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Abstract. [3] provided a detailed description of the mutual relation of formulas with finite propositional variables $p_1, \cdots, p_m$ in modal logic **S4**. It contains some lemmas whose detailed proofs are omitted. Here, we provide such detailed proofs.

1 Introduction

Here, we provide detailed proofs of Lemma 3.12, Lemma 4.15(2), and Theorem 5.4(2) in [3]. [3] treated the modal logic **S4** by using normal forms and exact models. In the present section, we introduce modal logic **S4**, exact models, and normal forms. The next three sections are devoted to giving detailed proofs of Lemma 3.12, Lemma 4.15(2), and Theorem 5.4(2) in [3], respectively.

We introduce **S4** as a sequent system.

Formulas are constructed from $\perp$ (contradiction) and the propositional variables $p_1, p_2, \cdots$ by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and $\Box$ (necessitation). We use upper case Latin letters, $A, B, C, \cdots$, with or without subscripts, for formulas. Also, we use Greek letters, $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^\Box$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. The *depth* $d(A)$ of a formula A is defined as

$d(p_i) = d(\perp) = 0$,
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
$d(\Box B) = d(B) + 1$.

Let **ENU** be an enumeration of the formulas. For a non-empty finite set Γ of formulas, the expressions $\bigwedge \Gamma$ and $\bigvee \Gamma$ denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \quad \text{and} \quad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

respectively, where $\{A_1, \cdots, A_n\} = \Gamma$ and A_i occurs earlier than A_{i+1} in **ENU**. Also, the expressions $\bigwedge \emptyset$ and $\bigvee \emptyset$ denote the formulas $\perp \supset \perp$ and $\perp$, respectively.

The set of propositional variables $p_1, \cdots, p_m$ $(m \geq 1)$ is denoted by $\mathbf{V}$ and the set of formulas constructed from $\mathbf{V}$ and $\perp$ is denoted by $\mathbf{F}$. Also, for any $n = 0, 1, \cdots$, we define $\mathbf{F}^n$ as $\mathbf{F}^n = \{A \in \mathbf{F} \mid d(A) \leq n\}$.

A *sequent* is the expression $(\Gamma \to \Delta)$. We often refer to $\Gamma \to \Delta$ as $(\Gamma \to \Delta)$ for brevity and refer to

$$A_1, \cdots, A_i, \Gamma_1, \cdots, \Gamma_j \to \Delta_1, \cdots, \Delta_k, B_1, \cdots, B_\ell$$

as

$$\{A_1, \cdots, A_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \cdots, B_\ell\}.$$

We use upper case Latin letters $X, Y, Z, \cdots$, with or without subscripts, for sequents. If $X = (\Gamma \to \Delta)$, then we sometimes refer to $\Gamma \overset{X}{\to} \Delta$ as $\Gamma \to \Delta$. The *antecedent* $\mathbf{ant}(\Gamma \to \Delta)$ and the *succedent* $\mathbf{suc}(\Gamma \to \Delta)$ of a sequent $\Gamma \to \Delta$ are defined as

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \to \Delta) = \Delta,$$

respectively. Also, for a sequent X and a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) \neq \emptyset \\ \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) = \emptyset \end{cases}$$

and

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

For a finite set S of formulas or sequents, the expression $\#(S)$ denotes the number of elements in S.

Ohnishi and Matsumoto [2] defined the system by adding the following two inference rules to the sequent system **LK** given by Gentzen [1] for the classical propositional logic:

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to) \qquad\qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box).$$

Here, we do not use $\neg$ as a primary connective, so we use the additional axiom $\bot \to$ instead of the inference rules $(\neg \to)$ and $(\to \neg)$. We write $X \in \mathbf{S4}$ if X is provable in $\mathbf{S4}$. We use $A \equiv B$ instead of $\to (A \supset B) \wedge (B \supset A) \in \mathbf{S4}$.

We introduce an exact model as a kind of Kripke models.

A *Kripke model* is a structure $\langle W, R, P\rangle$ where W is a non-empty set, R is a binary relation on W, and P is a mapping from the set of propositional variables to 2^W. We extend, as usual, the domain of P to include all formulas. We call P a *valuation* and a member of W a *world*. For a Kripke model $M = \langle W, R, P\rangle$, and for a world $\alpha \in W$, we often write $(M, \alpha) \models A$ and $M \models A$ instead of $\alpha \in P(A)$ and $P(A) = W$, respectively.

Let S be a set of formulas closed under $\supset$ and $\wedge$. We say that a Kripke model $M = \langle W, R, P\rangle$ is *exact for* S if the following two conditions hold:

- for any $A \in S$, $M \models A$ if and only if $\to A \in \mathbf{S4}$,
- $\{P(A) \mid A \in S\} = 2^W$.

Here, we note that 2^W is either finite or uncountable, whereas $\{P(A) \mid A \in S\}$ is at most countable. Therefore, from the second condition above, we find that W must be finite.

We also consider a Kripke model $M = \langle W, R, P\rangle$ satisfying the following two conditions:

- for any $A \in \mathbf{F}$, $M \models A$ if and only if $\to A \in \mathbf{S4}$,
- $\{\{\alpha\} \mid \alpha \in W\} \subseteq \{P(A) \mid A \in \mathbf{F}\}$.

If W is finite, then the second condition above is equivalent to the second condition for an exact model. Every exact model is finite, as stated above. Thus, there is no confusion if we refer to the above model as an *exact model for* $\mathbf{F}$.

[3] constructed normal forms as sequents corresponding to elementary disjunctions in classical propositional logic. Below, we introduce the set $\mathbf{ED}^n$ of such normal forms and some lemmas. By Lemma 1.4, we may say that $\mathbf{ED}^n$ is the set of normal forms. All of the lemmas below have been shown in [3].

Definition 1.1 The sets $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ of sequents are defined inductively as follows.

$\mathbf{G}(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,

$\mathbf{G}^*(0) = \emptyset$,

$\mathbf{G}(k+1) = \displaystyle\bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}(X)$,

$\mathbf{G}^*(k+1) = \{X \in \mathbf{G}(k+1) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y))^\Box$ implies $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, for any $Y \in \mathbf{G}(k+1)\}$,

where for any $X \in \mathbf{G}(k)$,

$\mathbf{next}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}(k)), \Gamma \cap \Delta = \emptyset, \mathbf{for}(X) \in \Delta\}$,

$\mathbf{prov}(X) = \{Y \in \mathbf{next}^+(X) \mid Y \in \mathbf{S4}\}$,

$\mathbf{next}(X) = \mathbf{next}^+(X) - \mathbf{prov}(X)$.

Definition 1.2

(1) We define the sets $\mathbf{ED}^n$, $\mathbf{G}^+(n)$, and $\mathbf{BG}(n)$ as

$$\mathbf{ED}^n = \mathbf{G}(n) \cup \bigcup_{i=0}^{n-1} \mathbf{G}^*(i),$$

$$\mathbf{G}^+(n) = \begin{cases} \mathbf{G}(0) & \text{if } n = 0 \\ \displaystyle\bigcup_{X \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)} \mathbf{next}^+(X) & \text{if } n > 0, \end{cases}$$

$$\mathbf{BG}(n) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i)).$$

(2) For any $X \in \mathbf{G}(n)$, we define the set $\mathbf{clus}(X)$ of sequents as

$$\mathbf{clus}(X) = \{Y \in \mathbf{G}(n) \mid (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box\}.$$

(3) For any $X \in \mathbf{G}^+(n)$ and for any k, we define the sequent $X(k)$ as

$$X(k) = (\mathbf{ant}(X) \cap \mathbf{BG}(k) \to \mathbf{suc}(X) \cap \mathbf{BG}(k)).$$

(4) For any sequent X and for any subset $\mathcal{S}$ of $\mathbf{G}(n)$, we define the sequent $\mathbf{n}(X, \mathcal{S})$ as

$$\mathbf{n}(X, \mathcal{S}) = (\Box\mathbf{for}(\mathbf{G}(n) - (\{X\} \cup \mathcal{S})), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\{X\} \cup \mathcal{S})).$$

We note that $\mathbf{n}(X, \mathcal{S}) \in \mathbf{next}^+(X)$ if $X \in \mathbf{G}(n)$.

Lemma 1.3
(1) *None of the members in* $\mathbf{G}(n)$ *is provable in* $\mathbf{S4}$.
(2) *For any* $X, Y \in \mathbf{ED}^n$, $X \neq Y$ *implies* $\mathbf{for}(X) \vee \mathbf{for}(Y) \in \mathbf{S4}$.
(3) *For any* $X \in \mathbf{G}^+(n)$, $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{BG}(n)$ *and* $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$.
(4) *For any* $X \in \mathbf{G}(n)$, $\bigwedge \mathbf{for}(\mathbf{next}(X)) \equiv \mathbf{for}(X)$.
(5) *For any* $X \in \mathbf{G}^*(n)$ *and for any* $Y \in \mathbf{clus}(X)$, $Y \in \mathbf{G}^*(n)$ *and* $\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in \mathbf{S4}$.

Lemma 1.4
(1) $\mathbf{F}^n/\equiv = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{ED}^n\}$.
(2) *For subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathbf{ED}^n$,

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \text{ if and only if } \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}.$$

Definition 1.5 For any $X \in \mathbf{G}(n)$, we define the sets $X\Downarrow$ inductively as follows:
(1) $X \in X\Downarrow$,
(2) if $Y \in \mathbf{next}(Z)$ for some $Z \in X\Downarrow - \bigcup_{i=1}^{\infty} \mathbf{G}^*(i)$, then $Y \in X\Downarrow$.

Here, we note that $X(k) = X$ if $k \geq n$.

Lemma 1.6 *Let* X *and* Y *be sequents in* $\mathbf{G}^+(n)$ *and* $\mathbf{G}(k)$, *respectively. Then*
(1) $n \neq 0$ *implies* $X(n-1) \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ *and* $X \in \mathbf{next}^+(X(n-1))$,
(2) $n > k$ *implies* $X(k) \in \mathbf{G}(k) - \mathbf{G}^*(k)$, $X \in X(k)\Downarrow \cup \mathbf{prov}(X(n-1))$ *and* $\Box\mathbf{for}(X(k)) \in \mathbf{suc}(X)$,
(3) *if* $X \in \mathbf{G}(n)$, *then the following three conditions are equivalent:*
  (3.1) $\mathbf{ant}(Y) \subseteq \mathbf{ant}(X)$ *and* $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$,
  (3.2) $n \geq k$ *and* $Y = X(k)$,
  (3.3) $X \in Y\Downarrow$.

Lemma 1.7 *Let* X *and* Y *be sequents in* $\mathbf{G}(n)$ *and* $\mathbf{G}^*(k)$, *respectively. If* $n \geq k$ *and* $(\mathbf{ant}(X(k)))^\Box = (\mathbf{ant}(Y))^\Box$, *then* $n = k$ *and* $X \in \mathbf{G}^*(n)$.

2 A construction of $\mathbf{ED}^n$ without the provability of S4

[3] constructed the sets $\mathbf{prov}_1(X)$, $\mathbf{prov}_2(X)$, and $\mathbf{prov}_3(X)$ for $X \in \mathbf{G}(n)$ without using the provability of **S4**; and proved that

$$\mathbf{prov}(X) = \mathbf{prov}_1(X) \cup \mathbf{prov}_2(X) \cup \mathbf{prov}_3(X)$$

for $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$. In other words, we obtain a construction of $\mathbf{ED}^n$ without the provability of **S4**. Lemma 3.12 in [3] is a lemma for the above result. In the present section, we provide a detailed proof of it.

Definition 2.1 For any $X \in \mathbf{G}(n)$, we define $\mathbf{prov}_1(X)$, $\mathbf{prov}_2(X)$ and $\mathbf{prov}_3(X)$ as follows:

$\mathbf{prov}_1(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box\}$,

$\mathbf{prov}_2(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(Y),$
$\Box\mathbf{for}(\{Z \in \mathbf{next}(Z_\ominus) \mid (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box\}) \subseteq \Gamma$ for some $Z_\ominus \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)\}$,

$\mathbf{prov}_3(X) = \{(\Box\mathbf{for}(Y), \Gamma \to \Delta, \Box\mathbf{for}(Z)) \in \mathbf{next}^+(X) \mid Y, Z \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(Z))^\Box\}$.

Lemma 2.2 *Let X_n and Y_n be sequents in $\mathbf{G}(n) - \mathbf{G}^*(n)$ and let X_{n+1} be a sequent in $\mathbf{next}^+(X_n) - (\mathbf{prov}_1(X_n) \cup \mathbf{prov}_2(X_n) \cup \mathbf{prov}_3(X_n))$. If $\Box\mathbf{for}(Y_n) \in \mathbf{suc}(X_{n+1})$, then*

$$Y_{n+1} \in \mathbf{next}^+(Y_n) - (\mathbf{prov}_1(Y_n) \cup \mathbf{prov}_2(Y_n) \cup \mathbf{prov}_3(Y_n)),$$

where

$Y_{n+1} = (\Gamma_Y, \mathbf{ant}(X_{n+1}) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y_n) \to \mathbf{suc}(Y_n), \Delta_Y)$,
$\Delta_Y = \{\Box\mathbf{for}(Z_n) \in \mathbf{suc}(X_{n+1}) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid (\mathbf{ant}(Y_n))^\Box \subseteq (\mathbf{ant}(Z_n))^\Box\}$,
$\Gamma_Y = \{\Box\mathbf{for}(Z_n) \in \mathbf{suc}(X_{n+1}) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid (\mathbf{ant}(Y_n))^\Box \not\subseteq (\mathbf{ant}(Z_n))^\Box\}$.

Definition 2.3 The saturation $\mathbf{sat}(X)$ of a sequent $X \in \mathbf{G}^+(n)$ is defined as follows:

$$\mathbf{sat}(X) = \begin{cases} X & \text{if } n = 0 \\ \Gamma_d, \Gamma_c, \mathbf{ant}(X), \{A \mid \Box A \in \mathbf{ant}(X)\} \to \mathbf{suc}(X), \Delta_c, \Delta_d, \Delta_f & \text{if } n > 0, \end{cases}$$

where

$\Gamma_c = \{\bigwedge\Sigma \mid \Sigma \subseteq \mathbf{ant}(X), \Sigma \subseteq \mathbf{BG}(n-1), \#(\Sigma) > 1\}$,
$\Gamma_d = \{\bigvee\Sigma \mid \Sigma \cap \mathbf{ant}(X) \neq \emptyset, \Sigma \subseteq \mathbf{BG}(n-1), \#(\Sigma) > 1\}$,
$\Delta_c = \{\bigwedge\Sigma \mid \Sigma \cap \mathbf{suc}(X) \neq \emptyset, \Sigma \subseteq \mathbf{BG}(n-1), \#(\Sigma) > 1\}$,
$\Delta_d = \{\bigvee\Sigma \mid \Sigma \subseteq \mathbf{suc}(X), \Sigma \subseteq \mathbf{BG}(n-1), \#(\Sigma) > 1\}$,
$\Delta_f = \{\mathbf{for}(X(\ell)) \mid \ell \leq n-1, \mathbf{ant}(X(\ell)) \neq \emptyset\}$.

We note that $\mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$ and $\mathbf{suc}(X) \subseteq \mathbf{suc}(\mathbf{sat}(X))$.

Lemma 2.4 (Lemma 3.12 in [3]) *Let $\mathcal{P}$ be a cut-free proof figure in **S4** whose end sequent is $\Phi \to \Psi$. Then for any $X_n \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and for any $X_{n+1} \in \mathbf{next}^+(X_n) - (\mathbf{prov}_1(X_n) \cup \mathbf{prov}_2(X_n) \cup \mathbf{prov}_3(X_n))$,*

$$(\Phi \to \Psi) \notin \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1})), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_{n+1}))\}.$$

Proof. We give a detailed proof. We use $\Gamma_c, \Gamma_d, \Delta_c, \Delta_d$, and Δ_f as in the above definition. Specifically,

$\Gamma_c = \{\bigwedge\Sigma \mid \Sigma \subseteq \mathbf{ant}(X_{n+1}), \Sigma \subseteq \mathbf{BG}(n), \#(\Sigma) > 1\}$,
$\Gamma_d = \{\bigvee\Sigma \mid \Sigma \cap \mathbf{ant}(X_{n+1}) \neq \emptyset, \Sigma \subseteq \mathbf{BG}(n), \#(\Sigma) > 1\}$,
$\Delta_c = \{\bigwedge\Sigma \mid \Sigma \cap \mathbf{suc}(X_{n+1}) \neq \emptyset, \Sigma \subseteq \mathbf{BG}(n), \#(\Sigma) > 1\}$,
$\Delta_d = \{\bigvee\Sigma \mid \Sigma \subseteq \mathbf{suc}(X_{n+1}), \Sigma \subseteq \mathbf{BG}(n), \#(\Sigma) > 1\}$,
$\Delta_f = \{\mathbf{for}(X(\ell)) \mid \ell \leq n, \mathbf{ant}(X(\ell)) \neq \emptyset\}$.

First, we show

$$\Box\mathbf{for}(X_n(k)) \notin \mathbf{ant}(X_{n+1}) \text{ for any } k \in \{0, \cdots, n\}. \tag{1}$$

By Lemma 1.6 and $X_{n+1}(k+1) \in \mathbf{next}^+((X_{n+1}(k+1))(k)) = \mathbf{next}^+(X_n(k))$, we have $\Box\mathbf{for}(X_n(k)) \in \mathbf{suc}(X_{n+1}(k+1)) \subseteq \mathbf{suc}(X_{n+1})$. Using Lemma 1.3(3), we obtain (1). Also, by Lemma 1.3(3), we have

$$\begin{array}{rl} & \{A_1 \vee A_2 \mid \Box(A_1 \vee A_2) \in \mathbf{ant}(X_{n+1})\} \\ = & \{\bigvee\Sigma \mid \Box\bigvee\Sigma \in \mathbf{ant}(X_{n+1}), \Sigma \subseteq \mathbf{BG}(n), \#(\Sigma) > 1\} \\ \subseteq & \Gamma_d \cup \{\bigvee\Sigma \mid \Box\bigvee\Sigma \in \mathbf{ant}(X_{n+1}), \Sigma \subseteq \mathbf{BG}(n), \#(\Sigma) > 1, \Sigma \cap \mathbf{ant}(X_{n+1}) = \emptyset\} \\ = & \Gamma_d \cup \{\bigvee\Sigma \mid \Box\bigvee\Sigma \in \mathbf{ant}(X_{n+1}), \Sigma \subseteq \mathbf{BG}(n), \#(\Sigma) > 1, \Sigma \subseteq \mathbf{suc}(X_{n+1})\} \\ = & \Gamma_d \cup \{\bigvee\Sigma \mid \Box\bigvee\Sigma = \Box\mathbf{for}(Y(k)) \in \mathbf{ant}(X_{n+1}) \text{ for some } k \in \{0, \cdots, n\}, \#(\Sigma) > 1, \Sigma \subseteq \mathbf{suc}(X_{n+1})\} \\ = & \Gamma_d \cup \{\bigvee\Sigma \mid \Box\bigvee\Sigma = \Box\mathbf{for}(X_n(k)) \in \mathbf{ant}(X_{n+1}) \text{ for some } k \in \{0, \cdots, n\}, \#(\Sigma) > 1\}, \end{array}$$

and using (1), we have

$$\{A_1 \vee A_2 \mid \Box(A_1 \vee A_2) \in \mathbf{ant}(X_{n+1})\} \subseteq \Gamma_d \tag{2}$$

In order to prove the lemma, we use an induction on $\mathcal{P}$.

Basis ($\mathcal{P}$ consists of an axiom). By Lemma 1.3(3), we have $\perp \notin \mathbf{ant}(\mathbf{sat}(X_{n+1}))$. We show $\mathbf{ant}(\mathbf{sat}(X_{n+1})) \cap \mathbf{suc}(\mathbf{sat}(X_{n+1})) = \emptyset$. Suppose that $A \in \mathbf{ant}(\mathbf{sat}(X_{n+1})) \cap \mathbf{suc}(\mathbf{sat}(X_{n+1}))$. We divide the cases.

The case that $A = p_i$. By $p_i \in \mathbf{suc}(\mathbf{sat}(X_{n+1}))$, we have $p_i \in \mathbf{suc}(X_n)$. Using Lemma 1.3(3), we have $p_i \notin \mathbf{ant}(X_n)$. Using $p_i \in \mathbf{ant}(\mathbf{sat}(X_{n+1}))$, we have $\Box p_i \in \mathbf{ant}(X_{n+1})$, and thus, $i = m = 1$. In other words, $\Box\mathbf{for}(X_n(0)) = \Box p_1 \in \mathbf{ant}(X_{n+1})$, which is in contradiction with (1).

The case that $A = A_1 \wedge A_2$. We have $A \in \Gamma_c \cap \Delta_c$, which is in contradiction with Lemma 1.3(3).

The case that $A = A_1 \vee A_2$. By (2), we have $A \in \Gamma_d \cap \Delta_d$, which is in contradiction with Lemma 1.3(3).

The case that $A = A_1 \supset A_2$. By $A \in \mathbf{suc}(\mathbf{sat}(X_{n+1}))$, we have $A = \mathbf{for}(X_n(k)) \in \Delta_f$. Using $A \in \mathbf{ant}(\mathbf{sat}(X_n(k)))$, we have $\Box A = \Box\mathbf{for}(X_n(k)) \in \mathbf{ant}(X_{n+1})$, which is in contradiction with (1).

The case that $A = \Box A_1$. We have $A \in \mathbf{ant}(X_{n+1}) \cap \mathbf{suc}(X_{n+1})$. Using $X_{n+1} \in \mathbf{next}^+(X_n)$, we have $A \in \mathbf{ant}(X_n) \cap \mathbf{suc}(X_n)$, which is in contradiction with Lemma 1.3(3).

Induction step ($\mathcal{P}$ has the inference rule I introducing the end sequent $\Phi \to \Psi$). We define **SAT** as

$$\mathbf{SAT} = \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1})), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_{n+1}))\}$$

and we suppose that

$$(\Phi \to \Psi) \in \mathbf{SAT}.$$

We divide the cases. The case that I is $(\to \Box)$ is shown in [3].[1] We show the other cases.

The case that I is either weakening rules or $(\to\supset)$ is clear.

The case that I is $(\wedge \to)$. I is of the form of

$$\frac{A_i, \Phi' \to \Psi}{A_1 \wedge A_2, \Phi' \to \Psi},$$

where $\{A_1 \wedge A_2\} \cup \Phi' = \Phi$. We note that $A_1 \wedge A_2 \in \Gamma_c$. Therefore, $A_1 \in \Gamma_c \cup \mathbf{ant}(X_{n+1}) \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1}))$ and $A_2 \in \mathbf{ant}(X_{n+1}) \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1}))$. Hence, the upper sequent of I belongs to **SAT**, which is in contradiction with the induction hypothesis.

[1] The condition (6)

$$\mathbf{for}(Y_k) = \mathbf{for}(Y(k)) = \mathbf{for}(Y_{n+1}(k)) \in \mathbf{suc}(\mathbf{sat}(Y_{n+1})) \tag{6}$$

in [3] is shown by

$$\mathbf{ant}(Y_k) \neq \emptyset \Rightarrow \mathbf{for}(Y_k) \in \Delta_f \text{ and } \mathbf{ant}(Y_k) = \emptyset \Rightarrow \mathbf{for}(Y_k) \in \Delta_d.$$

The case that I is $(\to \vee)$. We can show the lemma similarly to the above case.

The case that I is $(\to \wedge)$. I is of the form of

$$\frac{\Phi \to \Psi', A_1 \quad \Phi \to \Psi', A_2}{\Phi \to \Psi', A_1 \wedge A_2},$$

where $\{A_1 \wedge A_2\} \cup \Psi' = \Psi$. We note that $A_1 \wedge A_2 \in \Delta_c$. If $A_2 \in \mathbf{suc}(X_{n+1})$, then the right upper sequent of I belongs to **SAT**, which is in contradiction with the induction hypothesis. If $A_2 \notin \mathbf{suc}(X_{n+1})$, then $A_1 \in \Delta_c \cup \mathbf{suc}(X_{n+1}) \subseteq \mathbf{suc}(\mathbf{sat}(X_{n+1}))$, and hence, the left upper sequent of I belongs to **SAT**, which is in contradiction with the induction hypothesis.

The case that I is $(\to \vee)$. By (2), we can show the lemma similarly to the above case.

The case that I is $(\supset\to)$. I is of the form of

$$\frac{\Phi' \to \Psi, A_1 \quad A_2, \Phi' \to \Psi}{A_1 \supset A_2, \Phi' \to \Psi},$$

where $\{A_1 \supset A_2\} \cup \Phi' = \Phi$. We note that $\Box(A_1 \supset A_2) = \Box\mathbf{for}(Z) \in \mathbf{ant}(X_{n+1})$ for some $Z \in \mathbf{G}(k)$ $(k = 0, 1, \cdots, n)$. By (1), we have $Z \neq X_n(k)$. Using Lemma 1.3(3), we have either $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(X_n(k))$ or $\mathbf{suc}(Z) \not\subseteq \mathbf{suc}(X_n(k))$. If $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(X_n(k))$, then there exists a formula $B \in \mathbf{ant}(Z) \cap \mathbf{suc}(X_n(k))$. Therefore, we have either $A_1 = \bigwedge \mathbf{ant}(Z) \in \Delta_c$ or $A_1 = \bigwedge \mathbf{ant}(Z) = B \in \mathbf{suc}(X_n)$. Hence, the left upper sequent of I belongs to **SAT**, which is in contradiction with the induction hypothesis. If $\mathbf{suc}(Z) \not\subseteq \mathbf{suc}(X_n(k))$, then similarly, we have either $A_2 = \bigvee \mathbf{suc}(Z) \in \Gamma_d$ or $A_2 = \bigvee \mathbf{suc}(Z) \in \mathbf{ant}(X_n)$. Hence, the right upper sequent of I belongs to **SAT**, which is in contradiction with the induction hypothesis. $\dashv$

3 Exact models for $\mathbf{F}^n$

[3] introduced the Kripke model **EM**, an exact set $\mathcal{E}$ for $\mathbf{F}^n$, and the Kripke model $\mathbf{EM}_{\mathcal{E}}$, and proved that **EM** is the exact model for **F** and that the set $\{\mathbf{EM}_{\mathcal{E}} \mid \mathcal{E}$ is an exact set for $\mathbf{F}^n\}$ is the set of exact models for $\mathbf{F}^n$. Also, a finite method to list all exact models for $\mathbf{F}^n$ was given. Lemma 4.14(2) in [3] is a lemma for one of the above result. In the present section, we provide a detailed proof of it.

Definition 3.1 The Kripke model **EM** is defined as

$$\mathbf{EM} = \langle W_{\mathbf{E}}, R_{\mathbf{E}}, P_{\mathbf{E}} \rangle,$$

where $W_{\mathbf{E}} = \bigcup_{n=0}^{\infty} \mathbf{G}^*(n)$, $R_{\mathbf{E}} = \{(X, Y) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ or $X \in \mathbf{clus}(Y)\}$, and $P_{\mathbf{E}}(p_i) = \{X \mid p_i \in \mathbf{ant}(X)\}$.

Definition 3.2

(1) A set $\mathcal{E}$ is said to be exact for $\mathbf{F}^n$ if the following three conditions hold:

(1.1) $\bigcup_{i=0}^{n} \mathbf{G}^*(i) \subseteq \mathcal{E} \subseteq W_{\mathbf{E}}$,

(1.2) for any $X \in \mathbf{ED}^n$, $\#(X\Downarrow \cap \mathcal{E}) = 1$,

(1.3) for any $X \in \mathcal{E}$ and for any $Y \in W_{\mathbf{E}}$, $X R_{\mathbf{E}} Y$ implies $Y \in \mathcal{E}$.

(2) For an exact set $\mathcal{E}$ for $\mathbf{F}^n$, the Kripke model $\mathbf{EM}_{\mathcal{E}}$ is defined as

$$\mathbf{EM}_{\mathcal{E}} = \langle \mathcal{E}, R_{\mathcal{E}}, P_{\mathcal{E}} \rangle,$$

where $R_{\mathcal{E}} = R_{\mathbf{E}} \cap \mathcal{E}^2$ and $P_{\mathcal{E}}(p_i) = P_{\mathbf{E}}(p_i) \cap \mathcal{E}$.

Lemma 4.14(2) in [3] is a lemma for the following lemma.

Lemma 3.3

(1) *For any exact model M for $\mathbf{F}^n$, there exists an exact set $\mathcal{E}$ for $\mathbf{F}^n$ such that M is isomorphic to $\mathbf{EM}_{\mathcal{E}}$.*

(2) *Every exact set for $\mathbf{F}^n$ is a subset of* $\displaystyle\bigcup_{i=0}^{n+2\#(\mathbf{ED}^n - W_{\mathbf{E}})} \mathbf{G}^*(i)$.

Lemma 3.4 (Lemma 4.14 in [3]) *Let $M = \langle W, R, P\rangle$ be an exact model for $\mathbf{F}^n$.*

(1) *There exists one-to-one mapping f from W onto $\mathbf{ED}^n$ such that $P(\mathbf{for}(f(\alpha))) = W - \{\alpha\}$.*

(2) *There exist an exact set $\mathcal{E}$ for $\mathbf{F}^n$ and an isomorphism from M to $\mathbf{EM}_{\mathcal{E}}$.*

Proof. (1) was shown in [3]. We give a detailed proof of (2). First, we note that

(3) R is reflexive and transitive

from $M \models \Box A \supset A$ and $M \models \Box A \supset \Box\Box A$.

For any $\alpha \in W$, we define the set $\mathbf{clus}(\alpha)$ as $\mathbf{clus}(\alpha) = \{\beta \in W \mid \alpha R\beta, \beta R\alpha\}$. By an induction on $k \in \{n, n+1, n+2, \cdots\}$, we define a mapping g_k from W to the set of sequents as

$$g_k(\alpha) = \begin{cases} f(\alpha) & \text{if } k = n \\ g_{k-1}(\alpha) & \text{if } k > n \text{ and } g_{k-1}(\alpha) \in W_{\mathbf{E}} \\ \mathbf{n}(g_{k-1}(\alpha), \{g_{k-1}(\beta) \in \mathbf{G}(k-1) \mid \alpha R\beta\}) & \text{otherwise.} \end{cases}$$

Also, we define the set $\mathcal{E}_k$ as $\mathcal{E}_k = \{g_k(\alpha) \mid \alpha \in W\}$. Then we have the following six conditions:

(2.1) for any $\alpha \in W$, $g_k(\alpha) \in \mathbf{ED}^k$ and $P(\mathbf{for}(g_k(\alpha))) = W - \{\alpha\}$,

(2.2) $\mathbf{ED}^n \cap W_{\mathbf{E}} \subseteq \mathcal{E}_k \subseteq \mathbf{ED}^k$,

(2.3) for any $X \in \mathbf{ED}^n$, $\#(X\Downarrow \cap \mathcal{E}_k) = 1$,

(2.4) for any $\alpha \in W$ and for any $\beta \in \mathbf{clus}(\alpha)$, $(\mathbf{ant}(g_k(\alpha)))^{\Box} = (\mathbf{ant}(g_k(\beta)))^{\Box}$,

(2.5) for any $\alpha \in W$ and for any $X \in W_{\mathbf{E}}$, $g_k(\alpha) \in \mathcal{E}_k \cap W_{\mathbf{E}}$ and $g_k(\alpha) R_{\mathbf{E}} X$ imply $X \in \mathcal{E}_k$,

(2.6) $\mathcal{E}_k \not\subseteq W_{\mathbf{E}}$ implies $\#(\mathcal{E}_k - W_{\mathbf{E}}) > \#(\mathcal{E}_{k+2} - W_{\mathbf{E}})$.

We show (2.1) by an induction on k.

Basis($k = n$). Let α be a world in W. Then by (1), we have

$$g_n(\alpha) = f(\alpha) \in \mathbf{ED}^n \text{ and } P(\mathbf{for}(g_n(\alpha))) = P(\mathbf{for}(f(\alpha))) = W - \{\alpha\}.$$

Induction step($k > n$). Let α be a world in W.

If $g_{k-1}(\alpha) \in W_{\mathbf{E}}$, then by the induction hypothesis,

$$g_k(\alpha) = g_{k-1}(\alpha) \in \mathbf{ED}^{k-1} \cap W_{\mathbf{E}} \subseteq \mathbf{ED}^k$$

and

$$P(\mathbf{for}(g_k(\alpha))) = P(\mathbf{for}(g_{k-1}(\alpha))) = W - \{\alpha\}.$$

We assume that $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$. Then by the induction hypothesis, we have

$$g_{k-1}(\alpha) \in \mathbf{ED}^{k-1} - W_{\mathbf{E}} = \mathbf{G}(k-1) - \mathbf{G}^*(k-1) \tag{2.1.1}$$

and

$$P(\mathbf{for}(g_{k-1}(\beta))) = W - \{\beta\} \text{ for any } \beta \in W. \tag{2.1.2}$$

By (3), we have $\alpha R\alpha$, and using (2.1.1),

$$g_{k-1}(\alpha) \in \{g_{k-1}(\beta) \in \mathbf{G}(k-1) \mid \alpha R\beta\}.$$

Therefore, we have $\Box\mathbf{for}(g_{k-1}(\alpha)) \in \mathbf{suc}(g_k(\alpha))$. Using (2.1.1), we have $g_k(\alpha) \in \mathbf{next}^+(g_{k-1}(\alpha))$, and thus,

$$P(\mathbf{for}(g_k(\alpha))) = W - \{\alpha\} \text{ implies } g_k(\alpha) \in \mathbf{next}(g_{k-1}(\alpha)) \subseteq \mathbf{ED}^k.$$

Therefore, we have only to show

$$P(\mathbf{for}(g_k(\alpha))) = W - \{\alpha\}.$$

To show the above condition, it is sufficient to show the following four conditions:
(2.1.3) for any $\beta \in W - \{\alpha\}$, $(M, \beta) \models \mathbf{for}(g_k(\alpha))$,
(2.1.4) $(M, \alpha) \not\models \mathbf{for}(g_{k-1}(\alpha))$,
(2.1.5) for any $X \in \{g_{k-1}(\beta) \in \mathbf{G}(k-1) \mid \alpha R \beta\}$, $(M, \alpha) \not\models \Box\mathbf{for}(X)$,
(2.1.6) for any $X \in \mathbf{G}(k-1) - \{g_{k-1}(\beta) \in \mathbf{G}(k-1) \mid \alpha R \beta\}$, $(M, \alpha) \models \Box\mathbf{for}(X)$.
(2.1.3) is from (2.1.2) and the condition:

$$(M, \beta) \models \mathbf{for}(g_{k-1}(\alpha)) \text{ implies } (M, \beta) \models \mathbf{for}(g_k(\alpha)).$$

(2.1.4) and (2.1.5) are from (2.1.2).

We show (2.1.6). Suppose that $X \in \mathbf{G}(k-1) - \{g_{k-1}(\beta) \in \mathbf{G}(k-1) \mid \alpha R \beta\}$ and $\gamma \in \{\gamma \in W \mid \alpha R \gamma\}$. We want to show $(M, \gamma) \models \mathbf{for}(X)$. By (2.1.2), we have

$$(M, \gamma) \not\models \mathbf{for}(g_{k-1}(\gamma)). \tag{2.1.7}$$

Also, by the induction hypothesis, we have either

$$g_{k-1}(\gamma) \in \mathbf{G}(k-1) \text{ or } g_{k-1}(\gamma) \in \mathbf{G}^*(\ell) \text{ for some } \ell < k-1.$$

If $g_{k-1}(\gamma) \in \mathbf{G}(k-1)$, then by $X \in \mathbf{G}(k-1) - \{g_{k-1}(\beta) \in \mathbf{G}(k-1) \mid \alpha R \beta\}$ and $\alpha R \gamma$, we have $X \neq g_{k-1}(\gamma)$, and using (2.1.7), $(M, \gamma) \models \mathbf{for}(X)$. We assume that $g_{k-1}(\gamma) \in \mathbf{G}^*(\ell) \subseteq W_{\mathbf{E}}$. Then by $\ell < k-1$, we have $X(\ell) \notin W_{\mathbf{E}}$, and thus, $X(\ell) \neq g_{k-1}(\gamma)$. Using (2.1.7), we have $(M, \gamma) \models \mathbf{for}(X(\ell))$, and hence $(M, \gamma) \models \mathbf{for}(X)$.

We show (2.2). Clearly, $\mathcal{E}_k \subseteq \mathbf{ED}^k$ from (2.1). We show $\mathbf{ED}^n \cap W_{\mathbf{E}} \subseteq \mathcal{E}_k$ by an induction on k.

Basis($k = n$). By (1), we have

$$\mathbf{ED}^n \cap W_{\mathbf{E}} \subseteq \mathbf{ED}^n = \mathcal{E}_n.$$

Induction step($k > n$). By the induction hypothesis,

$$\mathbf{ED}^n \cap W_{\mathbf{E}} \subseteq \mathcal{E}_{k-1} \cap W_{\mathbf{E}}.$$

Also, we note that

$$g_{k-1}(\alpha) \in \mathcal{E}_{k-1} \cap W_{\mathbf{E}} \text{ implies } g_{k-1}(\alpha) = g_k(\alpha) \in \mathcal{E}_k.$$

Hence, we obtain

$$\mathbf{ED}^n \cap W_{\mathbf{E}} \subseteq \mathcal{E}_{k-1} \cap W_{\mathbf{E}} \subseteq \mathcal{E}_k.$$

We show (2.3) by an induction on k.

Basis($k = n$). By (1), we have

$$\text{for any } X \in \mathbf{ED}^n, \#(X\Downarrow \cap \mathcal{E}(n)) = \#(X\Downarrow \cap \mathbf{ED}^n) = \{X\}.$$

Induction step($k > n$). By the induction hypothesis, there exists $\alpha \in W$ such that $X\Downarrow \cap \mathcal{E}(k-1) = \{g_{k-1}(\alpha)\}$. We note that $g_{k-1}(\alpha) \in X\Downarrow$.

We show $X\Downarrow \cap \mathcal{E}(k) \supseteq \{g_k(\alpha)\}$. Clearly, $g_k(\alpha) \in \mathcal{E}(k)$. If $g_{k-1}(\alpha) \in W_{\mathbf{E}}$, then we have $g_k(\alpha) = g_{k-1}(\alpha) \in X\Downarrow$. If $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$, then by (2.2), we have $g_k(\alpha) \in \mathbf{next}(g_{k-1}(\alpha)) \subseteq X\Downarrow$.

We show $X\Downarrow \cap \mathcal{E}(k) \subseteq \{g_k(\alpha)\}$. Suppose that $Y \in X\Downarrow \cap \mathcal{E}(k)$. Then by (2.2), there exists $\beta \in W$ such that $Y = g_k(\beta) \in \mathbf{ED}^k$. If $g_{k-1}(\beta) \notin W_{\mathbf{E}}$, then by (2.2),

$$Y = g_k(\beta) \in \mathbf{next}(g_{k-1}(\beta)),$$

and using $Y \in X\Downarrow$ and Lemma 1.6, we have

$$g_{k-1}(\beta) = Y(k-1) \in X\Downarrow \cap \mathcal{E}(k-1) = \{g_{k-1}(\alpha)\}.$$

If $g_{k-1}(\beta) \in W_{\mathbf{E}}$, then by $Y \in X\Downarrow$, we also have

$$g_{k-1}(\beta) = Y \in X\Downarrow \cap \mathcal{E}(k-1) = \{g_{k-1}(\alpha)\}.$$

Therefore, in any case, we obtain $g_{k-1}(\beta) = g_{k-1}(\alpha)$. Also, by (1) and (2.2), we have

$$\alpha \neq \beta \Rightarrow f(\alpha) \neq f(\beta) \Rightarrow g_n(\alpha) \neq g_n(\beta) \Rightarrow g_{n+1}(\alpha) \neq g_{n+1}(\beta) \Rightarrow \cdots \Rightarrow g_{k-1}(\alpha) \neq g_{k-1}(\beta). \quad (2.3.1)$$

Hence, we obtain $\alpha = \beta$ and $Y = g_k(\beta) \in \{g_k(\alpha)\}$.

We show (2.4). By (2.1), we have $g_k(\alpha), g_k(\beta) \in \mathbf{ED}^k$. Therefore, $g_k(\alpha) \in G(\ell_1)$ and $g_k(\beta) \in G(\ell_2)$ for some $\ell_1, \ell_2 \in \{1, 2, \cdots, k\}$. Without loss of generality, we can assume that $\ell_1 \leq \ell_2$. Then for any $X \in \{X \mid \Box\mathbf{for}(X) \in \mathbf{BG}(\ell_1)\}$,

$$\begin{array}{rcll}
\Box\mathbf{for}(X) \in \mathbf{suc}(g_k(\alpha)) & \Leftrightarrow & (M, \alpha) \not\models \Box\mathbf{for}(X) & \text{by } (2.1) \\
& \Rightarrow & \text{there exists } \gamma \text{ such that } \alpha R \gamma \text{ and } (M, \gamma) \not\models \mathbf{for}(X) & \\
& \Rightarrow & (M, \beta) \not\models \Box\mathbf{for}(X) & \text{by (3) and } \beta R \alpha \\
& \Leftrightarrow & \Box\mathbf{for}(X) \in \mathbf{suc}((g_k(\beta))(\ell_1)) & \text{by } (2.1).
\end{array}$$

Similarly,

$$\Box\mathbf{for}(X) \in \mathbf{suc}((g_k(\beta))(\ell_1)) \Rightarrow \Box\mathbf{for}(X) \in \mathbf{suc}(g_k(\alpha)).$$

Using Lemma 1.3(3), we have

$$(\mathbf{ant}(g_k(\alpha)))^{\Box} = (\mathbf{ant}((g_k(\beta)(\ell_1)))^{\Box}. \quad (2.4.1)$$

If $\ell_1 = k$, then clearly, $\ell_1 = \ell_2$. If $\ell_1 < k$, then we have $g_k(\alpha) \in \mathbf{G}^*(\ell_1)$, and using (2.4.1) and Lemma 1.7, we have $\ell_1 = \ell_2$. Hence, we obtain (2.4). [2]

We show (2.5) by an induction on k. Suppose that
(2.5.1) $\alpha \in W$
(2.5.2) $X \in W_{\mathbf{E}}$,
(2.5.3) $g_k(\alpha) \in \mathcal{E}_k \cap W_{\mathbf{E}}$, and
(2.5.4) $g_k(\alpha) R_{\mathbf{E}} X$.
We want to show $X \in \mathcal{E}_k$. By (2.5.4), we have either
(4a) $\Box\mathbf{for}(X) \in \mathbf{suc}(g_k(\alpha))$ or
(4b) $g_k(\alpha) \in \mathbf{clus}(X)$.

Basis($k = n$). If (4a) holds, then by (2.1), we have $\Box\mathbf{for}(X) \in \mathbf{BG}(n)$, using (2.5.2), we have $X \in \mathbf{ED}^n = \mathcal{E}_n$. If (4b) holds, then by $g_n(\alpha) \in \mathcal{E}_n = \mathbf{ED}^n$, (2.5.2), and Lemma 1.7, we have $X \in \mathbf{ED}^n = \mathcal{E}_n$.

Induction step($k > n$). We divide the cases.

The case that $g_{k-1}(\alpha) \in W_{\mathbf{E}}$. We have $g_{k-1}(\alpha) = g_k(\alpha)$. Using the induction hypothesis, we have $X \in \mathcal{E}_{k-1}$. Therefore, there exists a world $\beta \in W$ such that $X = g_{k-1}(\beta)$, using (2.5.2), we have $X = g_{k-1}(\beta) = g_k(\beta) \in \mathcal{E}_k$.

The case that $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$ and that (4a) holds. We note that for any $i \in \{n+1, n+2, \cdots\}$,

$$g_i(\alpha) \in W_{\mathbf{E}} \text{ implies } g_{i+1}(\alpha) = g_i(\alpha) \in W_{\mathbf{E}}. \quad (2.5.7)$$

Using $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$, none of the sequents in $\{g_{k-1}(\alpha), g_{k-2}(\alpha), \cdots, g_n(\alpha)\}$ belongs to $W_{\mathbf{E}}$. Hence,

$$\begin{array}{rcl}
g_k(\alpha) & = & \mathbf{n}(g_{k-1}(\alpha), \mathcal{S}_{k-1}) \\
& = & \mathbf{n}(\mathbf{n}(g_{k-2}(\alpha), \mathcal{S}_{k-2}), \mathcal{S}_{k-1}) \\
& = & \cdots \\
& = & \mathbf{n}(\mathbf{n}(\cdots \mathbf{n}(g_n(\alpha), \mathcal{S}_n), \cdots \mathcal{S}_{k-2})\mathcal{S}_{k-1}),
\end{array}$$

[2] We can also show (2.4) by an induction on k. Basis($k = n$) can be shown as the case that $k = n$ in the proof of (2.4). Induction step($k > n$). By the induction hypothesis, we have

$$(\mathbf{ant}(g_{k-1}(\alpha)))^{\Box} = (\mathbf{ant}(g_{k-1}(\beta)))^{\Box}. \quad (2.4.2)$$

If both of $g_{k-1}(\alpha)$ and $g_{k-1}(\beta)$ belong to $W_{\mathbf{E}}$, then we obtain (2.4) from (2.4.2). If none of $g_{k-1}(\alpha)$ and $g_{k-1}(\beta)$ belongs to $W_{\mathbf{E}}$, then by (3), $\beta \in \mathbf{clus}(\alpha)$, and the definition of $g_k(\alpha)$, we obtain (2.4). The remaining case is that only one of $g_{k-1}(\alpha)$ and $g_{k-1}(\beta)$ belongs to $W_{\mathbf{E}}$, but this is in contradiction with Lemma 1.7.

where $\mathcal{S}_i = \{g_i(\beta) \in \mathbf{G}(i) \mid \alpha R \beta\}$). Using (4a), we have either $\Box\mathbf{for}(X) \in \mathbf{suc}(g_n(\alpha))$ or $X \in \mathcal{S}_i$ for some $i \in \{n, n+1, \cdots, k-1\}$. If $\Box\mathbf{for}(X) \in \mathbf{suc}(g_n(\alpha))$, then by (2.5.2), $X \in \mathbf{ED}^n = \mathcal{E}_n$. Therefore, in both cases, $X = g_i(\beta)$ for some $i \in \{n, n+1, \cdots, k-1\}$ and $\beta \in W$. Using (2.5.2), we have $X = g_i(\beta) \in W_{\mathbf{E}}$. Also,

$$\begin{array}{rcl} X = g_i(\beta) \in W_{\mathbf{E}} & \Rightarrow & X = g_i(\beta) = g_{i+1}(\beta) \in W_{\mathbf{E}} \\ & \Rightarrow & X = g_i(\beta) = g_{i+1}(\beta) = g_{i+2}(\beta) \in W_{\mathbf{E}} \\ & \Rightarrow & \cdots \\ & \Rightarrow & X = g_i(\beta) = g_{i+1}(\beta) = \cdots = g_{k-1}(\beta) \in W_{\mathbf{E}} \\ & \Rightarrow & X = g_i(\beta) = g_{i+1}(\beta) = \cdots = g_{k-1}(\beta) = g_k(\beta). \end{array}$$

Hence, we obtain $X = g_k(\beta) \in \mathcal{E}_k$.

The case that $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$ and that (4b) holds. This case is shown in [3].

We show (2.6). By (2.3.1),

$$\#(W) = \#(\mathcal{E}_n) = \#(\mathcal{E}_{n+1}) = \#(\mathcal{E}_{n+2}) = \cdots,$$

and using (2.5.7),

$$\#(\mathcal{E}_n - W_{\mathbf{E}}) \geq \#(\mathcal{E}_{n+1} - W_{\mathbf{E}}) \geq \#(\mathcal{E}_{n+2} - W_{\mathbf{E}}) \geq \cdots. \tag{2.6.1}$$

Also, [3] proved that there exists a world $\alpha \in W$ such that

$$g_k(\alpha) \in \mathcal{E}_k - W_{\mathbf{E}} \text{ and } g_{k+2}(\alpha) \in \mathcal{E}_{k+2} \cap W_{\mathbf{E}}.$$

Hence, we obtain (2.6).

By (2.6) and (2.6.1), there exists $\ell \in \{0, 1, \cdots, \#(\mathbf{ED}^n - W_{\mathbf{E}})\}$, such that

$$\begin{array}{rcll} \#(\mathbf{ED}^n - W_{\mathbf{E}}) & = & \#(\mathcal{E}_n - W_{\mathbf{E}}) & \\ & > & \#(\mathcal{E}_{n+2} - W_{\mathbf{E}}) & \\ & > & \#(\mathcal{E}_{n+4} - W_{\mathbf{E}}) & \\ & \vdots & & \\ & > & \#(\mathcal{E}_{n+2\ell} - W_{\mathbf{E}}) & (= 0) \\ & = & \#(\mathcal{E}_{n+2(\ell+1)} - W_{\mathbf{E}}) & (= 0) \\ & \vdots & & \\ & = & \#(\mathcal{E}_{n+2\#(\mathbf{ED}^n - W_{\mathbf{E}})} - W_{\mathbf{E}}) & (= 0). \end{array}$$

We consider the number $\kappa = n + 2\#(\mathbf{ED}^n - W_{\mathbf{E}})$. Then

$$\#(\mathcal{E}_\kappa - W_{\mathbf{E}}) = 0,$$

and using (2.2), we have

$$\mathbf{ED}^n \cap W_{\mathbf{E}} \subseteq \mathcal{E}_\kappa \subseteq W_{\mathbf{E}}. \tag{2.7}$$

Using (2.3) and (2.5), $\mathcal{E}_\kappa$ is an exact set for $\mathbf{F}^n$. Therefore, in order to prove the lemma, it is sufficient to show that g_κ is an isomorphism from M to $\mathbf{EM}_{\mathcal{E}_\kappa}$. Specifically, we have only to show the following three conditions:

(2.8) g_κ is one-to-one and $\{g_\kappa(\alpha) \mid \alpha \in W\} = \mathcal{E}_\kappa$,

(2.9) $\alpha R \beta$ if and only if $g_\kappa(\alpha) R_{\mathcal{E}_\kappa} g_\kappa(\beta)$,

(2.10) $\alpha \in P(p_i)$ if and only if $g_\kappa(\alpha) \in P_{\mathcal{E}_\kappa}(p_i)$.

Clearly, we obtain (2.8) from (2.3.1) and the definition of $\mathcal{E}_k$.

We show (2.9). By (2.7), we have $g_\kappa(\alpha) \in \mathbf{G}^*(\ell_1)$ and $g_\kappa(\beta) \in \mathbf{G}^*(\ell_2)$ for some $\ell_1, \ell_2 \in \{1, 2, \cdots, \kappa\}$. Also, by (2.1), we have

$$(M, \alpha) \not\models g_\kappa(\alpha) \tag{2.9.1}$$

and

$$(M, \beta) \not\models g_\kappa(\beta). \tag{2.9.2}$$

Suppose that $\alpha R\beta$. Then by (2.9.2), we have $(M,\alpha) \not\models \Box\mathbf{for}(g_\kappa(\beta))$. Therefore, if $\ell_1 > \ell_2$, then by (2.9.1), we have $\Box\mathbf{for}(g_\kappa(\beta)) \in \mathbf{suc}(g_\kappa(\alpha))$, and hence $g_\kappa(\alpha)R_{\mathcal{E}_\kappa}g_\kappa(\beta)$. We assume that $\ell_1 \leq \ell_2$. Then

$$\begin{array}{llll}\Box\mathbf{for}(X) \in \mathbf{ant}(g_\kappa(\alpha)) & \Rightarrow & (M,\alpha) \models \Box\mathbf{for}(X) \text{ and } \Box\mathbf{for}(X) \in \mathbf{BG}(\ell_1) & \text{by (2.9.1)} \\ & \Rightarrow & (M,\beta) \models \Box\mathbf{for}(X) \text{ and } \Box\mathbf{for}(X) \in \mathbf{BG}(\ell_1) & \text{by } \alpha R\beta \text{ and (3)} \\ & \Rightarrow & \Box\mathbf{for}(X) \in \mathbf{ant}(g_\kappa(\beta)) & \text{by (2.9.2) and } \ell_1 \leq \ell_2.\end{array}$$

Therefore, we have $(\mathbf{ant}(g_\kappa(\alpha)))^\Box \subseteq (\mathbf{ant}(g_\kappa(\beta)))^\Box$. Using $g_\kappa(\alpha) \in G^*(\ell_1)$, we have $(\mathbf{ant}(g_\kappa(\alpha)))^\Box = (\mathbf{ant}(g_\kappa(\beta)(\ell_1)))^\Box$, and using Lemma 1.7, we have $\ell_1 = \ell_2$ and $(\mathbf{ant}(g_\kappa(\alpha)))^\Box = (\mathbf{ant}(g_\kappa(\beta)))^\Box$. Hence, we obtain $g_\kappa(\alpha)R_{\mathcal{E}_\kappa}g_\kappa(\beta)$.

Suppose that $g_\kappa(\alpha)R_{\mathcal{E}(\kappa)}g_\kappa(\beta)$. If $\Box\mathbf{for}(g_\kappa(\beta)) \in \mathbf{suc}(g_\kappa(\alpha))$, then by (2.9.1), we have $(M,\alpha) \not\models \Box\mathbf{for}(g_\kappa(\beta))$. If $\ell_1 = \ell_2$ and $(\mathbf{ant}(g_\kappa(\alpha)))^\Box = (\mathbf{ant}(g_\kappa(\beta)))^\Box$, then by (2.9.1) and Lemma 1.3(5), we have $(M,\alpha) \not\models \Box\mathbf{for}(g_\kappa(\beta))$. Hence, in both cases, we have $(M,\alpha) \not\models \Box\mathbf{for}(g_\kappa(\beta))$, and using (2.1), we have $\alpha R\beta$.

We show (2.10). By (2.1), we have

$$\alpha \in P(p_i) \text{ if and only if } p_i \in \mathbf{ant}(g_\kappa(\alpha)).$$

Hence, we obtain (2.10). $\dashv$

4 The structure $\langle \mathbf{F}/\equiv, \leq \rangle$

[3] introduced the set **CNF** and a detailed description of the structure $\langle \mathbf{F}/\equiv, \leq \rangle$, where $[A] \leq [B] \Leftrightarrow A \to B \in \mathbf{S4}$. Theorem 5.4(2) in [3] is one of the description of the structure. In the present section, we provide a detailed proof of it.

Definition 4.1 We define the set **CNF**, and for any $\mathcal{S} \in 2^{\mathbf{ED}^n}$, we define $\mathbf{cnf}(\mathcal{S})$ and $\mathcal{S}\Downarrow$ as follows.

(1) $\mathbf{CNF}_k = \begin{cases} 2^{\mathbf{G}(0)} & \text{if } k = 0 \\ \{\mathcal{S} \in 2^{\mathbf{ED}^k} \mid \mathbf{next}(X) \not\subseteq \mathcal{S}, \mathbf{next}(X) \cap \mathcal{S} \neq \emptyset, \text{ for some } X \in \mathbf{G}(k-1) - W_{\mathbf{E}}\} & \text{if } k > 0, \end{cases}$

(2) $\mathbf{CNF} = \bigcup_{i=0}^{\infty} \mathbf{CNF}_i$,

(3) $\mathbf{cnf}(\mathcal{S}) = \begin{cases} \mathcal{S} & \text{if } \mathcal{S} \in \mathbf{CNF} \\ \mathbf{cnf}(\{X \in \mathbf{G}(n-1) \mid \mathbf{next}(X) \subseteq \mathcal{S}\}) & \text{if } \mathcal{S} \notin \mathbf{CNF}, \end{cases}$

(4) $\mathcal{S}\Downarrow = \bigcup_{X \in \mathcal{S}} (X\Downarrow)$.

Lemma 4.2 *For any* $\mathcal{S} \in 2^{\mathbf{ED}^n}$,

$$\bigwedge \mathbf{for}(\mathcal{S}) \equiv \bigwedge \mathbf{for}(\mathbf{cnf}(\mathcal{S})).$$

Theorem 4.3 (Theorem 5.4(2) in [3]) *For any* $\mathcal{S}_1 \in \mathbf{CNF}_\ell$ *and for any* $\mathcal{S}_2 \in \mathbf{CNF}_k$,

(1) $\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}$ *if and only if either* $\mathcal{S}_1 \subseteq \mathcal{S}_2\Downarrow$ *or both* $\mathcal{S}_1\Downarrow \cap \mathbf{ED}^k \subseteq \mathcal{S}_2$ *and* $\ell \leq k$,

(2) $\bigwedge \mathbf{for}(\mathcal{S}_1) \equiv \bigwedge \mathbf{for}(\mathcal{S}_2)$ *if and only if* $\mathcal{S}_1 = \mathcal{S}_2$.

Proof.

For (1). We divide the cases.

The case that $\ell \leq k$. By $\ell \leq k$, we have

$$\mathbf{cnf}(\mathcal{S}_1\Downarrow \cap \mathbf{ED}^k) = \mathcal{S}_1.$$

Using Lemma 4.2, we have

$$\bigwedge \mathbf{for}(\mathcal{S}_1\Downarrow \cap \mathbf{ED}^k) \equiv \bigwedge \mathbf{for}(\mathbf{cnf}(\mathcal{S}_1\Downarrow \cap \mathbf{ED}^k)) = \bigwedge \mathbf{for}(\mathcal{S}_1),$$

and using Lemma 1.4, we have

$$\begin{aligned} \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4} \quad &\Leftrightarrow \quad \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1 \Downarrow \cap \mathbf{ED}^k) \in \mathbf{S4} \\ &\Leftrightarrow \quad \mathcal{S}_1 \Downarrow \cap \mathbf{ED}^k \subseteq \mathcal{S}_2. \end{aligned} \tag{1.1}$$

Hence, we obtain the "only if" part. We also obtain the "if" part by (1.1) and

$$\begin{aligned} \mathcal{S}_1 \subseteq \mathcal{S}_2 \Downarrow \quad &\Rightarrow \quad \mathcal{S}_1 \Downarrow \subseteq (\mathcal{S}_2 \Downarrow) \Downarrow = \mathcal{S}_2 \Downarrow \\ &\Rightarrow \quad \mathcal{S}_1 \Downarrow \cap \mathbf{ED}^k \subseteq \mathcal{S}_2 \Downarrow \cap \mathbf{ED}^k = \mathcal{S}_2. \end{aligned}$$

The case that $\ell > k$. Similarly to the above case, we obtain

$$\begin{aligned} \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4} \quad &\Leftrightarrow \quad \bigwedge \mathbf{for}(\mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4} \\ &\Leftrightarrow \quad \mathcal{S}_1 \subseteq \mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell \\ &\Leftrightarrow \quad \mathcal{S}_1 \subseteq \mathcal{S}_2 \Downarrow . \end{aligned}$$

For (2). The "if" part is clear. We show the "only if" part. Suppose that $\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}$. Then by (1), one of the following four conditions holds:
(2.1) $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Downarrow$ and $\mathcal{S}_2 \subseteq \mathcal{S}_1 \Downarrow$,
(2.2) $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Downarrow$, $\mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell \subseteq \mathcal{S}_1$, and $\ell \geq k$,
(2.3) $\mathcal{S}_1 \Downarrow \cap \mathbf{ED}^k \subseteq \mathcal{S}_2$, $\ell \leq k$, and $\mathcal{S}_2 \subseteq \mathcal{S}_1 \Downarrow$,
(2.4) $\mathcal{S}_1 \Downarrow \cap \mathbf{ED}^k \subseteq \mathcal{S}_2$, $\mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell \subseteq \mathcal{S}_1$, and $\ell = k$.
We divide the cases.

The case that (2.1) holds. By $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Downarrow$, we have $\ell \geq k$; and by $\mathcal{S}_2 \subseteq \mathcal{S}_1 \Downarrow$, we have $\ell \leq k$. Therefore, we have $\ell = k$, and using Lemma 1.4, we obtain $\mathcal{S}_1 = \mathcal{S}_2$.

The case that (2.2) holds. By $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Downarrow$, we have $\mathcal{S}_1 \cap \mathbf{ED}^\ell \subseteq \mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell$. Using $\mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell \subseteq \mathcal{S}_1$,

$$\mathcal{S}_1 = \mathcal{S}_1 \cap \mathbf{ED}^\ell \subseteq \mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell \subseteq \mathcal{S}_1.$$

Therefore, we have $\mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell = \mathcal{S}_1$. Using $\ell \geq k$,

$$\mathcal{S}_2 = \mathbf{cnf}(\mathcal{S}_2 \Downarrow \cap \mathbf{ED}^\ell) = \mathbf{cnf}(\mathcal{S}_1) = \mathcal{S}_1.$$

The case that (2.3) holds can be shown similarly to the above case.

The case that (2.4) holds. By $\ell = k$ and Lemma 1.4, we obtain $\mathcal{S}_1 = \mathcal{S}_2$. $\dashv$

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