma-1}{c_2-c_1}, \\ \dfrac{c_2-c_1}{c_2}F^\circ\{A\}, & \text{if } F^\circ\{A\} < \dfrac{c_2+\gamma-1}{c_2-c_1}. \end{cases} \end{aligned} \tag{3.3}$$

Let f° and g be the density functions of F° and G with respect to a σ - finite measure μ (e.g. $\mu = F^\circ + G$), respectively, and let

$$A = \{x \in \mathbb{X} \mid c_1f^\circ(x) \le g(x) \le c_2f^\circ(x)\} \quad (A^c = \{x \in \mathbb{X} \mid g(x) > c_2f^\circ(x)\}).$$

Then it follows from (3.3) that if $F^\circ\{A\} \ge (c_2+\gamma-1)/(c_2-c_1)$, then

$$0 \le F^\circ\{A\} - \frac{1}{c_2}G\{A\} \le \frac{c_2+\gamma-1}{c_2} \le \frac{c_2-c_1}{c_2}F^\circ\{A\}$$

and if $F^\circ\{A\} < (c_2+\gamma-1)/(c_2-c_1)$, then

$$0 \le \frac{c_2+\gamma-1}{c_2} - \frac{c_2-c_1}{c_2}F^\circ\{A\} = \frac{c_2F^\circ(A^c) - \{1-\gamma-c_1F^\circ\{A\}\}}{c_2} \le \frac{c_2-c_1}{c_2}F^\circ\{A^c\}.$$

Therefore, there exist two functions $\phi_1(x)$ and $\phi_2(x)$ defined on A and A^c, respectively, such that

$$0 \le f^\circ(x) - \frac{1}{c_2}g(x) \le \phi_1(x) \le \frac{c_2-c_1}{c_2}f^\circ(x), \quad {}^\forall x \in A$$

and

$$0 \le \phi_2(x) \le \frac{c_2-c_1}{c_2}f^\circ(x), \quad {}^\forall x \in A^c,$$

and that if $F^\circ\{A\} \ge (c_2+\gamma-1)/(c_2-c_1)$, then

$$\int_A \phi_1(x)dx = \frac{c_2+\gamma-1}{c_2}, \quad \phi_2(x) \equiv 0, \ {}^\forall x \in A^c,$$

and if $F^\circ\{A\} < (c_2+\gamma-1)/(c_2-c_1)$, then

$$\phi_1(x) = \frac{c_2-c_1}{c_2} f^\circ(x), \ {}^\forall x \in A, \quad \int_{A^c} \phi_2(x)dx = \frac{c_2+\gamma-1}{c_2} - \frac{c_2-c_1}{c_2} F^\circ\{A\}.$$

Using these ϕ_1 and ϕ_2, we define a function ϕ on $\mathbb{X}$ such that

$$\phi(x) = \begin{cases} \phi_1(x), & \text{if } \ x \in A \\ \phi_2(x), & \text{if } \ x \in A^c. \end{cases}$$

Then we have

$$0 \le \phi(x) \le \frac{c_2-c_1}{c_2} f^\circ(x), \quad {}^\forall x \in \mathbb{X}$$

and

$$\int_{\mathbb{X}} \phi(x)\, d = \int_A \phi_1(x)\, dx + \int_{A^c} \phi_2(x)\, dx = \frac{c_2+\gamma-1}{c_2}.$$

Denoting

$$W\{B\} = \int_B \phi(x)\, dx, \quad {}^\forall B \in \mathscr{B}, \tag{3.4}$$

W is a measure with the density ϕ such that $W\{\mathbb{X}\} = (c_2+\gamma-1)/c_2$, which implies $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$. It follows from the definition of A that for ${}^\forall B \in \mathscr{B}$

$$\begin{aligned} & W\{B\} = W\{A\cap B\} + W\{A^c \cap B\} \\ & \ge F^\circ\{A\cap B\} - \frac{1}{c_2} G\{B\} + F^\circ\{A^c \cap B\} \\ & \ge F^\circ\{A\cap B\} - \frac{1}{c_2} F^\circ\{A\cap B\} \\ & \ge F^\circ\{B\} - \frac{1}{c_2} G\{B\}. \end{aligned}$$

When $\gamma \ne 0$, let

$$K\{B\} = \frac{1}{\gamma}\left\{G\{B\} - c_2(F^\circ\{B\} - W\{B\})\right\}, \quad {}^\forall B \in \mathscr{B}.$$

Then, it is clear that K is a probability measure on $(\mathbb{X}, \mathscr{B})$, and we have

$$G\{B\} = c_2(F^\circ\{B\} - W\{B\}) + \gamma K\{B\}, \quad {}^\forall B \in \mathscr{B}. \tag{3.5}$$

When $\gamma = 0$, any $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ in (2.1) has a density g and $G\{A\} = G\{\mathbb{X}\} = 1$. Let

$$\phi(x) = f^\circ(x) - \frac{1}{c_2} g(x), \quad {}^\forall x \in \mathbb{X}.$$

Then, using W given by (3.4) with this ϕ, we have

$$G\{B\} = c_2(F^\circ\{B\} - W\{B\}), \quad {}^\forall B \in \mathscr{B}. \tag{3.6}$$

Thus, the equations (3.5) and (3.6) imply that G belongs to $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ in (3.1).

Conversely, let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ in (3.1). Then, for any $B \in \mathscr{B}$ there exist $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$ and $K \in \mathscr{M}$ such that $G\{B\} = c_2(F^\circ\{B\} - W\{B\}) + \gamma K\{B\}$. Since $0 \le W\{B\} \le \{(c_2 - c_1)/c_2\}F^\circ\{B\}$, we have

$$c_2 F^\circ\{B\} + \gamma \ge G\{B\} \ge c_2 \left(F^\circ\{B\} - \frac{c_2 - c_1}{c_2} F^\circ\{B\} \right) = c_1 F^\circ\{B\}.$$

This implies that G belongs to $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ in (2.1). $\square$

Theorem 3.2 *For $0 < c_1 \le 1 - \gamma \le c_2 < \infty$, $c_1 \ne c_2$ and $0 \le \gamma < 1$, it holds that*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G = c_1(F^\circ + V) + \gamma K \in \mathscr{M} \mid V \in \mathscr{V}_{c_1,c_2,\gamma}(F^\circ),\ K \in \mathscr{M}\}, \tag{3.7}$$

where $\mathscr{V}_{c_1,c_2,\gamma}(F^\circ)$ is the set of all measures V on $(\mathbb{X}, \mathscr{B})$ such that $0 \le V\{A\} \le \{(c_2 - c_1)/c_1\}F^\circ\{A\}$ for ${}^\forall A \in \mathscr{B}$ and $V\{\mathbb{X}\} = (1 - \gamma - c_1)/c_1$.

Proof. We show that $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given in (3.7) is equal to that in (3.1) whenever $c_1 \ne 0$. Let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given in (3.1). Then we have $G = c_2(F^\circ - W) + \gamma K$. Hence,

$$G = c_1 \left(F^\circ + \frac{c_2 - c_1}{c_1} F^\circ - \frac{c_2}{c_1} W \right) + \gamma K.$$

Here, let $V = \{(c_2 - c_1)/c_1\}F^\circ - (c_2/c_1)W$. Then we have $G = c_1(F^\circ + V) + \gamma K$. Since $0 \le W\{A\} \le \{(c_2 - c_1)/c_2\}F^\circ\{A\}$ for ${}^\forall A \in \mathscr{B}$, it follows that $0 \le V(A) \le \{(c_2 - c_1)/c_1\}F^\circ\{A\}$ for ${}^\forall A \in \mathscr{B}$. It is obvious that V is a measure on $(\mathbb{X}, \mathscr{B})$ with $V\{\mathbb{X}\} = (1 - \gamma - c_1)/c_1$. Thus $V \in \mathscr{V}_{c_1,c_2,\gamma}(F^\circ)$, and hence $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given in (3.7).

Conversely, let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given in (3.7). Then, $G = c_1(F^\circ + V) + \gamma K$ is written as

$$G = c_2 \left(F^\circ - \frac{c_2 - c_1}{c_2} F^\circ + \frac{c_1}{c_2} V \right) + \gamma K.$$

Letting $W = \{(c_2 - c_1)/c_2\}F^\circ - (c_1/c_2)V$, we have $G = c_2(F^\circ - W) + \gamma K)$. From $V \in \mathscr{V}_{c_1,c_2,\gamma}(F^\circ)$, it is easy to see $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$, which implies $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given in (3.7). $\square$

Hereafter we consider the case of $\mathbb{X} = \mathbb{R}$, the real line. In this case we can express the (c_1, c_2, γ) - neighborhood $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ in the intuitively more understandable form by using density functions. Let F° be an absolutely continuous distribution function on $\mathbb{R}$ and let f° be a density function of F° (with respect to the Lebesgue measure). Also, let $\mathscr{M}_c (\subset \mathscr{M})$ be the set of all absolutely continuous distributions on $(\mathbb{R}, \mathscr{B})$.

Theorem 3.3 *For* $0 \le c_1 \le 1-\gamma \le c_2 < \infty$, $c_1 \ne c_2$ *and* $0 \le \gamma < 1$, *it holds that*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G = (1-\gamma)F + \gamma K \in \mathscr{M} \mid F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ),\ K \in \mathscr{M}\}, \tag{3.8}$$

where

$$\mathscr{F}_{c_1,c_2,\gamma}(F^\circ) = \{F \in \mathscr{M}_c \mid \frac{c_1}{1-\gamma} f^\circ \le f \le \frac{c_2}{1-\gamma} f^\circ\} \tag{3.9}$$

and f *is a density function of* F.

Proof. Let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given by (3.1). Then there exist $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$ and $K \in \mathscr{M}$ such that $G = c_2(F^\circ - W) + \gamma K$. Hence we have

$$G = (1-\gamma)\left\{\frac{c_2}{1-\gamma}(F^\circ - W)\right\} + \gamma K \equiv (1-\gamma)F + \gamma K,$$

where $F = \{c_2/(1-\gamma)\}(F^\circ - W)$. We note that

$$F\{R\} = \frac{c_2}{1-\gamma}\left(F^\circ\{R\} - W\{R\}\right) = \frac{c_2}{1-\gamma}\left(1 - \frac{c_2+\gamma-1}{c_2}\right) = 1.$$

Since F° and W are absolutely continuous, F is an absolutely continuous distribution. Also, since $0 \le W\{B\} \le \{(c_2-c_1)/c_2\}F^\circ\{B\}$ for any $B \in \mathscr{B}$, we have

$$\frac{c_2}{1-\gamma}F^\circ\{B\} \ge F\{B\} \ge \frac{c_2}{1-\gamma}\left(F^\circ\{B\} - \frac{c_2-c_1}{c_2}F^\circ\{B\}\right) = \frac{c_1}{1-\gamma}F^\circ\{B\}.$$

Therefore we obtain $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$, which implies $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given in (3.8).

Conversely, let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given in (3.8). Then there exists $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ such that $G = (1-\gamma)F + \gamma K$. Letting

$$W = F^\circ - \frac{1-\gamma}{c_2}F,$$

it is easily seen that $G = c_2(F^\circ - W) + \gamma K$ and $W \in \mathscr{W}_{c_1,c_2,\gamma}(F^\circ)$. This implies $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ given in (3.1). □

4 The stochastically smallest and largest distributions

In order to find stochastically smallest and largest distributions in (c_1, c_2, γ)-neighborhood, we first consider those in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$. Let F_L and F_R be the distributions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ defined by

$$F_L(x) = \begin{cases} \dfrac{c_2}{1-\gamma}F^\circ(x), & \text{if } x \le x_L, \\[2ex] \dfrac{c_1}{1-\gamma}F^\circ(x) + \left(1 - \dfrac{c_1}{1-\gamma}\right), & \text{if } x > x_L, \end{cases} \tag{4.1}$$

and

$$F_R(x) = \begin{cases} \dfrac{c_1}{1-\gamma} F^\circ(x), & \text{if } x \le x_R, \\ \dfrac{c_2}{1-\gamma} F^\circ(x) + \left(1 - \dfrac{c_2}{1-\gamma}\right), & \text{if } x > x_R, \end{cases} \tag{4.2}$$

where

$$x_L = (F^\circ)^{-1}\left(\frac{1-\gamma-c_1}{c_2-c_1}\right) \tag{4.3}$$

and

$$x_R = (F^\circ)^{-1}\left(\frac{c_2+\gamma-1}{c_2-c_1}\right), \tag{4.4}$$

respectively. Then we have the following lemma.

Lemma 4.1 *$F_L(x)$ and $F_R(x)$ given by (4.1) and (4.2) are stochastically smallest and largest distribution functions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$, respectively. That is, for any $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ it holds that*

$$F_R(x) \le F(x) \le F_L(x), \ {}^\forall x \in \mathbb{R}$$

Proof. Let

$$f_0(x) = \begin{cases} \dfrac{c_2}{1-\gamma} f^\circ, & \text{if } x \le x_0, \\ \dfrac{c_1}{1-\gamma} f^\circ, & \text{if } x > x_0, \end{cases}$$

and

$$f_1(x) = \begin{cases} \dfrac{c_1}{1-\gamma} f^\circ, & \text{if } x \le x_1, \\ \dfrac{c_2}{1-\gamma} f^\circ, & \text{if } x > x_1. \end{cases}$$

Then it follows from Theorem 3.3 that $f_0(x)$ and $f_1(x)$ are stochastically smallest and largest density functions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$, respectively, where x_0 and x_1 are given by those density conditions, that is,

$$1 = \int_{-\infty}^{\infty} f_0(x)dx = \frac{c_2}{1-\gamma}F^\circ(x_0) + \frac{c_1}{1-\gamma}(1-F^\circ(x_0)) = \frac{c_1}{1-\gamma} + \frac{c_2-c_1}{1-\gamma}F^\circ(x_0)$$

and

$$1 = \int_{-\infty}^{\infty} f_1(x)dx = \frac{c_1}{1-\gamma}F^\circ(x_1) + \frac{c_2}{1-\gamma}(1-F^\circ(x_1)) = \frac{c_2}{1-\gamma} + \frac{c_1-c_2}{1-\gamma}F^\circ(x_1).$$

Then we have

$$x_0 = (F^\circ)^{-1}\left(\frac{1-\gamma-c_1}{c_2-c_1}\right) \quad \text{and} \quad x_1 = (F^\circ)^{-1}\left(\frac{c_2+\gamma-1}{c_2-c_1}\right).$$

By the definitions (4.1) and (4.2) we obtain that the distribution functions of f_0 and f_1 are F_L and F_R, respectively, which completes the proof of the lemma. □

Theorem 4.1 *For any $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ it holds that*

$$(1-\gamma)F_R(x) \le G(x) \le (1-\gamma)F_L(x) + \gamma, \ {}^\forall x \in \mathbb{R}$$

Proof. Let G be any element of $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$. Then, by Theorem 3.3 there exist $F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ and $K \in \mathscr{M}$ such that $G = (1-\gamma)F + \gamma K$. By Lemma 4.1 we have

$$(1-\gamma)F_R(x) + \gamma K(x) \le G(x) \le (1-\gamma)F_L(x) + \gamma K(x), \ {}^\forall x \in \mathbb{R}.$$

Noting that K is a distribution function, we obtain

$$(1-\gamma)F_R(x) \le G(x) \le (1-\gamma)F_L(x) + \gamma, \ {}^\forall x \in \mathbb{R}. \quad \square$$

Remark 4.1 The stochastically smallest and largest distributions F_L and F_R given by (4.1) and (4.2) in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ are very useful to construct robust procedures. As one of their applications, the authors are studying the problem of constructing robust nonparametric confidence intervals and tests for the median of an unknown distribution F°under the (c_1, c_2, γ) - neighborhood. They have obtained some results which include those in Yohai and Zamar (2004) and Ando, Kakiuchi and Kimura (2009).

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