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Abstract. Here, we give a finite method to list all exact models for the set of formulas, which have finite modal degree n and contain only propositional variables $p_1, \cdots, p_m$ in modal logic **K4**. The method is obtained by modifying the method in [Sas10a] for modal logic **S4**.

1 Introduction

In the present section, we introduce formulas, the modal logic **K4**, and exact models; and also describe the motivation of the present paper.

Formulas are constructed from $\bot$ (contradiction) and the propositional variables $p_1, p_2, \cdots$ by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and $\Box$ (necessitation). We use upper case Latin letters, $A, B, C, \cdots$, with or without subscripts, for formulas. Also, we use Greek letters, $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. The *modal degree* $d(A)$ of a formula A is defined as

$d(p_i) = d(\bot) = 0,$
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\},$
$d(\Box B) = d(B) + 1.$

The set of propositional variables $p_1, \cdots, p_m$ ($m \geq 1$) is denoted by $\mathbf{V}$ and the set of formulas constructed from $\mathbf{V}$ and $\bot$ is denoted by $\mathbf{F}$. Also, for any $n = 0, 1, \cdots$, we define $\mathbf{F}(n)$ as $\mathbf{F}(n) = \{A \in \mathbf{F} \mid d(A) \leq n\}$. In the present paper, we mainly treat the set $\mathbf{F}(n)$.

We define the modal logic **K4** as a sequent system obtained by adding an inference rule to the sequent system **LK** introduced in Gentzen [Gen35] for the classical propositional logic.

A *sequent* is the expression $(\Gamma \to \Delta)$. We often refer to $\Gamma \to \Delta$ as $(\Gamma \to \Delta)$ for brevity and refer to

$$A_1, \cdots, A_i, \Gamma_1, \cdots, \Gamma_j \to \Delta_1, \cdots, \Delta_k, B_1, \cdots, B_\ell$$

as

$$\{A_1, \cdots, A_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \cdots, B_\ell\}.$$

We use upper case Latin letters $X, Y, Z, \cdots$, with or without subscripts, for sequents. If $X = (\Gamma \to \Delta)$, then we sometimes refer to $\Gamma \overset{X}{\to} \Delta$ as $\Gamma \to \Delta$. The *antecedent* $\mathbf{ant}(\Gamma \to \Delta)$ and the *succedent* $\mathbf{suc}(\Gamma \to \Delta)$ of a sequent $\Gamma \to \Delta$ are defined as

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \to \Delta) = \Delta,$$

respectively. Also, for a sequent X and a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) \neq \emptyset \\ \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) = \emptyset \end{cases}$$

and

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

For a finite set $\mathcal{S}$ of sequents, the expression $\#(S)$ denotes the number of elements in $\mathcal{S}$.

By **K4**, we mean the system obtained by adding the inference rule

$$\frac{\Gamma, \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\Box).$$

to **LK**. Here, we do not use $\neg$ as a primary connective, so we use the additional axiom $\bot \to$ instead of the inference rules $(\neg \to)$ and $(\to \neg)$. We write $X \in \mathbf{K4}$ if X is provable in **K4**.

We use $A \equiv B$ instead of $\to (A \supset B) \wedge (B \supset A) \in \mathbf{K4}$. Also, for any two equivalence classes $[A]$ and $[B]$ in $\mathbf{F}/\equiv$, we use $[A] \leq [B]$ instead of $A \to B \in \mathbf{K4}$. Thus, the structure $\langle \mathbf{F}(n)/\equiv, \leq \rangle$ expresses the mutual relation of formulas. Exact models are useful to clarify this structure.

We introduce an exact model, which is a kind of Kripke models.

A *Kripke model* is a structure $\langle W, R, P \rangle$ where W is a non-empty set, R is a binary relation on W, and P is a mapping from the set of propositional variables to 2^W. We extend, as usual, the domain of P to include all formulas. We call P a *valuation* and a member of W a *world*. For a Kripke model $M = \langle W, R, P \rangle$, and for a world $\alpha \in W$, we often write $(M, \alpha) \models A$ and $M \models A$ instead of $\alpha \in P(A)$ and $P(A) = W$, respectively. The following lemma is described in several articles (for example in Chagrov and Zakharyaschev [CZ97]):

Lemma 1.1 *$\to A \in \mathbf{K4}$ if and only if $M \models A$ for any transitive Kripke models.*

Let S be a set of formulas closed under $\supset$ and $\wedge$. We say that a Kripke model $M = \langle W, R, P \rangle$ is *exact for S* if the following two conditions hold:

- for any $A \in S$, $M \models A$ if and only if $\to A \in \mathbf{K4}$,
- $\{P(A) \mid A \in S\} = 2^W$.

This model was introduced in de Bruijn [Bru75].

The purpose of the present paper is to give a finite method to list all exact models for **K4**. Because exact models are useful to clarify the structure $\langle \mathbf{F}(n)/\equiv, \leq \rangle$, which expresses the mutual relation of formulas. Specifically, the following lemma holds.

Lemma 1.2 *Let $\langle W, R, P \rangle$ be an exact model for $\mathbf{F}(n)$.*

(1) *The mapping P^* from $\mathbf{F}(n)/\equiv$ to 2^W defined as*

$$P^*([A]) = P(A)$$

is an isomorphism and the structure $\langle \mathbf{F}(n)/\equiv, \leq \rangle$ is isomorphic to the structure $\langle 2^W, \subseteq \rangle$.

(2) *If a mapping f from W to $\mathbf{F}(n)$ satisfies*

$$P(f(\alpha)) = W - \{\alpha\},$$

then

- $\mathbf{F}(n)/\equiv = \{ \bigwedge_{\alpha \in W'} f(\alpha) \mid W' \in W \}$,
- *for any subsets W_1 and W_2 of W, $W_1 \subseteq W_2$ if and only if $\bigwedge_{\alpha \in W_2} f(\alpha) \to \bigwedge_{\alpha \in W_1} f(\alpha) \in \mathbf{K4}$,*
- *for any $A \in \mathbf{F}(n)$, $A \equiv \bigwedge_{(M,\alpha) \not\models A} f(\alpha)$.*

For the earlier works on this topic, we can refer to [Sas10a]. Also, [Sas10a] give a method to list all exact models in the modal logic **S4**. [Sas10a] defined the set $\mathbf{ED_{S4}}(n)$ of sequents, whose members behave like an elementary disjunction

$$p_1^* \vee \cdots \vee p_m^* \qquad (p_i^* \in \{p_i, p_i \supset \bot\}).$$

In other words, $\mathbf{ED_{S4}}(n)$ satisfies

- $\mathbf{F}(n)/\equiv_{\mathbf{S4}}=\{[\bigwedge \mathbf{for}(\mathcal{S})] \mid \mathcal{S} \subseteq \mathbf{ED}_{\mathbf{S4}}(n)\}$,
- for any subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}_{\mathbf{S4}}(n)$, $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}$.

[Sas10a] used it to construct exact models. By modifying this construction, [Sas10b] defined the set $\mathbf{ED}_L(n)$ for a normal modal logic L containing $\mathbf{K4}$ and proved the corresponding properties to the above two. Here, we construct exact models using $\mathbf{ED}_{\mathbf{K4}}(n)$ in [Sas10b]. In the next section, we introduce $\mathbf{ED}_{\mathbf{K4}}(n)$ in [Sas10b]. In section 3, we give a finite method to list all exact models for $\mathbf{F}(n)$ in $\mathbf{K4}$.

2 Constructions of $\mathrm{ED}_L(n)$

In [Sas10b], a construction of the set $\mathbf{ED}_L(n)$ for a normal modal logic L containing $\mathbf{K4}$ was given. Also, another construction of the set $\mathbf{ED}_{\mathbf{K4}}(n)$ was given. The former depends on L-provability, but latter doesn't. Here, we introduce these two constructions in the case that $L = \mathbf{K4}$. Therefore, we omit the subscript L and $\mathbf{K4}$ from $\mathbf{ED}$ and the other notations in [Sas10b].

We introduce the first construction as follows.

Definition 2.1 The sets $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ of sequents are defined inductively as follows.
$\mathbf{G}(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,
$\mathbf{G}^*(0) = \emptyset$,
$\mathbf{G}(k+1) = \bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}(X)$,
$\mathbf{G}^*(k+1) = \{X \in \mathbf{G}(k+1) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y))^\Box$ implies $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, for any $Y \in \mathbf{G}(k+1)\}$,
where for any $X \in \mathbf{G}(k)$,
$\mathbf{next}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}(k)), \Gamma \cap \Delta = \emptyset\}$,
$\mathbf{prov}(X) = \{Y \in \mathbf{next}^+(X) \mid Y \in L\}$,
$\mathbf{next}(X) = \mathbf{next}^+(X) - \mathbf{prov}(X)$.

Definition 2.2 We define the set $\mathbf{ED}(n)$ as

$$\mathbf{ED}(n) = \mathbf{G}(n) \cup \bigcup_{i=0}^{n-1} \mathbf{G}^*(i).$$

Lemma 2.3
(1) $\mathbf{F}(n)/\equiv = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{ED}(n)\}$.
(2) *For subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathbf{ED}(n)$,

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \textit{ if and only if } \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in L.$$

We introduce the second construction as follows.

Definition 2.4 We define the set $\mathbf{G}^+(n)$ as

$$\mathbf{G}^+(n) = \begin{cases} \mathbf{G}(0) & \text{if } n = 0 \\ \bigcup_{X \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)} \mathbf{next}^+(X) & \text{if } n > 0, \end{cases}$$

Let X be a sequent in $\mathbf{G}^+(n+1)$. Then there exists only one sequent $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$ such that $X \in \mathbf{next}^+(Y)$. We refer to $X_\ominus$ as this sequent Y. We note that $X_\ominus \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and $X \in \mathbf{next}(X_\ominus)$.

Definition 2.5 We define the sets $\mathbf{G}^\circ(n)$ and $\mathbf{G}^\bullet(n)$ as

$$\mathbf{G}^\circ(n) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{X \in \mathbf{G}^*(n) \mid \Box\mathbf{for}(X_\ominus) \in \mathbf{suc}(X)\} & \text{if } n > 0, \end{cases}$$

$$\mathbf{G}^\bullet(n) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{X \in \mathbf{G}^*(n) \mid \Box\mathbf{for}(X_\ominus) \in \mathbf{ant}(X)\} & \text{if } n > 0. \end{cases}$$

Definition 2.6 Let X, $Y_\oplus$, and Z be sequents in $\mathbf{G}(n)$, $\mathbf{G}(n+1)$, and $\mathbf{G}^+(n)$, respectively. We define two sets $\mathbf{next}(X, Y_\oplus)$ and $\mathbf{pclus}(Z)$ as

$$\mathbf{next}(X, Y_\oplus) = \{X_\oplus \in \mathbf{next}(X) \mid (\mathbf{ant}(Y_\oplus))^\Box \subseteq (\mathbf{ant}(X_\oplus))^\Box\},$$

$$\mathbf{pclus}(Z) = \{Y \in \mathbf{G}^+(n) \mid (\mathbf{ant}(Z))^\Box = (\mathbf{ant}(Y))^\Box\}.$$

Definition 2.7 For any $X \in \mathbf{G}(0)$, we define $\mathbf{pr}_0(X)$, $\mathbf{pr}_1(X)$, $\mathbf{pr}_2(X)$, $\mathbf{pr}_3(X)$ and $\mathbf{pr}_4(X)$ as

$$\mathbf{pr}_0(X) = \mathbf{pr}_1(X) = \mathbf{pr}_2(X) = \mathbf{pr}_3(X) = \mathbf{pr}_4(X) = \emptyset.$$

For any $X \in \mathbf{G}(n+1)$, we define $\mathbf{pr}_0(X)$, $\mathbf{pr}_1(X)$, $\mathbf{pr}_2(X)$, $\mathbf{pr}_3(X)$ and $\mathbf{pr}_4(X)$ as follows:

$\mathbf{pr}_0(X) = \{(\Box\mathbf{for}(Y_\ominus), \Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n)\}$,

$\mathbf{pr}_1(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box\}$,

$\mathbf{pr}_2(X) = \{(\Box\mathbf{for}(\mathbf{next}(Z', X)), \Gamma \to \Delta, \Box\mathbf{for}(Z')) \in \mathbf{next}^+(X) \mid Z' \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)\}$,

$\mathbf{pr}_3(X) = \{(\Box\mathbf{for}(\mathbf{next}(Z', Y)), \Gamma \to \Delta, \Box\mathbf{for}(\Gamma_1 \xrightarrow{Y} \Delta_1, \Box\mathbf{for}(Z'))) \in \mathbf{next}^+(X)$
$\quad \mid Y \in \mathbf{G}(n), Z' \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)\}$,

$\mathbf{pr}_4(X) = \{(\Box\mathbf{for}(Y), \Gamma \to \Delta, \Box\mathbf{for}(Z)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}^\circ(n), Z \in \mathbf{pclus}(Y)\}$.

Lemma 2.8 *For any* $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\mathbf{prov}(X) = \mathbf{pr}_0(X) \cup \mathbf{pr}_1(X) \cup \mathbf{pr}_2(X) \cup \mathbf{pr}_3(X) \cup \mathbf{pr}_4(X).$$

By the above lemma, we obtain a construction of $\mathbf{ED}(n)$, which does not depend on $\mathbf{K4}$-provability.

In [Sas10b], there are also useful notations and lemmas for our investigations. Below, we show such notations and lemmas.

Definition 2.9
(1) We define the set $\mathbf{BG}(n)$ as

$$\mathbf{BG}(n) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i)).$$

(2) For any $X \in \mathbf{G}^+(n)$ and for any k, we define $X(k)$ as

$$X(k) = (\mathbf{ant}(X) \cap \mathbf{BG}(k) \to \mathbf{suc}(X) \cap \mathbf{BG}(k)).$$

Here, we note that $X(k) = X$ if $k \geq n$; and that $X(n-1) = X_\ominus$ if $n \neq 0$. Also, we note the following.

Remark 2.10
(1) *None of the members in* $\mathbf{G}(n)$ *is provable in* $\mathbf{K4}$.
(2) *For any* $X, Y \in \mathbf{ED}(n)$, $X \neq Y$ *implies* $\mathbf{for}(X) \vee \mathbf{for}(Y) \in \mathbf{K4}$.
(3) *For any* $X \in \mathbf{G}^+(n)$, $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{BG}(n)$ *and* $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$.

Lemma 2.11 *Let* X *and* Y *be sequents in* $\mathbf{G}(n)$ *satisfying* $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. *Then*
(1) $X \in \mathbf{G}^*(n)$ *if and only if* $Y \in \mathbf{G}^*(n)$,
(2) $Y \in \mathbf{G}^\circ(n)$ *implies* $\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in L$.

Definition 2.12 Let X be a sequent and let $\mathcal{S}$ be a subset of $\mathbf{G}(n)$. We define the sequent $\mathbf{n}(X, \mathcal{S})$ as

$$\mathbf{n}(X, \mathcal{S}) = (\Box\mathbf{for}(\mathbf{G}(n) - \mathcal{S}), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\mathcal{S})).$$

We note that $\mathbf{n}(X, \mathcal{S}) \in \mathbf{next}^+(X)$ if $X \in \mathbf{G}(n)$.

Lemma 2.13 *Let X and Y be sequents in $\mathbf{G}(n) - \mathbf{G}^*(n)$ and let $X_\oplus$ be a sequent in $\mathbf{next}(X)$. If $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_\oplus)$, then*

$$\mathbf{n}(Y, \mathcal{S}) \in \mathbf{next}(Y),$$

where

$$\mathcal{S} = \begin{cases} \{Z \in \mathbf{G}(n) \mid \Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus)\} & \text{if } n = 0 \\ \{Z \in \mathbf{G}(n) \mid \Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus), \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(Y), (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box\} & \text{if } n > 0. \end{cases}$$

Definition 2.14 For any $X \in \mathbf{G}(n)$, we define the sets $X\Downarrow$ inductively as follows:

(1) $X \in X\Downarrow$,

(2) if $Z \in X\Downarrow - \bigcup_{i=1}^{\infty} \mathbf{G}^*(i)$, then $Z_\oplus \in X\Downarrow$ for any $Z_\oplus \in \mathbf{next}(Z)$.

Lemma 2.15 *Let X be a sequent in $\mathbf{G}(n)$. Then*

(1) $n > k$ implies $X(k) \in \mathbf{G}(k) - \mathbf{G}^*(k)$ and $X \in X(k)\Downarrow$,

(2) for any $Y \in \mathbf{G}(k)$, the following three conditions are equivalent:

(2.1) $\mathbf{ant}(Y) \subseteq \mathbf{ant}(X)$ and $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$,

(2.2) $n \geq k$ and $Y = X(k)$,

(2.3) $X \in Y\Downarrow$.

Lemma 2.16 *Let X and Y_k be sequents in $\mathbf{G}(n) - \mathbf{G}^*(n)$ and $\mathbf{G}(k)$, respectively, and $X_\oplus$ be sequents in $\mathbf{next}(X)$. If $n \geq k$ and $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_\oplus)$, then there exists a sequent $Y \in \mathbf{ED}(n)$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_\oplus)$ and $Y \in Y_k\Downarrow$.*

Also, by Lemma 2.8, we have the following lemma.

Lemma 2.17 *Let X be a sequent in $\mathbf{G}^+(n)$ and let Y be a subset of $\mathbf{pclus}(X)$. Then*

$$X \in \mathbf{G}(n) \text{ if and only if } Y \in \mathbf{G}(n).$$

Proof. If $n = 0$, then the lemma is clear from $\mathbf{G}^+(0) = \mathbf{G}(0)$. We assume that $n > 0$. It is observed easily that

$$X \in \mathbf{pr}_0(X_\ominus) \text{ if and only if } Y \in \mathbf{pr}_0(Y_\ominus),$$
$$X \in \mathbf{pr}_1(X_\ominus) \text{ if and only if } Y \in \mathbf{pr}_1(Y_\ominus),$$
$$X \in \mathbf{pr}_2(X_\ominus) \text{ if and only if } Y \in \mathbf{pr}_2(Y_\ominus),$$
$$X \in \mathbf{pr}_3(X_\ominus) \text{ if and only if } Y \in \mathbf{pr}_3(Y_\ominus),$$
$$X \in \mathbf{pr}_4(X_\ominus) \text{ if and only if } Y \in \mathbf{pr}_4(Y_\ominus).$$

Using Lemma 2.8, we obtain the lemma. ⊣

3 Exact models for $\mathbf{F}(n)$

In the present section, we give a finite method to list all exact models for $\mathbf{F}(n)$ in $\mathbf{K4}$.

First, we introduce the Kripke model $\mathbf{EM}$, an exact set $\mathcal{E}$ for $\mathbf{F}(n)$, and the Kripke model $\mathbf{EM}_\mathcal{E}$. Here, we use $\mathbf{EM}$ to investigate exact models for $\mathbf{F}(n)$. For an exact set $\mathcal{E}$ for $\mathbf{F}(n)$, the Kripke model $\mathbf{EM}_\mathcal{E}$ is shown to be exact for $\mathbf{F}(n)$.

Definition 3.1 The Kripke model $\mathbf{EM}$ is defined as

$$\mathbf{EM} = \langle W_\mathbf{E}, R_\mathbf{E}, P_\mathbf{E} \rangle,$$

where

$W_\mathbf{E} = \bigcup_{n=0}^{\infty} \mathbf{G}^*(n)$,

$R_\mathbf{E} = \{(X, Y) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ or both $X \in \mathbf{pclus}(Y)$ and $Y \in \bigcup_{n=0}^{\infty} \mathbf{G}^\circ(n)\}$, and

$P_\mathbf{E}(p_i) = \{X \mid p_i \in \mathbf{ant}(X)\}$.

Definition 3.2
(1) A set $\mathcal{E}$ is said to be exact for $\mathbf{F}(n)$ if the following three conditions hold:
(1.1) $\mathbf{ED}(n) \cap W_{\mathbf{E}} \subseteq \mathcal{E} \subseteq W_{\mathbf{E}}$,
(1.2) for any $X \in \mathbf{ED}(n)$, $\#(X\Downarrow \cap \mathcal{E}) = 1$,
(1.3) for any $X \in \mathcal{E}$ and for any $Y \in W_{\mathbf{E}}$, $XR_{\mathbf{E}}Y$ implies $Y \in \mathcal{E}$.

(2) For an exact set $\mathcal{E}$ for $\mathbf{F}(n)$, the Kripke model $\mathbf{EM}_{\mathcal{E}}$ is defined as

$$\mathbf{EM}_{\mathcal{E}} = \langle \mathcal{E}, R_{\mathcal{E}}, P_{\mathcal{E}} \rangle,$$

where $R_{\mathcal{E}} = R_{\mathbf{E}} \cap \mathcal{E}^2$ and $P_{\mathcal{E}}(p_i) = P_{\mathbf{E}}(p_i) \cap \mathcal{E}$.

Definition 3.3 For any $\ell \in \{0, 1, 2, \cdots\}$, we define the number $\kappa(\ell)$ as
- $\kappa(0) = n$,
- $\kappa(\ell + 1) = \kappa(\ell) + \#(\mathbf{G}(\kappa(\ell))) - 1$.

The main purpose in the present section is to prove the following theorem.

Theorem 3.4
(1) *For any exact set $\mathcal{E}$ for $\mathbf{F}(n)$, $\mathbf{EM}_{\mathcal{E}}$ is an exact model for $\mathbf{F}(n)$.*
(2) *For any exact model M for $\mathbf{F}(n)$, there exists an exact set $\mathcal{E}$ for $\mathbf{F}(n)$ such that M is isomorphic to $\mathbf{EM}_{\mathcal{E}}$.*
(3) *Every exact set for $\mathbf{F}(n)$ is a subset of* $\displaystyle\bigcup_{i=0}^{\kappa(\#(\mathbf{ED}(n)-W_{\mathbf{E}}))} \mathbf{G}^*(i)$,
(4) *Let $\mathcal{E}$ be an exact set for $\mathbf{F}(n)$. Then for any $A \in \mathbf{F}(n)$,*

$$A \equiv \bigwedge \{\mathbf{for}(X(n)) \mid X \in \mathcal{E}, (\mathbf{EM}_{\mathcal{E}}, X) \not\models A\}.$$

The above theorem has the following meaning. By (1), (2), and (3), we can list all exact models for $\mathbf{F}(n)$. By (4), for each exact model for $\mathbf{F}(n)$, we obtain a finite method to find a subset $\mathcal{S}$ of $\mathcal{E}$ such that $A \equiv \bigwedge \mathbf{for}(\mathcal{S})$ for a given formula $A \in \mathbf{F}(n)$.

To prove the theorem, we need some lemmas. (1) and (4) will be proved by Lemma 3.12. (2) will be proved by Lemma 3.19. (3) can be shown by the proof of Lemma 3.19(2).

Lemma 3.5 *Let X and Y_k be sequents in $\mathbf{G}(n)$ and $\mathbf{G}^*(k)$, respectively. If $n \geq k$ and $(\mathbf{ant}(X(k)))^{\Box} = (\mathbf{ant}(Y_k))^{\Box}$, then $n = k$ and $X \in \mathbf{G}^*(n)$.*

Proof. By Lemma 2.15 and Lemma 2.11(1), we have $X(k) \in \mathbf{G}^*(k)$. Hence, we obtain the lemma. ⊣

Lemma 3.6 *Let X, Y, and Z be sequent in $\mathbf{G}(n_1)$, $\mathbf{G}(n_2)$, and $\mathbf{G}(n_3)$, respectively. If $\Box\mathbf{for}(X) \in \mathbf{suc}(Y)$ and $\Box\mathbf{for}(Y) \in \mathbf{suc}(Z)$, then $\Box\mathbf{for}(X) \in \mathbf{suc}(Z)$.*

Proof. From $\Box\mathbf{for}(X) \in \mathbf{suc}(Y)$ and $\Box\mathbf{for}(Y) \in \mathbf{suc}(Z)$, we have $(\Box\mathbf{for}(X) \to \mathbf{suc}(Z)) \in \mathbf{K4}$ and $n_1 < n_3$. Hence, we obtain $\Box\mathbf{for}(X) \in \mathbf{suc}(Z)$. ⊣

Lemma 3.7
(1) *$R_{\mathbf{E}}$ is transitive.*
(2) *For any exact set $\mathcal{E}$ for $\mathbf{F}(n)$, $R_{\mathcal{E}}$ is transitive.*

Proof. We only show (1). Suppose that $XR_{\mathbf{E}}Y$, $YR_{\mathbf{E}}Z$, $X \in \mathbf{G}(n_1)$, $Y \in \mathbf{G}(n_2)$, and $Z \in \mathbf{G}(n_3)$. Then we have either one of the following four conditions:

(1.1) $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ and $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y)$,

(1.2) $\Box\mathbf{for}(Y) \in \mathbf{suc}(X)$, $n_2 = n_3$, $(\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(Z))^{\Box}$, and $Z \in \mathbf{G}^{\circ}(n_3)$,

(1.3) $n_1 = n_2$, $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$, $Y \in \mathbf{G}^{\circ}(n_2)$, and $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y)$,

(1.4) $n_1 = n_2 = n_3$, $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(Z))^{\Box}$, $Y \in \mathbf{G}^{\circ}(n_2)$, and $Z \in \mathbf{G}^{\circ}(n_3)$.

If (1.1) holds, then by Lemma 3.6, we obtain $XR_{\mathbf{E}}Z$. If (1.2) holds, then by Lemma 2.11, we have

$$\Box\mathbf{for}(Z) \to \mathbf{for}(Y) \in \mathbf{K4},\ \Box\mathbf{for}(Y) \in \mathbf{suc}(X), \text{ and } n_1 > n_2 = n_3,$$

and hence, we obtain $\Box\mathbf{for}(Z) \in \mathbf{suc}(X)$. If (1.3) holds, then we obtain $\Box\mathbf{for}(Z) \in (\mathbf{suc}(Y))^{\Box} = (\mathbf{suc}(X))^{\Box}$. If (1.4) holds, then clearly, we obtain $XR_{\mathbf{E}}Z$. ⊣

Lemma 3.8 *Let $X_{\oplus}$ and Y be sequents in $\mathbf{G}^*(n+1)$ and $\mathbf{G}(n) - \mathbf{G}^*(n)$, respectively. If $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_{\oplus})$, then $(\mathbf{ant}(X_{\oplus}))^{\Box} \cap \mathbf{BG}(n) = (\mathbf{ant}(Y))^{\Box}$.*

Proof. We define $\mathcal{S}$ as in Lemma 2.13. Then by Lemma 2.13, we have $\mathbf{n}(Y, \mathcal{S}) \in \mathbf{next}(Y) \subseteq \mathbf{G}(n+1)$. We note that $(\mathbf{ant}(X_{\oplus}))^{\Box} \subseteq (\mathbf{ant}(\mathbf{n}(Y,\mathcal{S})))^{\Box}$. Using $X_{\oplus} \in \mathbf{G}^*(n+1)$, we have $(\mathbf{ant}(X_{\oplus}))^{\Box} = (\mathbf{ant}(\mathbf{n}(Y,\mathcal{S})))^{\Box}$. Hence, $(\mathbf{ant}(X_{\oplus}))^{\Box} \cap \mathbf{BG}(n) = (\mathbf{ant}(\mathbf{n}(Y,\mathcal{S})))^{\Box} \cap \mathbf{BG}(n) = (\mathbf{ant}(Y))^{\Box}$. ⊣

The following two lemmas are concerned with **EM**, but as a corollary, we have the same results with $\mathbf{EM}_{\mathcal{E}}$.

Lemma 3.9 *For any $X_n \in \mathbf{G}^*(n)$ and for any $Y_k \in \mathbf{G}(k)$,*

(1) *$(\mathbf{EM}, X_n) \not\models p_i$ if and only if $p_i \in \mathbf{suc}(X_n)$,*

(2) *$(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k)$ if and only if either one of the following two conditions holds:*

(2.1) *$k < n$ and $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_n)$,*

(2.2) *$k = n$, $(\mathbf{ant}(Y_k))^{\Box} = (\mathbf{ant}(X_n))^{\Box}$ and $Y_k \in \mathbf{G}^{\circ}(k)$.*

Proof. From the definition of $P_{\mathbf{E}}$, we obtain (1). We show (2) by an induction on $n + k$. A proof of Basis is included in Induction step.

Induction step. We first note that for any $X' \in \mathbf{G}^*(n')$ and for any $Y' \in \mathbf{G}(k')$, if $n' \leq n$ and $k' \leq \min\{n', k\}$, then the following four conditions are equivalent:

- $(\mathbf{EM}, X') \not\models \mathbf{for}(Y')$,
- $(\mathbf{EM}, X') \models A$ for any $A \in \mathbf{ant}(Y')$; and $(\mathbf{EM}, X') \not\models B$ for any $B \in \mathbf{suc}(Y')$,
- $\mathbf{ant}(Y') \subseteq \mathbf{ant}(X')$ and $\mathbf{suc}(Y') \subseteq \mathbf{suc}(X')$,
- $X' \in Y'\Downarrow$.

The equivalence between the second one and the third one is from (1) and the induction hypothesis. The equivalence between the third and the fourth is from Lemma 2.15.

We show the "only if" part. Suppose that $(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k)$. Then there exist a number ℓ and a sequent $Z_\ell \in \mathbf{G}^*(\ell)$ such that $X_n R_{\mathbf{E}} Z_\ell$ and $(\mathbf{EM}, Z_\ell) \not\models \mathbf{for}(Y_k)$. By $X_n R_{\mathbf{E}} Z_\ell$, we have either one of the following two conditions:

$$\ell < n \text{ and } \Box\mathbf{for}(Z_\ell) \in \mathbf{suc}(X_n), \tag{3.1}$$

$$\ell = n, (\mathbf{ant}(X_n))^{\Box} = (\mathbf{ant}(Z_\ell))^{\Box} \text{ and } Z_\ell \in \mathbf{G}^{\circ}(\ell). \tag{3.2}$$

Therefore, we have $\ell \leq n$. Also, by $(\mathbf{EM}, Z_\ell) \not\models \mathbf{for}(Y_k)$, we have $(\mathbf{EM}, Z_\ell) \not\models \mathbf{for}(Y_k(\ell))$. Using $\ell \leq n$ and the equivalence we noted first, we have $Z_\ell \in Y_k(\ell)\Downarrow$. Using Lemma 2.15, we have either one of the following two conditions:

$$k < \ell \text{ and } Y_k = Z_\ell(k) \tag{4.1}$$

$$k \geq \ell \text{ and } Y_k(\ell) = Z_\ell. \tag{4.2}$$

We divide the cases.

The case that (4.1) and (3.1) hold. Clearly, we have $k < n$. By (4.1), we have

$$\Box\mathbf{for}(Y_k) \to \Box\mathbf{for}(Z_\ell) \in \mathbf{K4}.$$

Using (3.1), we obtain (2.1).

The case that (4.1) and (3.2) hold. Clearly, we have $k < n$. By (4.1), we have

$$\Box\mathbf{for}(Y_k) \to \Box\mathbf{for}(Z_\ell(\ell-1)) \in \mathbf{K4}.$$

Also, by (3.2), we have

$$\Box\mathbf{for}(Z_\ell(\ell-1)) \in (\mathbf{suc}(Z_\ell))^\Box = (\mathbf{suc}(X_n))^\Box.$$

Hence, we obtain (2.1).

The case that (4.2) holds. By Lemma 3.5, we have

$$k = \ell \text{ and } Y_k = Z_\ell.$$

Hence, we have that (3.1) implies (2.1) and that (3.2) implies (2.2).

We show the "if" part.

Suppose that (2.1) holds. Then by Lemma 2.16 and Lemma 2.8, there exists a sequent $Y \in \mathbf{ED}(n-1)$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_n)$ and $Y \in Y_k\Downarrow$. We divide the cases.

The case that $Y \in W_\mathbf{E} \cap \mathbf{ED}(n-1)$, By $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_n)$, we have

$$X_n R_\mathbf{E} Y.$$

Also, by $Y \in Y_k\Downarrow$ and the equivalence we noted first, we have

$$(\mathbf{EM}, Y) \not\models \mathbf{for}(Y_k).$$

Hence,

$$(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k).$$

The case that $Y \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$. By Lemma 3.8, we have $(\mathbf{ant}(X_n))^\Box \cap \mathbf{BG}(n-1) = (\mathbf{ant}(Y))^\Box$. We define Z_n as

$$Z_n = \mathbf{n}(Y, \{Z \in \mathbf{G}(n-1) \mid \Box\mathbf{for}(Z) \in \mathbf{suc}(X_n)\}).$$

Then we have $Z_n \in \mathbf{next}^+(Y) \subseteq \mathbf{G}^+(n)$ and $(\mathbf{ant}(X_n))^\Box = (\mathbf{ant}(Z_n))^\Box$. Using $X_n \in \mathbf{G}^*(n)$ and Lemma 2.17, we have $Z_n \in \mathbf{next}(Y) \subseteq \mathbf{G}(n)$, and using Lemma 2.11(1), $Z_n \in \mathbf{G}^*(n)$. Also, by $Y \in Y_k\Downarrow$ and Lemma 2.15, we have $\mathbf{ant}(Y_k) \subseteq \mathbf{ant}(Y) \subseteq \mathbf{ant}(Z_n)$ and $\mathbf{suc}(Y_k) \subseteq \mathbf{suc}(Y) \subseteq \mathbf{suc}(Z_n)$. Using the equivalence we noted first, we have $(\mathbf{EM}, Z_n) \not\models \mathbf{for}(Y_k)$. Using $(\mathbf{ant}(X_n))^\Box = (\mathbf{ant}(Z_n))^\Box$ and $\Box\mathbf{for}(Z_n(n-1)) = \Box\mathbf{for}(Y) \in \mathbf{suc}(X_n) \cap \Box\mathbf{for}(\mathbf{G}(n-1)) \subseteq \mathbf{suc}(Z_n)$, we have $X_n R_\mathbf{E} Z_n$, and hence, we obtain $(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k)$.

Suppose that (2.2) holds. Then we have

$$Y_k \in \mathbf{G}^*(n) \text{ and } X_n R_\mathbf{E} Y_k.$$

By the equivalence we noted first, we have

$$(\mathbf{EM}, Y_k) \not\models \mathbf{for}(Y_k).$$

Hence, we obtain

$$(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k).$$

⊣

Lemma 3.10 *For any $X_n \in \mathbf{G}^*(n)$ and for any $Y_k \in \mathbf{G}(k)$,*

$$(\mathbf{EM}, X_n) \not\models \mathbf{for}(Y_k) \textit{ if and only if } X_n \in Y_k\Downarrow.$$

Proof. By Lemma 3.9, we have

$$\text{for any } A \in \mathbf{BG}(n), (\mathbf{EM}, X_n) \models A \text{ if and only if } A \in \mathbf{ant}(X_n). \tag{1}$$

Therefore, if $n > k$, then we obtain the lemma as follows.

$$\begin{array}{llll} X_n \in Y_k\Downarrow & \Leftrightarrow & \mathbf{ant}(Y_k) \subseteq \mathbf{ant}(X_n) \text{ and } \mathbf{suc}(Y_k) \subseteq \mathbf{suc}(X_n) & \text{by Lemma 2.15} \\ & \Leftrightarrow & \text{for any } A \in \mathbf{BG}(k), A \in \mathbf{ant}(Y_k) \text{ if and only if } A \in \mathbf{ant}(X_n), & \\ & \Leftrightarrow & \text{for any } A \in \mathbf{BG}(k), (\mathbf{EM}, X_n) \models A \text{ if and only if } A \in \mathbf{ant}(Y_k) & \text{by (1)} \\ & \Leftrightarrow & (\mathbf{EM}, X_n) \not\models \mathbf{for}(Y_k). & \end{array}$$

If $n \leq k$, then we obtain the lemma as follows.

$$\begin{array}{llll} X_n \in Y_k\Downarrow & \Leftrightarrow & n = k \text{ and } X_n = Y_k & \text{by Lemma 2.15} \\ & \Rightarrow & \text{for any } A \in \mathbf{BG}(k), (\mathbf{EM}, X_n) \models A \text{ if and only if } A \in \mathbf{ant}(Y_k) & \text{by (1)} \\ & \Leftrightarrow & (\mathbf{EM}, X_n) \not\models \mathbf{for}(Y_k). & \end{array}$$

$$\begin{array}{lll} & (\mathbf{EM}, X_n) \not\models \mathbf{for}(Y_k) & \\ \Leftrightarrow & \text{for any } A \in \mathbf{BG}(k), (\mathbf{EM}, X_n) \models A \text{ if and only if } A \in \mathbf{ant}(Y_k) & \\ \Rightarrow & \text{for any } A \in \mathbf{BG}(n), A \in \mathbf{ant}(Y_k) \text{ if and only if } A \in \mathbf{ant}(X_n) & \text{by (1)} \\ \Leftrightarrow & \mathbf{ant}(X_n) = \mathbf{ant}(Y_k(n)) & \\ \Leftrightarrow & n = k \text{ and } X_n = Y_k & \text{by } X_n \in \mathbf{G}^*(n) \\ \Leftrightarrow & X_n \in Y_k\Downarrow . & \end{array}$$

⊣

Corollary 3.11 *Let $\mathcal{E}$ be an exact set for $\mathbf{F}(n)$. Then for any $X_\ell \in \mathbf{G}^*(\ell) \cap \mathcal{E}$ and for any $Y_k \in \mathbf{G}(k)$,*

$$(\mathbf{EM}_\mathcal{E}, X_\ell) \not\models \mathbf{for}(Y_k) \textit{ if and only if } X_\ell \in Y_k\Downarrow .$$

Lemma 3.12 *Let $\mathcal{E}$ be an exact set for $\mathbf{F}(n)$. Then*
(1) *for any $A \in \mathbf{F}(n)$, $\mathbf{EM}_\mathcal{E} \models A$ if and only if $\to A \in \mathbf{K4}$,*
(2) *for any $X, Y \in \mathcal{E}$, $(\mathbf{EM}_\mathcal{E}, X) \not\models \mathbf{for}(Y(n))$ if and only if $X = Y$,*
(3) *$\{P_\mathcal{E}(A) \mid A \in \mathbf{F}(n)\} = 2^\mathcal{E}$,*
(4) *for any $X \in \mathcal{E}$, $P_\mathcal{E}(\mathbf{for}(X(n))) = \mathcal{E} - \{X\}$.*

Proof. For (1). The "if" part is shown by Lemma 1.1 and Lemma 3.7. We show the "only if" part. Suppose that $\to A \notin \mathbf{K4}$. Then by Theorem 2.11, $A \equiv \bigwedge \mathbf{for}(\mathcal{S})$ for some $\mathcal{S} \in 2^{\mathbf{ED}(n)} - \{\emptyset\}$. Therefore, there exists $X \in \mathcal{S}$, and by Definition 3.2(1.2), there exists $Y \in X\Downarrow \cap \mathcal{E} \subseteq W_\mathbf{E}$. Using Corollary 3.11, we have $(\mathbf{EM}_\mathcal{E}, Y) \not\models \mathbf{for}(X)$. Hence, we obtain $(\mathbf{EM}_\mathcal{E}, Y) \not\models A$.

For (2). The "if" part is clear from Corollary 3.11. We show the "only if" part. Suppose that $(\mathbf{EM}_\mathcal{E}, X) \not\models \mathbf{for}(Y(n))$. Then by Corollary 3.11, we have

$$X \in Y(n)\Downarrow \cap \mathcal{E}.$$

We also have

$$Y \in Y(n)\Downarrow \cap \mathcal{E}.$$

By Definition 3.2(1.2),

$$\#(Y(n)\Downarrow \cap \mathcal{E}) = 1,$$

Hence, we obtain $X = Y$.

For (3). By (2), we have that $\mathcal{S} \in 2^\mathcal{E}$ implies $P_\mathcal{E}((\bigwedge \mathbf{for}(\{X(n) \mid X \in \mathcal{S}\})) \supset \bot) = \mathcal{S}$, and hence, we obtain (3).

For (4). Clear from (2). ⊣

From (1) and (3) of the above lemma, we obtain Theorem 3.4(1). From (4) of the above lemma and Lemma 1.2(2), we obtain Theorem 3.4(4).

Next, we prove Theorem 3.4(2).

Definition 3.13 For a sequent $X \in \mathbf{G}(n)$, we define the set $\mathbf{antg}(X)$ as follows.

$$\mathbf{antg}(X) = \{Y \in \mathbf{G}(n) \mid (\mathbf{ant}(X))^{\Box} \subsetneq (\mathbf{ant}(Y))^{\Box}\}.$$

Lemma 3.14 *Let X be a sequent in $\mathbf{G}(n) - \mathbf{G}^*(n)$ $(n > 0)$ and let $\mathcal{S}$ be a subset of $\mathbf{pclus}(X)$ satisfying $\mathbf{n}(X, \mathcal{S}) \in \mathbf{next}(X)$. Then*

(1) $Y_{\oplus} \in \mathbf{antg}(\mathbf{n}(X, \mathcal{S}))$ *implies* $Y_{\oplus}(n) \in \mathbf{antg}(X)$,

(2) $Y_{\oplus} \in \mathbf{antg}(\mathbf{n}(X, \mathcal{S}))$ *implies* $Y_{\oplus} = \mathbf{n}(Y_{\oplus}(n), \emptyset)$,

(3) $\#(\mathbf{antg}(X)) > \#(\mathbf{antg}(\mathbf{n}(X, \mathcal{S})))$.

Proof. For (1). By $Y_{\oplus} \in \mathbf{antg}(\mathbf{n}(X, \mathcal{S}))$, we have

$$(\mathbf{ant}(\mathbf{n}(X, \mathcal{S})))^{\Box} \subsetneq (\mathbf{ant}(Y_{\oplus}))^{\Box}, \tag{1.1}$$

and thus,

$$(\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Y_{\oplus}(n)))^{\Box}.$$

Therefore, we have only to show

$$(\mathbf{ant}(X))^{\Box} \neq (\mathbf{ant}(Y_{\oplus}(n)))^{\Box}.$$

Suppose that

$$(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y_{\oplus}(n)))^{\Box}. \tag{1.2}$$

Then by (1.1), we have

$$(\mathbf{ant}(\mathbf{n}(X, \mathcal{S})))^{\Box} \cap \Box\mathbf{for}(\mathbf{G}(n)) \subsetneq (\mathbf{ant}(Y_{\oplus}))^{\Box} \cap \Box\mathbf{for}(\mathbf{G}(n)).$$

Therefore, there exists a sequent Z such that

$$\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_{\oplus}) \cap \mathbf{suc}(\mathbf{n}(X, \mathcal{S})) \cap \Box\mathbf{for}(\mathbf{G}(n)) = \mathbf{ant}(Y_{\oplus}) \cap \Box\mathbf{for}(\mathcal{S}). \tag{1.3}$$

By $\mathbf{n}(X, \mathcal{S}) \in \mathbf{next}(X)$ and Lemma 2.8, we have $\mathbf{n}(X, \mathcal{S}) \notin \mathbf{pr}_0(X)$, and using $\Box\mathbf{for}(Z) \in \mathbf{suc}(\mathbf{n}(X, \mathcal{S}))$ and (1.2),

$$\Box\mathbf{for}(Z_{\ominus}) \in (\mathbf{suc}(X))^{\Box} = (\mathbf{suc}(Y_{\oplus}(n)))^{\Box} \subseteq \mathbf{suc}(Y_{\oplus}).$$

By $Y_{\oplus} \in \mathbf{G}(n+1)$ and Lemma 2.8, we have $Y_{\oplus} \notin \mathbf{pr}_2(Y_{\oplus}(n))$. Therefore,

$$\Box\mathbf{for}(\mathbf{next}(Z_{\ominus}, Y_{\oplus}(n))) \cap \mathbf{suc}(Y_{\oplus}) \neq \emptyset.$$

Also, by (1.1), we have $(\mathbf{suc}(Y_{\oplus}))^{\Box} \subseteq (\mathbf{suc}(\mathbf{n}(X, \mathcal{S})))^{\Box}$, and thus,

$$\emptyset \neq \Box\mathbf{for}(\mathbf{next}(Z_{\ominus}, Y_{\oplus}(n))) \cap \mathbf{suc}(Y_{\oplus}) \subseteq \Box\mathbf{for}(\mathbf{next}(Z_{\ominus}, Y_{\oplus}(n))) \cap \Box\mathbf{for}(\mathcal{S}).$$

By (1.3), we have $Z \in \mathcal{S} \subseteq \mathbf{pclus}(X)$, and thus,

$$\emptyset \neq \Box\mathbf{for}(\mathbf{next}(Z_{\ominus}, Y_{\oplus}(n))) \cap \mathbf{suc}(Y_{\oplus}) \subseteq \Box\mathbf{for}(\mathbf{next}(Z_{\ominus}, Y_{\oplus}(n))) \cap \Box\mathbf{for}(\mathcal{S}) = \{\Box\mathbf{for}(Z)\}.$$

Therefore, $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_{\oplus})$, which is in contradiction with (1.3). Hence, we obtain (1).

For (2). By $Y_{\oplus} \in \mathbf{antg}(\mathbf{n}(X, \mathcal{S}))$, it is observed easily that $Y_{\oplus} = \mathbf{n}(Y_{\oplus}(n), \mathcal{S}')$ for a subset $\mathcal{S}'$ of $\mathcal{S}$. Suppose that $Z \in \mathcal{S}'$. Then we have $Z \in \mathcal{S}' \subseteq \mathcal{S} \subseteq \mathbf{pclus}(X)$, and using (1.1), $(\mathbf{ant}(Z))^{\Box} = (\mathbf{ant}(X))^{\Box} \subsetneq (\mathbf{ant}(Y_{\oplus}(n)))^{\Box}$. Hence, $Y_{\oplus} \in \mathbf{pr}_1(Y_{\oplus}(n))$, which is in contradiction with Lemma 2.8. Hence, $\mathcal{S}' = \emptyset$.

For (3). By (1) and (2), we have the following two conditions:

- $Y_{\oplus} \in \mathbf{antg}(\mathbf{n}(X, \mathcal{S}))$ implies $Y_{\oplus}(n) \in \mathbf{antg}(X)$,
- $Y_{\oplus}, Z_{\oplus} \in \mathbf{antg}(\mathbf{n}(X, \mathcal{S}))$ and $Y_{\oplus} \neq Z_{\oplus}$ imply $Y_{\oplus}(n) \neq Z_{\oplus}(n)$.

Also, it is observed easily that $\mathbf{antg}(X) \cap \mathbf{G}^*(n) \neq \emptyset$. Hence, we obtain (3). $\dashv$

Definition 3.15 Let $\mathcal{S}$ be a subset of $\mathbf{G}(n)$. We define the subset $\mathcal{S}^k$ $(k = 1, 2, \cdots)$ of $\mathbf{G}(n + k - 1)$ as

- $\mathcal{S}^1 = \mathcal{S}$,
- $\mathcal{S}^{k+1} = \{\mathbf{n}(Y, \mathcal{S}^k) \in \mathbf{G}(n + k) \mid Y \in \mathcal{S}^k - W_{\mathbf{E}}\}$.

Lemma 3.16 *For any subset $\mathcal{S}$ of $\mathbf{G}(n)$, $(\mathcal{S}^2)^k = \mathcal{S}^{k+1}$.*

Proof. Clearly, we have

$$(\mathcal{S}^2)^1 = \mathcal{S}^2 = \mathcal{S}^{1+1}.$$

Also, if $(\mathcal{S}^2)^k = \mathcal{S}^{k+1}$, then

$$\begin{aligned}(\mathcal{S}^2)^{k+1} &= \{\mathbf{n}(Y, (\mathcal{S}^2)^k) \in \mathbf{G}(n+k+1) \mid Y \in (\mathcal{S}^2)^k - W_{\mathbf{E}}\} \\ &= \{\mathbf{n}(Y, \mathcal{S}^{k+1}) \in \mathbf{G}(n+k+1) \mid Y \in \mathcal{S}^{k+1} - W_{\mathbf{E}}\} \\ &= \mathcal{S}^{k+2}.\end{aligned}$$

⊣

Definition 3.17 Let X be a sequent in $\mathbf{ED}(n)$ $(n > 0)$ and let $\mathcal{S}$ be a subset of $\mathbf{pclus}(X)$. We define the sequent $\mathbf{n}^k(X, \mathcal{S}) \in \mathbf{ED}(n+k) \cup \{\perp \to\}$ $(k = 0, 1, \cdots)$ as

- $\mathbf{n}^0(X, \mathcal{S}) = X$,
- $$\mathbf{n}^{k+1}(X,\mathcal{S}) = \begin{cases} \mathbf{n}(\mathbf{n}^k(X,\mathcal{S}), \mathcal{S}^{k+1}) & \text{if } \mathbf{n}^k(X,\mathcal{S}) \in \mathbf{G}(n+k) - \mathbf{G}^*(n+k) \\ & \quad \text{and } \mathbf{n}(\mathbf{n}^k(X,\mathcal{S}), \mathcal{S}^{k+1}) \in \mathbf{next}(\mathbf{n}^k(X,\mathcal{S})) \\ \perp \to & \text{if } \mathbf{n}^k(X,\mathcal{S}) \in \mathbf{G}(n+k) - \mathbf{G}^*(n+k) \\ & \quad \text{and } \mathbf{n}(\mathbf{n}^k(X,\mathcal{S}), \mathcal{S}^{k+1}) \in \mathbf{prov}(\mathbf{n}^k(X,\mathcal{S})) \\ \mathbf{n}^k(X,\mathcal{S}) & \text{if } \mathbf{n}^k(X,\mathcal{S}) \in \displaystyle\bigcup_{i=0}^{n+k} \mathbf{G}^*(i) \\ \perp \to & \text{if } \mathbf{n}^k(X,\mathcal{S}) = (\perp \to). \end{cases}$$

Lemma 3.18 *Let X be a sequent in $\mathbf{G}(n) - \mathbf{G}^*(n)$ $(n > 0)$ and let $\mathcal{S}$ be a subset of $\mathbf{pclus}(X)$. Then*
(1) *$\mathbf{n}^1(X, \mathcal{S}) \in \mathbf{G}(n+1)$ implies $\mathcal{S}^2 \subseteq \mathbf{pclus}(\mathbf{n}^1(X, \mathcal{S}))$ and $\mathbf{n}^k(\mathbf{n}^1(X, \mathcal{S}), \mathcal{S}^2) = \mathbf{n}^{k+1}(X, \mathcal{S})$,*
(2) *there exists $k \in \{1, 2, \cdots, \#(\mathbf{antg}(X))\}$ such that $\mathbf{n}^k(X, \mathcal{S}) \in \mathbf{G}^*(n+k) \cup \{\perp \to\}$.*

Proof.
For (1). For brevity, we define X^2 as

$$X^2 = \mathbf{n}^1(X, \mathcal{S}).$$

Suppose that $X^2 (= \mathbf{n}^1(X, \mathcal{S})) \in \mathbf{G}(n+1)$. Then we have

$$X^2 = \mathbf{n}^1(X, \mathcal{S}) = \mathbf{n}(\mathbf{n}^0(X, \mathcal{S}), \mathcal{S}^1) = \mathbf{n}(X, \mathcal{S}) \in \mathbf{next}(X).$$

Hence,

$$\mathcal{S}^2 = \{\mathbf{n}(Y, \mathcal{S}) \in \mathbf{G}(n+1) \mid Y \in \mathcal{S} - W_{\mathbf{E}}\} \subseteq \mathbf{pclus}(\mathbf{n}(X, \mathcal{S})) = \mathbf{pclus}(X^2).$$

In order to show

$$\mathbf{n}^k(X^2, \mathcal{S}^2) = \mathbf{n}^{k+1}(X, \mathcal{S}), \tag{1.1}$$

we use an induction on k. Basis$(k = 0)$ is clear.

Induction step$(k > 0)$. We divide the cases.

The case that $\mathbf{n}^{k-1}(X^2, \mathcal{S}^2) \in \mathbf{G}(n+k) - \mathbf{G}^*(n+k)$ and $\mathbf{n}(\mathbf{n}^{k-1}(X^2, \mathcal{S}^2), (\mathcal{S}^2)^k) \in \mathbf{next}(\mathbf{n}^{k-1}(X^2, \mathcal{S}^2))$. By the induction hypothesis and Lemma 3.16, we have

$$\mathbf{n}^k(X, \mathcal{S}) = \mathbf{n}^{k-1}(X^2, \mathcal{S}^2) \in \mathbf{G}(n+k) - \mathbf{G}^*(n+k)$$

and

$$\mathbf{n}(\mathbf{n}^k(X, \mathcal{S}), \mathcal{S}^{k+1}) = \mathbf{n}(\mathbf{n}^{k-1}(X^2, \mathcal{S}^2), (\mathcal{S}^2)^k) \in \mathbf{next}(\mathbf{n}^{k-1}(X^2, \mathcal{S}^2)) = \mathbf{next}(\mathbf{n}^k(X, \mathcal{S})).$$

Hence,

$$\mathbf{n}^k(X^2, \mathcal{S}^2) = \mathbf{n}(\mathbf{n}^{k-1}(X^2, \mathcal{S}^2), (\mathcal{S}^2)^k) = \mathbf{n}(\mathbf{n}^k(X, \mathcal{S}), \mathcal{S}^{k+1}) = \mathbf{n}^{k+1}(X, \mathcal{S}).$$

The case that $\mathbf{n}^{k-1}(X^2, \mathcal{S}^2) \in \mathbf{G}(n+k) - \mathbf{G}^*(n+k)$ and $\mathbf{n}(\mathbf{n}^{k-1}(X^2, \mathcal{S}^2), (\mathcal{S}^2)^k) \in \mathbf{prov}(\mathbf{n}^{k-1}(X^2, \mathcal{S}^2))$. By the induction hypothesis and Lemma 3.16, we have

$$\mathbf{n}^k(X, \mathcal{S}) = \mathbf{n}^{k-1}(X^2, \mathcal{S}^2) \in \mathbf{G}(n+k) - \mathbf{G}^*(n+k)$$

and

$$\mathbf{n}(\mathbf{n}^k(X,\mathcal{S}),\mathcal{S}^{k+1}) = \mathbf{n}(\mathbf{n}^{k-1}(X^2,\mathcal{S}^2),(\mathcal{S}^2)^k) \in \mathbf{prov}(\mathbf{n}^{k-1}(X^2,\mathcal{S}^2)) = \mathbf{prov}(\mathbf{n}^k(X,\mathcal{S})).$$

Hence,

$$\mathbf{n}^k(X^2,\mathcal{S}^2) = (\perp\to) = \mathbf{n}^{k+1}(X,\mathcal{S}).$$

The case that $\mathbf{n}^{k-1}(X^2,\mathcal{S}^2) \in \bigcup_{i=0}^{n+k} \mathbf{G}^*(i)$. By the induction hypothesis, we have

$$\mathbf{n}^k(X,\mathcal{S}) = \mathbf{n}^{k-1}(X^2,\mathcal{S}^2) \in \bigcup_{i=0}^{n+k} \mathbf{G}^*(i).$$

Hence,

$$\mathbf{n}^k(X^2,\mathcal{S}^2) = \mathbf{n}^{k-1}(X^2,\mathcal{S}^2) = \mathbf{n}^k(X,\mathcal{S}) = \mathbf{n}^{k+1}(X,\mathcal{S}).$$

The case that $\mathbf{n}^{k-1}(X^2,\mathcal{S}^2) = (\perp\to)$. By the induction hypothesis, we have

$$\mathbf{n}^k(X,\mathcal{S}) = \mathbf{n}^{k-1}(X^2,\mathcal{S}^2) = (\perp\to).$$

Hence,

$$\mathbf{n}^k(X^2,\mathcal{S}^2) = (\perp\to) = \mathbf{n}^{k+1}(X,\mathcal{S}).$$

For (2). If $X^2(=\mathbf{n}^1(X,\mathcal{S})) = (\perp\to)$, then (2) is clear. We assume that $X^2 \neq (\perp\to)$. Then we have

$$X^2 = \mathbf{n}^1(X,\mathcal{S}) = \mathbf{n}(X,\mathcal{S}) \in \mathbf{next}(X) \subseteq \mathbf{G}(n+1). \tag{2.1}$$

Using Lemma 3.14(3),

$$\#(\mathbf{antg}(X)) > \#(\mathbf{antg}(X^2)) \tag{2.2}$$

We use an induction on $\#(\mathbf{antg}(X))$.

Basis($\#(\mathbf{antg}(X)) = 1$). By (2.2), we have $\#(\mathbf{antg}(X^2)) = 0$, and using (2.1), we obtain $X^2 \in \mathbf{G}^*(n+1)$.

Induction step($\#(\mathbf{antg}(X)) > 1$). If $X^2 \in \mathbf{G}^*(n+1)$, then (2) is clear. Using (2.1), we can assume that

$$X^2 \in \mathbf{G}(n+1) - \mathbf{G}^*(n+1).$$

Also, by (2.1) and (1), we have

$$\mathcal{S}^2 \subseteq \mathbf{pclus}(X^2) \text{ and } \mathbf{n}^k(X^2,\mathcal{S}^2) = \mathbf{n}^{k+1}(X,\mathcal{S}).$$

Using (2.2) and the induction hypothesis, there exists $\ell \in \{1,2,\cdots,\#(\mathbf{antg}(X^2))\}$ such that

$$\mathbf{n}^{\ell+1}(X,\mathcal{S}) = \mathbf{n}^\ell(X^2,\mathcal{S}^2) \in \mathbf{G}^*(n+\ell+1) \cup \{\perp\to\}.$$

Also, using (2.2),

$$\ell+1 \in \{1,2,\cdots,\#(\mathbf{antg}(X^2))+1\} \subseteq \{1,2,\cdots,\#(\mathbf{antg}(X))\}.$$

$\dashv$

Lemma 3.19 *Let $M = \langle W,R,P\rangle$ be an exact model for $\mathbf{F}(n)$.*

(1) *There exists one-to-one mapping f from W onto $\mathbf{ED}(n)$ such that $P(\mathbf{for}(f(\alpha))) = W - \{\alpha\}$.*

(2) *There exist an exact set $\mathcal{E}$ for $\mathbf{F}(n)$ and an isomorphism from M to $\mathbf{EM}_\mathcal{E}$.*

Proof. First, we note that

(3) R is transitive.

For (1). Let α be a world in W. Then by Lemma 1.2 and Lemma 2.3, there exists only one sequent X in $\mathbf{ED}(n)$ such that $(M, \alpha) \not\models \mathbf{for}(X)$. Therefore, we can define a mapping f as $f(\alpha) = X$; and f satisfies the conditions.

For (2). For any $\alpha \in W$, we define the set $\mathbf{clus}(\alpha)$ as $\mathbf{clus}(\alpha) = \{\beta \in W \mid \alpha R \beta, \beta R \alpha\}$. By an induction on $k \in \{n, n+1, n+2, \cdots\}$, we define a mapping g_k from W to the set of sequents as

$$g_k(\alpha) = \begin{cases} f(\alpha) & \text{if } k = n \\ g_{k-1}(\alpha) & \text{if } k > n \text{ and } g_{k-1}(\alpha) \in W_{\mathbf{E}} \\ \mathbf{n}(g_{k-1}(\alpha), \{g_{k-1}(\beta) \in \mathbf{G}(k-1) \mid \alpha R \beta\}) & \text{otherwise.} \end{cases}$$

Also, we define the set $\mathcal{E}(k)$ as $\mathcal{E}(k) = \{g_k(\alpha) \mid \alpha \in W\}$. Then by (1), we have

(2.0) $\mathbf{ED}(n) = \mathcal{E}(n)$.

Also, we have the following six conditions:

(2.1) for any $\alpha \in W$, $g_k(\alpha) \in \mathbf{ED}(k)$ and $P(\mathbf{for}(g_k(\alpha))) = W - \{\alpha\}$,

(2.2) $\mathbf{ED}(n) \cap W_{\mathbf{E}} \subseteq \mathcal{E}(k) \subseteq \mathbf{ED}(k)$,

(2.3) for any $X \in \mathbf{ED}(n)$, $\#(X\Downarrow \cap \mathcal{E}(k)) = 1$,

(2.4) for any $\alpha \in W$ and for any $\beta \in \mathbf{clus}(\alpha)$, $(\mathbf{ant}(g_k(\alpha)))^{\Box} = (\mathbf{ant}(g_k(\beta)))^{\Box}$,

(2.5) for any $\alpha \in W$ and for any $X \in W_{\mathbf{E}}$, $g_k(\alpha) \in \mathcal{E}(k) \cap W_{\mathbf{E}}$ and $g_k(\alpha) R_{\mathbf{E}} X$ imply $X \in \mathcal{E}(k)$,

(2.6) $\mathcal{E}(k) \not\subseteq W_{\mathbf{E}}$ implies $\#(\mathcal{E}(k) - W_{\mathbf{E}}) > \#(\mathcal{E}(k + \#(\mathbf{G}(k)) - 1) - W_{\mathbf{E}})$.

The first four conditions are shown similarly to the proof of Lemma 3.4 in [Sas10c], which is an extended proof of Lemma 4.14 in [Sas10a]. We note the following two things.

[Sas10a] and [Sas10c] treat the modal logic **S4**. Therefore, definition of $\mathbf{next}^{+}(X)$ is different from ours. To show (2.1), we show that

$$g_{k-1}(\alpha) \notin W_{\mathbf{E}} \text{ implies } g_k(\alpha) \in \mathbf{next}^{+}(g_{k-1}(\alpha)). \tag{2.1.1}$$

In order to show (2.1.1), [Sas10c] has to show $\Box\mathbf{for}(g_{k-1}(\alpha)) \in \mathbf{suc}(g_k(\alpha))$, but in our case, we don't have to show it and we can directly obtain (2.1.1).

To show (2.4), [Sas10c] uses Lemma 1.7 in [Sas10c] for **S4**. Here, we use Lemma 3.5 instead of Lemma 1.7 in [Sas10c].

We show (2.5) by an induction on k. Suppose that

(2.5.1) $\alpha \in W$

(2.5.2) $X \in W_{\mathbf{E}}$,

(2.5.3) $g_k(\alpha) \in \mathcal{E}(k) \cap W_{\mathbf{E}}$, and

(2.5.4) $g_k(\alpha) R_{\mathbf{E}} X$.

We want to show $X \in \mathcal{E}(k)$. By (2.5.4), we have either

(4a) $\Box\mathbf{for}(X) \in \mathbf{suc}(g_k(\alpha))$ or

(4b) $(\mathbf{ant}(g_k(\alpha)))^{\Box} = (\mathbf{ant}(X))^{\Box}$ and $X \in \mathbf{G}^{\circ}(\ell)$ for some ℓ.

Basis($k = n$). If (4a) holds, then by (2.0), we have $\Box\mathbf{for}(X) \in \mathbf{BG}(n)$, using (2.5.2) and (2.0), we have $X \in \mathbf{ED}(n) = \mathcal{E}(n)$. If (4b) holds, then by (2.5.2), (2.0), (2.5.3), and Lemma 3.5, we have $\ell \leq n$ and $X \in \mathbf{G}^{\circ}(\ell) \subseteq \mathbf{ED}(n) = \mathcal{E}(n)$.

Induction step($k > n$). We only show the case that both $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$ and (4b) hold. The other case can be shown similarly to the proof of Lemma 3.4 in [Sas10c].

By (2.2) and $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$, we have $g_{k-1}(\alpha) \in \mathbf{G}(k-1) - \mathbf{G}^{*}(k-1)$, and using (2.2) and (2.5.3), we have $g_k(\alpha) \in \mathbf{G}^{*}(k)$. Using (4b) and Lemma 3.5, we have $\ell = k$ and $X \in \mathbf{G}^{\circ}(k)$, and therefore, we have $\Box\mathbf{for}(X(k-1)) \in (\mathbf{suc}(X))^{\Box} = (\mathbf{suc}(g_k(\alpha)))^{\Box}$. Considering the form of $g_k(\alpha)$, there exists a world $\beta \in W$ such that $X(k-1) = g_{k-1}(\beta)$ and $\alpha R \beta$. Also, considering (3) and the forms of $g_k(\alpha)$ and $g_k(\beta)$, we have $(\mathbf{suc}(g_k(\beta)))^{\Box} \subseteq (\mathbf{suc}(g_k(\alpha)))^{\Box}$. Using $g_k(\alpha) \in \mathbf{G}^{*}(k)$ and (4b), we have $(\mathbf{ant}(g_k(\beta)))^{\Box} = (\mathbf{ant}(g_k(\alpha)))^{\Box} = (\mathbf{ant}(X))^{\Box}$. Using $X(k-1) = g_{k-1}(\beta)$, we obtain $X = g_k(\beta) \in \mathcal{E}(k)$.

We show (2.6). Let i be a number in $\{n, n+1, n+2, \cdots\}$. Then we have

$$\text{for any } \alpha \in W, \quad g_i(\alpha) \in W_{\mathbf{E}} \Rightarrow g_{i+1}(\alpha)(= g_i(\alpha)) \in W_{\mathbf{E}}; \tag{2.6.1}$$

and by (1) and (2.2), we have

$$\text{for any } \alpha, \beta \in W, \quad \alpha \neq \beta \Rightarrow g_n(\alpha) \neq g_n(\beta) \Rightarrow \cdots \Rightarrow g_i(\alpha) \neq g_i(\beta) \Rightarrow g_{i+1}(\alpha) \neq g_{i+1}(\beta) \Rightarrow \cdots. \tag{2.6.2}$$

By $\mathcal{E}(k) \not\subseteq W_{\mathbf{E}}$ and (2.2), there exists a world $\alpha \in W$ such that

$$g_k(\alpha) \in \mathcal{E}(k) \cap (\mathbf{G}(k) - \mathbf{G}^*(k)) \tag{2.6.3}$$

and

$$g_k(\beta) \in \mathbf{ED}(k) \cap W_{\mathbf{E}} \text{ for any } \beta \in \{\gamma \in W \mid \alpha R \gamma, \gamma \not\!R \alpha\}.$$

Therefore,

$$\text{for any } \beta \in \{\gamma \in W \mid \alpha R \gamma, \gamma \not\!R \alpha\}, \mathbf{ED}(k) \cap W_{\mathbf{E}} \ni g_k(\beta) = g_{k+1}(\beta) = \cdots = g_{k+\ell}(\beta) = \cdots. \tag{2.6.4}$$

We define $\mathcal{S}$ as

$$\mathcal{S} = \{g_k(\beta) \in \mathbf{G}(k) \mid \beta \in \mathbf{clus}(\alpha)\}.$$

Using (2.4), we have $\mathcal{S} \subseteq \mathbf{pclus}(g_k(\alpha))$. Therefore, using (2.6.3) and Lemma 3.18(2),

$$\mathbf{n}^{\ell}(g_k(\alpha), \mathcal{S}) \in \mathbf{G}^*(k+\ell) \cup \{\perp \rightarrow\} \text{ for some } \ell \in \{1, 2, \cdots, \#(\mathbf{antg}(g_k(\alpha)))\}. \tag{2.6.5}$$

We show
(2.6.6) $\mathcal{S}^{\ell} = \{g_{k+\ell-1}(\beta) \in \mathbf{G}(k+\ell-1) \mid \beta \in \mathbf{clus}(\alpha)\}$ and
(2.6.7) $g_{k+\ell-1}(\alpha) = \mathbf{n}^{\ell-1}(g_k(\alpha), \mathcal{S})$
for $\ell \in \{1, 2, \cdots\}$. We use an induction on ℓ. Clearly, we have Basis($\ell = 1$).
Induction step ($\ell > 1$). By the induction hypothesis, we have

$$\mathcal{S}^{\ell-1} = \{g_{k+\ell-2}(\beta) \in \mathbf{G}(k+\ell-2) \mid \beta \in \mathbf{clus}(\alpha)\} \tag*{(2.6.6)$'$}$$

$$g_{k+\ell-2}(\alpha) = \mathbf{n}^{\ell-2}(g_k(\alpha), \mathcal{S}). \tag*{(2.6.7)$'$}$$

Therefore, we obtain (2.6.6) as follows.

$$\begin{array}{lll}
\mathcal{S}^{\ell} & = & \{\mathbf{n}(Y, \mathcal{S}^{\ell-1}) \in \mathbf{G}(k+\ell-1) \mid Y \in \mathcal{S}^{\ell-1} - W_{\mathbf{E}}\} \\
& = & \{\mathbf{n}(Y, \mathcal{S}^{\ell-1}) \in \mathbf{G}(k+\ell-1) \mid Y \in \{g_{k+\ell-2}(\beta) \in \mathbf{G}(k+\ell-2) \mid \beta \in \mathbf{clus}(\alpha)\} - W_{\mathbf{E}}\} \quad \text{(by (2.6.6)}') \\
& = & \{\mathbf{n}(g_{k+\ell-2}(\beta), \mathcal{S}^{\ell-1}) \in \mathbf{G}(k+\ell-1) \mid g_{k+\ell-2}(\beta) \in \mathbf{G}(k+\ell-2) - W_{\mathbf{E}}, \beta \in \mathbf{clus}(\alpha)\} \\
& = & \{\mathbf{n}(g_{k+\ell-2}(\beta), \{g_{k+\ell-2}(\gamma) \in \mathbf{G}(k+\ell-2) \mid \gamma \in \mathbf{clus}(\alpha)\}) \in \mathbf{G}(k+\ell-1) \\
& & \quad \mid g_{k+\ell-2}(\beta) \in \mathbf{G}(k+\ell-2) - W_{\mathbf{E}}, \beta \in \mathbf{clus}(\alpha)\} \quad \text{(by (2.6.6)}') \\
& = & \{\mathbf{n}(g_{k+\ell-2}(\beta), \{g_{k+\ell-2}(\gamma) \in \mathbf{G}(k+\ell-2) \mid \alpha R \gamma\}) \in \mathbf{G}(k+\ell-1) \\
& & \quad \mid g_{k+\ell-2}(\beta) \in \mathbf{G}(k+\ell-2) - W_{\mathbf{E}}, \beta \in \mathbf{clus}(\alpha)\} \quad \text{(by (2.6.4))} \\
& = & \{\mathbf{n}(g_{k+\ell-2}(\beta), \{g_{k+\ell-2}(\gamma) \in \mathbf{G}(k+\ell-2) \mid \beta R \gamma\}) \in \mathbf{G}(k+\ell-1) \\
& & \quad \mid g_{k+\ell-2}(\beta) \in \mathbf{G}(k+\ell-2) - W_{\mathbf{E}}, \beta \in \mathbf{clus}(\alpha)\} \quad \text{(by (3))} \\
& = & \{g_{k+\ell-1}(\beta) \in \mathbf{G}(k+\ell-1) \mid \beta \in \mathbf{clus}(\alpha)\}.
\end{array}$$

We show (2.6.7). By (2.6.7)$'$ and (2.2), we have

$$\mathbf{n}^{\ell-2}(g_k(\alpha), \mathcal{S}) = g_{k+\ell-2}(\alpha) \in \mathbf{ED}(k+\ell-2).$$

If $\mathbf{n}^{\ell-2}(g_k(\alpha), \mathcal{S}) = g_{k+\ell-2}(\alpha) \in \mathbf{G}(k+\ell-2) - \mathbf{G}^*(k+\ell-2)$, then

$$\begin{array}{lll}
g_{k+\ell-1}(\alpha) & = & \mathbf{n}(g_{k+\ell-2}(\alpha), \{g_{k+\ell-2}(\beta) \in \mathbf{G}(k+\ell-2) \mid \alpha R \beta\}) \\
& = & \mathbf{n}(g_{k+\ell-2}(\alpha), \{g_{k+\ell-2}(\beta) \in \mathbf{G}(k+\ell-2) \mid \beta \in \mathbf{clus}(\alpha)\}) \quad \text{(by (2.6.4))} \\
& = & \mathbf{n}(g_{k+\ell-2}(\alpha), \mathcal{S}^{\ell-1}) \quad \text{(by (2.6.6))} \\
& = & \mathbf{n}(\mathbf{n}^{\ell-2}(g_k(\alpha), \mathcal{S}), \mathcal{S}^{\ell-1}) \quad \text{(by (2.6.7)}') \\
& = & \mathbf{n}^{\ell-1}(g_k(\alpha), \mathcal{S}) \quad \text{(by (2.1)} : g_{k+\ell-1}(\alpha) \in \mathbf{ED}(k+\ell-1)).
\end{array}$$

If $\mathbf{n}^{\ell-2}(g_k(\alpha),\mathcal{S}) = g_{k+\ell-2}(\alpha) \in W_{\mathbf{E}}$, then

$$\begin{array}{rcll} g_{k+\ell-1}(\alpha) & = & g_{k+\ell-2}(\alpha) & \\ & = & \mathbf{n}^{\ell-2}(g_k(\alpha),\mathcal{S}) & \text{(by (2.6.7)}')\\ & = & \mathbf{n}^{\ell-1}(g_k(\alpha),\mathcal{S}). & \end{array}$$

Hence, we obtain (2.6.7).

By (2.6.5), (2.6.7), and (2.1), we have

$$g_{k+\ell}(\alpha) \in \mathbf{G}^*(k+\ell) \text{ for some } \ell \in \{1,2,\cdots,\#(\mathbf{antg}(g_k(\alpha)))\}.$$

Also, we have

$$\mathbf{ant}(g_k(\alpha)) \notin \mathbf{antg}(g_k(\alpha)),$$

and using (2.1),

$$\#(\mathbf{antg}(g_k(\alpha))) \leq \#(\mathbf{G}(k)) - 1.$$

Therefore,

$$g_{k+\ell}(\alpha) \in \mathbf{G}^*(k+\ell) \text{ for some } \ell \in \{1,2,\cdots,\#(\mathbf{G}(k))-1\}.$$

Using (2.6.3),

$$g_k(\alpha) \notin W_{\mathbf{E}} \text{ and } g_{k+\ell}(\alpha) \in W_{\mathbf{E}}.$$

Using (2.6.1) and (2.6.2),

$$\begin{array}{rcl} \#(\mathcal{E}(k) - W_{\mathbf{E}}) & = & \#(\mathcal{E}(k+1) - \{g_{k+1}(\gamma) \mid g_k(\gamma) \in W_{\mathbf{E}}\}) \\ & = & \cdots \\ & = & \#(\mathcal{E}(k+\ell) - \{g_{k+\ell}(\gamma) \mid g_k(\gamma) \in W_{\mathbf{E}}\}) \\ & > & \#(\mathcal{E}(k+\ell) - W_{\mathbf{E}}) \\ & \geq & \cdots \\ & \geq & \#(\mathcal{E}(k+\#(\mathbf{G}(k))-1) - W_{\mathbf{E}}). \end{array}$$

Hence, we obtain (2.6).

We consider the number $\kappa = \kappa(\#(\mathbf{ED}(n) - W_{\mathbf{E}}))$. Then by (2.0), (2.6), and (2.6.1), there exists $\ell \in \{0,1,\cdots,\#(\mathbf{ED}(n) - W_{\mathbf{E}})\}$, such that

$$\begin{array}{cll} & \#(\mathbf{ED}(n) - W_{\mathbf{E}}) & \\ = & \#(\mathcal{E}(\kappa(0)) - W_{\mathbf{E}}) & (= \#(\mathcal{E}(n) - W_{\mathbf{E}})) \\ > & \#(\mathcal{E}(\kappa(1)) - W_{\mathbf{E}}) & (= \#(\mathcal{E}(\kappa(0) + \#(\mathbf{G}(\kappa(0))) - 1) - W_{\mathbf{E}})) \\ > & \#(\mathcal{E}(\kappa(2)) - W_{\mathbf{E}}) & (= \#(\mathcal{E}(\kappa(1) + \#(\mathbf{G}(\kappa(1))) - 1) - W_{\mathbf{E}})) \\ \vdots & & \\ > & \#(\mathcal{E}(\kappa(\ell)) - W_{\mathbf{E}}) & (=0) \\ = & \#(\mathcal{E}(\kappa(\ell+1)) - W_{\mathbf{E}}) & (=0) \\ \vdots & & \\ = & \#(\mathcal{E}(\kappa(\#(\mathbf{ED}(n) - W_{\mathbf{E}})) - W_{\mathbf{E}}) & (=0) \\ = & \#(\mathcal{E}(\kappa) - W_{\mathbf{E}}) & (=0). \end{array}$$

Using (2.2), we have

$$\mathbf{ED}(n) \cap W_{\mathbf{E}} \subseteq \mathcal{E}(\kappa) \subseteq W_{\mathbf{E}}. \tag{2.7}$$

Using (2.3) and (2.5), $\mathcal{E}(\kappa)$ is an exact set for $\mathbf{F}(n)$. Therefore, in order to prove the lemma, it is sufficient to show that g_κ is an isomorphism from M to $\mathbf{EM}_{\mathcal{E}(\kappa)}$. Specifically, we have only to show the following three conditions:

(2.8) g_κ is one-to-one and $\{g_\kappa(\alpha) \mid \alpha \in W\} = \mathcal{E}(\kappa)$,

(2.9) $\alpha R \beta$ if and only if $g_\kappa(\alpha) R_{\mathcal{E}(\kappa)} g_\kappa(\beta)$,

(2.10) $\alpha \in P(p_i)$ if and only if $g_\kappa(\alpha) \in P_{\mathcal{E}(\kappa)}(p_i)$.
Clearly, we obtain (2.8) from (2.6.2) and the definition of $\mathcal{E}(k)$. (2.10) is shown similarly to the proof in [Sas10c].

We show (2.9). By (2.7), we have $g_\kappa(\alpha) \in \mathbf{G}^*(\ell_1)$ and $g_\kappa(\beta) \in \mathbf{G}^*(\ell_2)$ for some $\ell_1, \ell_2 \in \{1, 2, \cdots, \kappa\}$.

Suppose that $\alpha R \beta$. If $\ell_1 > \ell_2$, we obtain (2.9), similarly to the proof in [Sas10c]. We assume that $\ell_1 \leq \ell_2$. Similarly to the proof in [Sas10c], we have

$$\ell_1 = \ell_2 \text{ and } (\mathbf{ant}(g_\kappa(\alpha)))^\Box = (\mathbf{ant}(g_\kappa(\beta)))^\Box.\text{[1]} \tag{2.8.1}$$

Also, by (2.1), we have

$$(M, \beta) \not\models \mathbf{for}(g_\kappa(\beta)_\ominus).$$

Using $\alpha R \beta$, (2.1), and $\ell_1 = \ell_2$, we have

$$\Box\mathbf{for}(g_\kappa(\beta)_\ominus) \in \mathbf{suc}(g_\kappa(\alpha)).$$

Using (2.8.1), we have

$$\Box\mathbf{for}(g_\kappa(\beta)_\ominus) \in \mathbf{suc}(g_\kappa(\beta)).$$

Hence, $g_\kappa(\beta) \in \mathbf{G}^\circ(\ell_2)$, and using (2.8.1), we obtain $g_\kappa(\alpha) R_{\mathcal{E}(\kappa)} g_\kappa(\beta)$.

Suppose that $g_\kappa(\alpha) R_{\mathcal{E}(\kappa)} g_\kappa(\beta)$. If $\Box\mathbf{for}(g_\kappa(\beta)) \in \mathbf{suc}(g_\kappa(\alpha))$, then by (2.1), we have $(M, \alpha) \not\models \Box\mathbf{for}(g_\kappa(\beta))$, and using (2.1) again, we have $\alpha R \beta$. We assume that $\ell_1 = \ell_2$, $(\mathbf{ant}(g_\kappa(\alpha)))^\Box = (\mathbf{ant}(g_\kappa(\beta)))^\Box$, and $g_\kappa(\beta) \in \mathbf{G}^\circ(\ell_2)$. By (2.1), we have $(M, \alpha) \not\models \mathbf{for}(g_\kappa(\alpha))$. Using Lemma 2.11, we have $(M, \alpha) \not\models \Box\mathbf{for}(g_\kappa(\beta))$, and using (2.1) again, we have $\alpha R \beta$. $\dashv$

From the above lemma, we obtain Theorem 3.4(2). Also, by the proof of (2) of the above lemma, we obtain Theorem 3.4(3).

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[1] We need Lemma 3.5 for (2.8.1).