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Abstract. In [Sas10], two constructions of normal forms in modal logic **S4** were given. The first construction depends on **S4**-provability, but the second one does not. Here, we extend the first construction to normal modal logics containing the modal logic **K4**. We also modify the second construction and give a construction of normal forms in **K4**, which does not depend on **K4**-provability.

1 Introduction

In the present section, we introduce formulas and normal modal logics, and also describe the purpose of the present paper.

1.1 Formulas

Formulas are constructed from $\bot$ (contradiction) and the propositional variables $p_1, p_2, \cdots$ by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and $\Box$ (necessitation). We use upper case Latin letters, $A, B, C, \cdots$, with or without subscripts, for formulas. Also, we use Greek letters, $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. The *depth* $d(A)$ of a formula A is defined as

$d(p_i) = d(\bot) = 0$,
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
$d(\Box B) = d(B) + 1$.

Let **ENU** be an enumeration of the formulas. For a non-empty finite set Γ of formulas, the expressions $\bigwedge \Gamma$ and $\bigvee \Gamma$ denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \quad \text{and} \quad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

respectively, where $\{A_1, \cdots, A_n\} = \Gamma$ and A_i occurs earlier than A_{i+1} in **ENU**. Also, the expressions $\bigwedge \emptyset$ and $\bigvee \emptyset$ denote the formulas $\bot \supset \bot$ and $\bot$, respectively.

The set of propositional variables $p_1, \cdots, p_m$ ($m \geq 1$) is denoted by **V** and the set of formulas constructed from **V** and $\bot$ is denoted by **F**. Also, for any $n = 0, 1, \cdots$, we define $\mathbf{F}(n)$ as $\mathbf{F}(n) = \{A \in \mathbf{F} \mid d(A) \leq n\}$. In the present paper, we treat the set $\mathbf{F}(n)$.

1.2 Normal modal logics

A normal modal logic is a set of formulas containing all tautologies and the axiom

$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$

and closed under modus ponens, substitution, and necessitation ($A/\Box A$). By **K4**, we mean the smallest normal modal logic containing the axiom

$4 : \Box p \supset \Box\Box p$.

For a normal modal logic L, we use $A \equiv_L B$ instead of $(A \supset B) \wedge (B \supset A) \in L$.

In order to treat normal modal logics, we use sequent systems obtained by adding axioms or inference rules to the sequent system **LK** given by Gentzen [Gen35].

A *sequent* is the expression $(\Gamma \to \Delta)$. We often refer to $\Gamma \to \Delta$ as $(\Gamma \to \Delta)$ for brevity and refer to

$$A_1, \cdots, A_i, \Gamma_1, \cdots, \Gamma_j \to \Delta_1, \cdots, \Delta_k, B_1, \cdots, B_\ell$$

as

$$\{A_1, \cdots, A_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \cdots, B_\ell\}.$$

We use upper case Latin letters $X, Y, Z, \cdots$, with or without subscripts, for sequents. If $X = (\Gamma \to \Delta)$, then we sometimes refer to $\Gamma \overset{X}{\to} \Delta$ as $\Gamma \to \Delta$. The *antecedent* $\mathbf{ant}(\Gamma \to \Delta)$ and the *succedent* $\mathbf{suc}(\Gamma \to \Delta)$ of a sequent $\Gamma \to \Delta$ are defined as

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \to \Delta) = \Delta,$$

respectively. Also, for a sequent X and a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) \neq \emptyset \\ \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) = \emptyset \end{cases}$$

and

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

Here, we do not use $\neg$ as a primary connective, so we use the additional axiom $\bot \to$ instead of the inference rules $(\neg \to)$ and $(\to \neg)$.

For a sequent system L and for a sequent X, we write $X \in L$ if X is provable in L.

It is known that a sequent system for **K4** is obtained by adding the inference rule

$$\frac{\Gamma, \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\Box)$$

to **LK**. In other words,

$$\to A \text{ is provable in the above system if and only if } A \in \mathbf{K4}.$$

Therefore, we can identify the above system with **K4** and call the above system **K4**. The system **K4** enjoys cut-elimination theorem:

Lemma 1.1 *If $X \in \mathbf{K4}$, then there exists a cut-free proof figure for X in* **K4**.

Also, for any sequent system L, the following two conditions are equivalent:

- L is a sequent system for a normal modal logic containing **K4**,
- L satisfies the inference rule $(\Box)$;

and thus, we treat a sequent system satisfying the second conditions above as a normal modal logic containing **K4**.

1.3 The purpose

Let L be a normal modal logic containing **K4**. The purpose of the paper is to construct normal forms in L, which behave like an elementary disjunctions

$$p_1^* \vee \cdots \vee p_m^* \qquad (p_i^* \in \{p_i, p_i \supset \bot\})$$

in the classical propositional logic. Specifically, we do the following tasks.

(I) We construct a finite set $\mathbf{ED}_L(n)$ of sequents satisfying

  (I-1) $\mathbf{F}(n)/\equiv_L = \{[\bigwedge \mathbf{for}(\mathcal{S})] \mid \mathcal{S} \subseteq \mathbf{ED}_L(n)\}$,

  (I-2) for any subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}_L(n)$, $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in L$.

(II) For a formula $A \in \mathbf{F}(n)$, we give a finite way to find a subset $\mathcal{S}$ of $\mathbf{ED}_L(n)$ in (I) such that $A \equiv_L \bigwedge \mathbf{for}(\mathcal{S})$,

(III) Without using **K4**-provability, we give the set $\mathbf{ED}_{\mathbf{K4}}(n)$ in (I).

Two tasks (I) and (II) prove that a member of $\mathbf{ED}(n)$ and a formula $\bigwedge \mathbf{for}(\mathcal{S})$ behave like an elementary disjunction

$$p_1^* \vee \cdots \vee p_m^* \qquad (p_i^* \in \{p_i, p_i \supset \bot\})$$

and a principal conjunctive normal form in the classical propositional logic, respectively.

In [Sas10], Tasks (I) and (III) have been done in the case $L = \mathbf{S4}$. We extend it to other normal modal logics containing $\mathbf{K4}$. In [Sas10], Task (III) for $\mathbf{S4}$ has also been done. For the other earlier works, we can consult [Sas10].

2 A construction of $\mathbf{ED}(n)$

In the present section, we construct the set $\mathbf{ED}_L(n)$ satisfying the conditions (I-1) and (I-2) in subsection 1.3 for a normal modal logic L containing $\mathbf{K4}$.

Definition 2.1 Let L be a normal modal logic containing $\mathbf{K4}$. The sets $\mathbf{G}_L(n)$ and $\mathbf{G}^*_L(n)$ of sequents are defined inductively as follows.

$\mathbf{G}_L(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,

$\mathbf{G}^*_L(0) = \emptyset$,

$\mathbf{G}_L(k+1) = \displaystyle\bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}_L(X)$,

$\mathbf{G}^*_L(k+1) = \{X \in \mathbf{G}_L(k+1) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y))^\Box$ implies $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, for any $Y \in \mathbf{G}_L(k+1)\}$,

where for any $X \in \mathbf{G}_L(k)$,

$\mathbf{next}^+_L(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}_L(k)), \Gamma \cap \Delta = \emptyset\}$,

$\mathbf{prov}_L(X) = \{Y \in \mathbf{next}^+_L(X) \mid Y \in L\}$,

$\mathbf{next}_L(X) = \mathbf{next}^+_L(X) - \mathbf{prov}_L(X)$.

Definition 2.2 We define the sets $\mathbf{ED}_L(n)$ and $\mathbf{G}^+_L(n)$ as

$$\mathbf{ED}_L(n) = \mathbf{G}_L(n) \cup \bigcup_{i=0}^{n-1} \mathbf{G}^*_L(i),$$

$$\mathbf{G}^+_L(n) = \begin{cases} \mathbf{G}_L(0) & \text{if } n = 0 \\ \displaystyle\bigcup_{X \in \mathbf{G}_L(n-1) - \mathbf{G}^*_L(n-1)} \mathbf{next}^+_L(X) & \text{if } n > 0, \end{cases}$$

Let X be a sequent in $\mathbf{G}^+_L(n+1)$. Then there exists only one sequent $Y \in \mathbf{G}_L(n) - \mathbf{G}^*_L(n)$ such that $X \in \mathbf{next}^+_L(Y)$. We refer to $X_\ominus$ as this sequent Y. We note that $X_\ominus \in \mathbf{G}_L(n) - \mathbf{G}^*_L(n)$ and $X \in \mathbf{next}_L(X_\ominus)$.

Definition 2.3 We define the sets $\mathbf{G}^\circ_L(n)$ and $\mathbf{G}^\bullet_L(n)$ as

$$\mathbf{G}^\circ_L(n) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{X \in \mathbf{G}^*_L(n) \mid \Box\mathbf{for}(X_\ominus) \in \mathbf{suc}(X)\} & \text{if } n > 0, \end{cases}$$

$$\mathbf{G}^\bullet_L(n) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{X \in \mathbf{G}^*_L(n) \mid \Box\mathbf{for}(X_\ominus) \in \mathbf{ant}(X)\} & \text{if } n > 0. \end{cases}$$

If there is no confusion, we omit the subscript L from $\mathbf{G}_L(n)$, $\mathbf{ED}_L(n)$, $\equiv_L$, and so on.

Example 2.4 We list members of $\mathbf{G}(n)$, $\mathbf{G}^\bullet(n+1)$, and $\mathbf{G}^\circ(n+1)$ in the case that $L = \mathbf{K4}$, $m = 1$ and $n = 0, 1$. We use $(\quad)^\bullet$ and $(\quad)^\circ$ for a sequent in $\mathbf{G}^\bullet(n)$. $\mathbf{G}^\circ(n)$.

$\mathbf{G}(0) = \{T, F\}$,

$\mathbf{next}^+(T) = \mathbf{next}(T) = \{T1^\bullet, T2, T3, T4\}$, $\quad \mathbf{next}^+(F) = \mathbf{next}(F) = \{F1^\bullet, F2, F3, F4\}$,

$\mathbf{G}(1) = \{T1^\bullet, T2, T3, T4, F1^\bullet, F2, F3, F4\}$,

$\mathbf{G}^\bullet(1) = \{T1^\bullet, F1^\bullet\}$, $\quad \mathbf{G}^\circ(1) = \emptyset$,

$\mathbf{next}(T2) = \{T2.1^\bullet, T2.2^\bullet, T2.3\}$,
$\mathbf{next}(T3) = \{T3.1^\bullet, T3.2^\circ, T3.3\}$,
$\mathbf{next}(T4) \ni T4.1^\circ$,
$\mathbf{next}(F2) = \{F2.1^\bullet, F2.2^\circ, F2.3\}$,
$\mathbf{next}(F3) = \{F3.1^\bullet, F3.2^\bullet, F3.3\}$,
$\mathbf{next}(F4) \ni F4.1^\circ$,
$\mathbf{G}^\bullet(2) = \{T2.1^\bullet, T2.2^\bullet, T3.1^\bullet, F2.1^\bullet, F3.1^\bullet, F3.2^\bullet\}$,
$\mathbf{G}^\circ(2) = \{T3.2^\circ, T4.1^\circ, F2.2^\circ, F4.1^\circ\}$,
where

$$
\begin{array}{lcrcl}
T & = & (p_1 & \to &), \\
F & = & (& \to & p_1), \\
T1^\bullet & = & (\Box\mathbf{for}(\{T,F\}), \mathbf{ant}(T) & \to & \mathbf{suc}(T)), \\
T2 & = & (\Box\mathbf{for}(T), \mathbf{ant}(T) & \to & \mathbf{suc}(T), \Box\mathbf{for}(F)), \\
T3 & = & (\Box\mathbf{for}(F), \mathbf{ant}(T) & \to & \mathbf{suc}(T), \Box\mathbf{for}(T)), \\
T4 & = & (\mathbf{ant}(T) & \to & \mathbf{suc}(T), \Box\mathbf{for}(\{T,F\})), \\
F1^\bullet & = & (\Box\mathbf{for}(\{T,F\}), \mathbf{ant}(F) & \to & \mathbf{suc}(F)), \\
F2 & = & (\Box\mathbf{for}(T), \mathbf{ant}(F) & \to & \mathbf{suc}(F), \Box\mathbf{for}(F)), \\
F3 & = & (\Box\mathbf{for}(F), \mathbf{ant}(F) & \to & \mathbf{suc}(F), \Box\mathbf{for}(T)), \\
F4 & = & (\mathbf{ant}(F) & \to & \mathbf{suc}(F), \Box\mathbf{for}(\{T,F\})), \\
T2.1^\bullet & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{F1^\bullet\}), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(F1^\bullet)), \\
T2.2^\bullet & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{F2\}), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(F2)), \\
T2.3 & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{F1^\bullet, F2\}), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(\{F1^\bullet, F2\})), \\
T3.1^\bullet & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{T1^\bullet\}), \mathbf{ant}(T3) & \to & \mathbf{suc}(T3), \Box\mathbf{for}(T1^\bullet)), \\
T3.2^\circ & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{T3\}), \mathbf{ant}(T3) & \to & \mathbf{suc}(T3), \Box\mathbf{for}(T3)), \\
T3.3 & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{\{T1^\bullet, T3\}), \mathbf{ant}(T3) & \to & \mathbf{suc}(T3), \Box\mathbf{for}(\{T1^\bullet, T3\})), \\
T4.1^\circ & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{\{T4, F4\}\}), \mathbf{ant}(T4) & \to & \mathbf{suc}(T4), \Box\mathbf{for}(\{T4, F4\})), \\
F2.1^\bullet & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{F1^\bullet\}), \mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(F1^\bullet)), \\
F2.2^\circ & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{F2\}), \mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(F2)), \\
F2.3 & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{F1^\bullet, F2\}), \mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(\{F1^\bullet, F2\})), \\
F3.1^\bullet & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{T1^\bullet\}), \mathbf{ant}(F3) & \to & \mathbf{suc}(F3), \Box\mathbf{for}(T1^\bullet)), \\
F3.2^\bullet & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{T3\}), \mathbf{ant}(F3) & \to & \mathbf{suc}(F3), \Box\mathbf{for}(T3)), \\
F3.3 & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{T1^\bullet, T2\}), \mathbf{ant}(F3) & \to & \mathbf{suc}(F3), \Box\mathbf{for}(\{T1^\bullet, T3\})), \\
F4.1^\circ & = & (\Box\mathbf{for}(\mathbf{G}(1) - \{\{T4, F4\}\}), \mathbf{ant}(F4) & \to & \mathbf{suc}(F4), \Box\mathbf{for}(\{T4, F4\})).
\end{array}
$$

By an induction on n, we can show the following lemma.

Lemma 2.5
(1) *None of the members in* $\mathbf{G}(n)$ *is provable in* L.
(2) *For any* $X, Y \in \mathbf{ED}(n)$, $X \neq Y$ *implies* $\mathbf{for}(X) \vee \mathbf{for}(Y) \in L$.
(3) *For any* $X \in \mathbf{G}^+(n)$, $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i))$ *and* $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$.

The main purpose in the present section is to prove the following two theorems.

Theorem 2.6
(1) $\bot \equiv \bigwedge \mathbf{for}(\mathbf{ED}(n))$.
(2) $p_i \equiv \bigwedge \mathbf{for}(\{X \in \mathbf{ED}(n) \mid p_i \in \mathbf{suc}(X)\})$.
(3) *For any subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathbf{ED}(n)$,
  (3.1) $\bigwedge \mathbf{for}(\mathcal{S}_1) \wedge \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}(\mathcal{S}_1 \cup \mathcal{S}_2)$,
  (3.2) $\bigwedge \mathbf{for}(\mathcal{S}_1) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}(\mathcal{S}_1 \cap \mathcal{S}_2)$,
  (3.3) $\bigwedge \mathbf{for}(\mathcal{S}_1) \supset \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}((\mathbf{ED}(n) - \mathcal{S}_1) \cap \mathcal{S}_2)$.
(4) *For any subset* $\mathcal{S}$ *of* $\mathbf{ED}(k)$, $\Box \bigwedge \mathbf{for}(\mathcal{S}) \equiv \bigwedge \mathbf{for}(\mathcal{S}_1 \cup \mathcal{S}_2)$, *where*

$$\mathcal{S}_1 = \bigcup_{X \in \mathcal{S}} \{Y \in \mathbf{ED}(k+1) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y)\},$$

$$\mathcal{S}_2 = \bigcup_{i=1}^{k} \bigcup_{X \in \mathcal{S} \cap \mathbf{G}^\circ(i)} \{Y \in \mathbf{G}^*(i) \mid (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box\}.$$

We note that $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathbf{ED}(k+1)$ for $\mathcal{S}_1$ and $\mathcal{S}_2$ in the above (4).

Theorem 2.7
(1) $\mathbf{F}(n)/\equiv = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{ED}(n)\}$.
(2) *For subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathbf{ED}(n)$,

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \textit{ if and only if } \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in L.$$

By the above theorem, the conditions (I-1) and (I-2) in subsection 1.3 are shown. Theorem 2.6 provide a finite way described in (II) in subsection 1.3.

Theorem 2.7(1) and Theorem 2.7(2) can be shown by Theorem 2.6 and Lemma 2.5, respectively. In order to prove Theorem 2.6, especially (4), we need some lemmas.

Lemma 2.8 *Let* Σ, Γ *and* Δ *be finite sets of formulas. Then*

$$\{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L.$$

Proof. We use an induction on the number $\#(\Sigma)$ of elements in Σ. If $\Sigma = \emptyset$, then the lemma is clear from

$$\{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\} = \{\mathbf{for}(\Gamma \to \Delta)\}.$$

Suppose that $A \in \Sigma$. Then by the induction hypothesis,

$$\{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma - \{A\}, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L.$$

Therefore,

$$\Box A, \{\mathbf{for}(\Box\Phi, \Box A, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma - \{A\}, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L$$

and

$$\Box A \supset \bot, \{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box A, \Box\Psi) \mid \Phi \cup \Psi = \Sigma - \{A\}, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L.$$

Hence,

$$\Box A \vee (\Box A \supset \bot), \{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}, \Gamma \to \Delta \in L.$$

Since $\Box A \vee (\Box A \supset \bot) \in L$, we obtain the lemma. ⊣

Corollary 2.9 *For any* $X, Y \in \mathbf{G}(n)$,
(1) $\mathbf{for}(\mathbf{next}(X)) \to \mathbf{for}(X) \in L$,
(2) $\bigwedge \mathbf{for}(\mathbf{next}(X)) \equiv_L \mathbf{for}(X)$,
(3) $\{\mathbf{for}(Z) \mid Z \in \mathbf{next}(X), \Box\mathbf{for}(Y) \in \mathbf{suc}(Z)\}, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y) \in L$.

Proof. We can show (1) and (3) by considering the case that $(\Sigma, \Gamma, \Delta) = (\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(X), \mathbf{suc}(X))$ and the case that $(\Sigma, \Gamma, \Delta) = (\mathbf{for}(\mathbf{G}(n) - \{Y\}), \mathbf{ant}(X), \mathbf{suc}(X) \cup \{\Box\mathbf{for}(Y)\})$ in Lemma 2.8, respectively. (2) is clear from (1). ⊣

Lemma 2.10 *Let* X *and* Y *be sequents in* $\mathbf{G}(n)$ *satisfying* $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$. *Then*
(1) $(\to \mathbf{for}(X), \Box\mathbf{for}(Y)) \in L$,
(2) *for any* $X_\oplus \in \mathbf{next}^+(X)$, $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_\oplus)$ *implies* $X_\oplus \in L$.

Proof. [Sas10] proved the lemma in the case $L = \mathbf{S4}$. The other cases can also be shown similarly. $\dashv$

Lemma 2.11 *Let X and Y be sequents in $\mathbf{G}(n)$ satisfying $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$. Then*
(1) *$X \in \mathbf{G}^*(n)$ if and only if $Y \in \mathbf{G}^*(n)$,*
(2) *$Y \in \mathbf{G}^{\circ}(n)$ implies $\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in L$.*

Proof. (1) is clear from the definition of $\mathbf{G}^*(n)$. We show (2). By Corollary 2.9(1) and ($\Box$),

$$\Box\mathbf{for}(\mathbf{next}(Y_{\ominus})) \to \Box\mathbf{for}(Y_{\ominus}) \in L.$$

By $Y \in \mathbf{G}^{\circ}(n)$, $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$, and Lemma 2.5(3), we have $\Box\mathbf{for}(Y_{\ominus}) \in (\mathbf{suc}(Y))^{\Box} = (\mathbf{suc}(X))^{\Box}$. Therefore,

$$\Box\mathbf{for}(\mathbf{next}(Y_{\ominus})) \to \mathbf{for}(X) \in L.$$

From this, we can show the lemma, similarly to Lemma 2.11 in [Sas10]. $\dashv$

Definition 2.12 For any $X \in \mathbf{G}^*(n)$ $(n > 0)$, we define the set $\mathbf{pclus}(X)$ as

$$\mathbf{pclus}(X) = \{Y \in \mathbf{G}(n) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}\}.$$

Lemma 2.13 *For any $X \in \mathbf{G}(n)$,*
(1) *$\Phi(X) \to \Box\mathbf{for}(X) \in L$,*
(2) *$X \notin \mathbf{G}^*(n)$ implies $(\Phi^{\triangle}(X) \to \Box\mathbf{for}(X)) \in L$,*
(3) *$X \in \mathbf{G}^*(n)$ implies $(\mathbf{for}(\mathbf{pclus}(X)), \Phi^{\triangle}(X) \to \Box\mathbf{for}(X)) \in L$,*
(4) *$X \in \mathbf{G}^{\circ}(n)$ implies $(\mathbf{for}(\mathbf{pclus}(X)), \Phi^{\triangle}(X) \to \Box\mathbf{for}(X)) \in L$,*
(5) *$X \in \mathbf{G}^{\bullet}(n)$ implies $\Phi^{\triangle}(X) \to \Box\mathbf{for}(X)) \in L$,*
where $\Phi(X) = \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(X))^{\Box}\}$ and $\Phi^{\triangle}(X) = \Phi(X) - \mathbf{G}^*(n)$.

Proof. We first show that
(6) $X \notin \mathbf{G}^*(n)$ implies $\Phi(X) = \Phi^{\triangle}(X)$.
Suppose that $\mathbf{for}(Y) \in \Phi(X)$. Then we have $(\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(X))^{\Box}$. Using $X \notin \mathbf{G}^*(n)$, we have $Y \notin \mathbf{G}^*(n)$. Hence, we obtain (6).

We show (1) by an induction on n.

Basis ($n = 0$) is clear from $\Phi(X) = \mathbf{for}(\mathbf{G}(0))$ and $\mathbf{for}(\mathbf{G}(0)) \to \in L$.

Induction step ($n > 0$). By the induction hypothesis,

$$\Phi(X_{\ominus}) \to \Box\mathbf{for}(X_{\ominus}) \in L.$$

Using $X_{\ominus} \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ and (6),

$$\Phi^{\triangle}(X_{\ominus}) \to \Box\mathbf{for}(X_{\ominus}) \in L.$$

Therefore,

$$\Phi^{\triangle}(X_{\ominus}) \to \Box\mathbf{for}(X) \in L. \tag{1.1}$$

In order to prove (1), we show

$$\Sigma, \Phi(X) \to \Box\mathbf{for}(X) \in L \tag{1.2}$$

for any subset Σ of $\mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1))$. In order to show (1.2), we use an induction on $\#(\mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1)) - \Sigma)$. We show Basis ($\Sigma = \mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1))$) and Induction step ($\Sigma \subsetneq \mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1))$) simultaneously. We define the set Ψ as

$$\Psi = \bigcup_{Y_{n-1} \in \Phi^{\triangle}(X_{\ominus})} \mathbf{for}(\{Y \in \mathbf{next}^{+}(Y_{n-1}) \mid \mathbf{ant}(Y) \cap \Box\mathbf{for}(\mathbf{G}(n-1)) = \Sigma\}).$$

We note that $\Psi - \{Z \mid Z \in L\} \subseteq \Phi(X)$ and for any $\mathbf{for}(Y) \in \Psi$,

$$\begin{aligned}\mathbf{suc}(Y) \cap \Box\mathbf{for}(\mathbf{G}(n-1)) &= \Box\mathbf{for}(\mathbf{G}(n-1)) - \Sigma \\ &= ((\mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1))) \cup (\mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1)))) - \Sigma \\ &= (\mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1))) \cup (\mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1)) - \Sigma).\end{aligned}$$

It is easily seen that

$$A \to \Box\mathbf{for}(X) \in L \quad \text{for any } A \in \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1)). \tag{1.3}$$

Also, by the induction hypothesis, we have

$$A, \Sigma, \Phi(X) \to \Box\mathbf{for}(X) \in L \quad \text{for any } A \in \mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1)) - \Sigma. \tag{1.4}$$

By (1.1), (1.3) and (1.4), we have

$$\Sigma, \Phi(X), \Psi \to \Box\mathbf{for}(X) \in L.$$

Using $\Psi - \{Z \mid Z \in L\} \subseteq \Phi(X)$, we have (1.2). Considering the case that $\Sigma = \emptyset$, we obtain (1).

(2) is clear from (6) and (1).

(3) can be shown similarly to Lemma 2.14(3) in [Sas10].

(4) is clear from (3).

We show (5). If $n = 0$, then (5) is clear. So, we assume $n > 0$. Suppose that $Y \in \mathbf{pclus}(X)$. Then $\Box\mathbf{for}(X_\ominus) \in (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. Therefore,

$$\mathbf{ant}(Y) \to \mathbf{suc}(Y), \Box\mathbf{for}(X) \in L$$

and thus,

$$\to \mathbf{for}(Y) \vee \Box\mathbf{for}(X) \in L. \tag{5.1}$$

On the other hand, by (3),

$$\{\mathbf{for}(Y) \vee \Box\mathbf{for}(X) \mid Y \in \mathbf{pclus}(X)\}, \Phi^\triangle(X) \to \Box\mathbf{for}(X) \in L.$$

Using (5.1), we obtain (5). ⊣

Lemma 2.14 *For any $X_n \in \mathbf{G}(n)$ and for any $k \in \{1, 2, \cdots\}$,*

$$\Box\mathbf{for}(X_n) \equiv \begin{cases} \bigwedge \mathbf{for}(\mathbf{pclus}(X_n)) \wedge \bigwedge \Phi & \text{if } X_n \in \mathbf{G}^\circ(n) \\ \bigwedge \Phi & \text{if } X_n \notin \mathbf{G}^\circ(n). \end{cases}$$

where $\Phi = \{\mathbf{for}(Y) \mid Y \in \mathbf{ED}(n+k), \Box\mathbf{for}(X_n) \in \mathbf{suc}(Y)\}$.

Proof. We only show the case that $k = 1$. The other cases can be shown by an induction on k. It is sufficient to show the following four conditions:

(1) $\Box\mathbf{for}(X_n) \to \bigwedge \Phi \in L$,

(2) $\Box\mathbf{for}(X_n) \to \bigwedge \mathbf{for}(\mathbf{pclus}(X_n)) \in L$ if $X_n \in \mathbf{G}^\circ(n)$,

(3) $\Phi \to \Box\mathbf{for}(X_n) \in L$ if $X_n \notin \mathbf{G}^\circ(n)$,

(4) $\mathbf{for}(\mathbf{pclus}(X_n)), \Phi \to \Box\mathbf{for}(X_n) \in L$ if $X_n \in \mathbf{G}^\circ(n)$.

(1) is clear. (2) is also clear from Lemma 2.11(2). We show (3) and (4). By Corollary 2.9(3), for any $Z_n \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\{\mathbf{for}(Z_{n+1}) \mid Z_{n+1} \in \mathbf{next}(Z_n), \Box\mathbf{for}(X_n) \in \mathbf{suc}(Z_{n+1})\} \to \mathbf{for}(Z_n), \Box\mathbf{for}(X_n) \in L.$$

Therefore,

$$\{\mathbf{for}(Z_{n+1}) \mid Z_{n+1} \in \mathbf{G}(n+1), \Box\mathbf{for}(X_n) \in \mathbf{suc}(Z_{n+1})\} \to \bigwedge \mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)), \Box\mathbf{for}(X_n) \in L.$$

Using Lemma 2.13(2) and Lemma 2.13(5); and Lemma 2.13(4), we obtain (3); and (4), respectively. ⊣

By Corollary 2.9(2), we obtain Theorem 2.6(1). By Lemma 2.14 and Lemma 2.11(1), we obtain Theorem 2.6(4). Theorem 2.6(2) and Theorem 2.6(3) can be shown easily.

3 A construction of $\mathbf{ED_{K4}}(n)$ without K4-provability

Let L be a normal modal logic containing **K4**. In Definition 2.1, we use L-provability to define $\mathbf{prov}_L(X)$ for $X \in \mathbf{G}(n)$. In the present section, we only consider the case that $L = \mathbf{K4}$ and we give the set $\mathbf{prov_{K4}}(X)$ without using **K4**-provability. First, we define five sets $\mathbf{pr}_0(X)$, $\mathbf{pr}_1(X)$, $\mathbf{pr}_2(X)$, $\mathbf{pr}_3(X)$, and $\mathbf{pr}_4(X)$ for $X \in \mathbf{G}_L(n)$, and prove that $\mathbf{prov_{K4}}(X) = \mathbf{pr}_0(X) \cup \mathbf{pr}_1(X) \cup \mathbf{pr}_2(X) \cup \mathbf{pr}_3(X) \cup \mathbf{pr}_4(X)$. We only use L-provability to define $\mathbf{prov}_L(X)$ in Definition 2.1. Therefore, if the definition of the above five sets does not depend on **K4**-provability, then we obtain a construction of $\mathbf{ED_{K4}}(n)$, which does not depend on **K4**-provability.

Definition 3.1 For any $X \in \mathbf{G}_L(0)$, we define $\mathbf{pr}_0(X)$, $\mathbf{pr}_1(X)$, $\mathbf{pr}_2(X)$, $\mathbf{pr}_3(X)$ and $\mathbf{pr}_4(X)$ as

$$\mathbf{pr}_0(X) = \mathbf{pr}_1(X) = \mathbf{pr}_2(X) = \mathbf{pr}_3(X) = \mathbf{pr}_4(X) = \emptyset.$$

For any $X \in \mathbf{G}_L(n+1)$, we define $\mathbf{pr}_0(X)$, $\mathbf{pr}_1(X)$, $\mathbf{pr}_2(X)$, $\mathbf{pr}_3(X)$ and $\mathbf{pr}_4(X)$ as follows:

$$\mathbf{pr}_0(X) = \{(\Box\mathbf{for}(Y_\ominus), \Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}_L^+(X) \mid Y \in \mathbf{G}_L(n)\},$$

$$\mathbf{pr}_1(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}_L^+(X) \mid Y \in \mathbf{G}_L(n), (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box\},$$

$$\mathbf{pr}_2(X) = \{(\Box\mathbf{for}(\mathbf{next}(Z', X)), \Gamma \to \Delta, \Box\mathbf{for}(Z')) \in \mathbf{next}_L^+(X) \mid Z' \in \mathbf{G}_L(n-1) - \mathbf{G}_L^*(n-1)\},$$

$$\begin{aligned}\mathbf{pr}_3(X) = \{&(\Box\mathbf{for}(\mathbf{next}(Z', Y)), \Gamma \to \Delta, \Box\mathbf{for}(\Gamma_1 \xrightarrow{Y} \Delta_1, \Box\mathbf{for}(Z'))) \in \mathbf{next}_L^+(X) \\ &\mid Y \in \mathbf{G}_L(n), Z' \in \mathbf{G}_L(n-1) - \mathbf{G}_L^*(n-1)\},\end{aligned}$$

$$\mathbf{pr}_4(X) = \{(\Box\mathbf{for}(Y), \Gamma \to \Delta, \Box\mathbf{for}(Z)) \in \mathbf{next}_L^+(X) \mid Y \in \mathbf{G}_L^\circ(n), Z \in \mathbf{pclus}(Y)\},$$

where $\mathbf{next}(Z', Y) = \{Z \in \mathbf{next}_L(Z') \mid (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box\}$.

Theorem 3.2 *For any $X \in \mathbf{G_{K4}}(n) - \mathbf{G^*_{K4}}(n)$,*

$$\mathbf{prov_{K4}}(X) = \mathbf{pr}_0(X) \cup \mathbf{pr}_1(X) \cup \mathbf{pr}_2(X) \cup \mathbf{pr}_3(X) \cup \mathbf{pr}_4(X).$$

To prove Theorem 3.2, we provide some preparations. From now on, we only treat the case that $L = \mathbf{K4}$ and omit the subscript **K4** from $\mathbf{prov_{K4}}(X)$, $\mathbf{ED_{K4}}(n)$, $\mathbf{G_{K4}}(n)$, and so on.

Lemma 3.3 *For any $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,*
(0) $\mathbf{pr}_0(X) \subseteq \mathbf{prov}(X)$,
(1) $\mathbf{pr}_1(X) \subseteq \mathbf{prov}(X)$,
(2) $\mathbf{pr}_2(X) \subseteq \mathbf{prov}(X)$,
(3) $\mathbf{pr}_3(X) \subseteq \mathbf{prov}(X)$,
(4) $\mathbf{pr}_4(X) \subseteq \mathbf{prov}(X)$.

Proof. (0) is clear. (1), (2), (3), and (4) can be shown similarly to Lemma 3.3 in [Sas10] using Lemma 2.10, Corollary 2.9(1), and Lemma 2.11(2). ⊣

Lemma 3.4 *Let X and Y be sequents in $\mathbf{G}(n) - \mathbf{G}^*(n)$ and let $X_\oplus$ be a sequent in $\mathbf{next}^+(X) - (\mathbf{pr}_0(X) \cup \mathbf{pr}_1(X) \cup \mathbf{pr}_2(X) \cup \mathbf{pr}_3(X) \cup \mathbf{pr}_4(X))$. If $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_\oplus)$, then*

$$Y_\oplus \in \mathbf{next}^+(Y) - (\mathbf{pr}_0(Y) \cup \mathbf{pr}_1(Y) \cup \mathbf{pr}_2(Y) \cup \mathbf{pr}_3(Y) \cup \mathbf{pr}_4(Y)),$$

where

$$Y_\oplus = (\Gamma_Y, \mathbf{ant}(X_\oplus) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Delta_Y),$$

$$\Delta_Y = \begin{cases} \mathbf{suc}(X_\oplus) \cap \Box\mathbf{for}(\mathbf{G}(n)) & \text{if } n = 0 \\ \{\Box\mathbf{for}(Z) \in \mathbf{suc}(X_\oplus) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box, \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(Y)\} & \text{if } n > 0 \end{cases},$$

$$\Gamma_Y = (\mathbf{suc}(X_\oplus) \cap \Box\mathbf{for}(\mathbf{G}(n))) - \Delta_Y.$$

Proof. From the definition of Δ_Y, it is observed easily that $Y_\oplus \in \mathbf{next}^+(Y) - (\mathbf{pr}_0(Y) \cup \mathbf{pr}_1(Y))$.

We show $Y_\oplus \notin \mathbf{pr}_2(Y)$. Suppose that $Y_\oplus \in \mathbf{pr}_2(Y)$. Then there exists a sequent $Z_{n-1} \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $Y_\oplus$ is of the form of

$$\cdots, \Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y)) \xrightarrow{Y_\oplus} \Box\mathbf{for}(Z_{n-1}), \cdots.$$

Using $\mathbf{ant}(Y_\oplus) = \Gamma_Y \cup (\mathbf{ant}(X_\oplus) \cap \Box\mathbf{for}(\mathbf{G}(n))) \cup \mathbf{ant}(Y)$,

$$\Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y)) \cap \mathbf{suc}(X_\oplus) \subseteq \Gamma_Y.$$

Also, we have $\Box\mathbf{for}(Z_{n-1}) \in \mathbf{suc}(Y)$, and using the definitions of $\mathbf{next}(Z_{n-1}, Y)$ and Δ_Y,

$$\Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y)) \cap \mathbf{suc}(X_\oplus) \subseteq \Delta_Y.$$

Hence,

$$\Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y)) \cap \mathbf{suc}(X_\oplus) \subseteq \Gamma_Y \cap \Delta_Y = \emptyset,$$

and therefore,

$$\Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y)) \subseteq \mathbf{ant}(X_\oplus).$$

Hence, $X_\oplus$ is of the form of

$$\cdots, \Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y)) \xrightarrow{X_\oplus} \Box\mathbf{for}(\cdots \xrightarrow{Y} \Box\mathbf{for}(Z_{n-1}), \cdots), \cdots,$$

which is in contradiction with $X_\oplus \notin \mathbf{pr}_3(X)$.

We show $Y_\oplus \notin \mathbf{pr}_3(Y)$. Suppose that $Y_\oplus \in \mathbf{pr}_3(Y)$. Then there exist sequents $Y' \in \mathbf{G}(n)$ and $Z_{n-1} \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $Y_\oplus$ is of the form of

$$\cdots, \Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y')) \xrightarrow{Y_\oplus} \Box\mathbf{for}(\cdots \xrightarrow{Y'} \Box\mathbf{for}(Z_{n-1}), \cdots), \cdots.$$

Similarly to the proof of $Y_\oplus \notin \mathbf{pr}_2(Y)$,

$$\Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y')) \cap \mathbf{suc}(X_\oplus) \subseteq \Gamma_Y, \tag{1}$$

$$\Box\mathbf{for}(Y') \in \Delta_Y \subseteq \mathbf{suc}(X_\oplus), \tag{2}$$

and

$$\Box\mathbf{for}(Z_{n-1}) \in (\mathbf{suc}(Y'))^\Box. \tag{3}$$

By (2), we have $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Y'))^\Box$, and using (3),

$$\Box\mathbf{for}(Z_{n-1}) \in (\mathbf{suc}(Y))^\Box. \tag{4}$$

Using the definition of $\mathbf{next}(Z_{n-1}, Y')$ and $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Y'))^\Box$,

$$\Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y')) \cap \mathbf{suc}(X_\oplus) \subseteq \Delta_Y.$$

Using (1), we have

$$\Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y')) \cap \mathbf{suc}(X_\oplus) = \emptyset \text{ and } \Box\mathbf{for}(\mathbf{next}(Z_{n-1}, Y')) \subseteq \mathbf{ant}(X_\oplus).$$

Using (2), we have $X_\oplus \in \mathbf{pr}_3(X)$.

We show $Y_\oplus \notin \mathbf{pr}_4(Y)$. Suppose that $Y_\oplus \in \mathbf{pr}_4(Y)$. Then by $X_\oplus \notin \mathbf{pr}_4(X)$, there exist sequents $Z \in \mathbf{G}^\circ(n)$ and $Z' \in \mathbf{pclus}(Z)$ such that $\Box\mathbf{for}(Z) \in \Gamma_Y$ and $\Box\mathbf{for}(Z') \in \Delta_Y$. By $\Box\mathbf{for}(Z') \in \Delta_Y$, we have $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z'))^\Box$, and using $Z' \in \mathbf{pclus}(Z)$, we have $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box$. Using $\Box\mathbf{for}(Z) \in \Gamma_Y$, we have $n > 0$ and $\Box\mathbf{for}(Z_\ominus) \in \mathbf{ant}(Y) \subseteq (\mathbf{ant}(Z))^\Box$, which is in contradiction with $Z \in \mathbf{G}^\circ(n)$. $\dashv$

By Lemma 2.5(3), every sequent in $\mathbf{G}^+(n)$ consists of the members in the set $\mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i))$. We call this set the *base of the sequents in* $\mathbf{G}^+(n)$ and use it to define one of the notions.

Definition 3.5

(1) We define the set $\mathbf{BG}(n)$ as

$$\mathbf{BG}(n) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i)).$$

(2) For any $X \in \mathbf{G}^+(n)$ and for any k, we define $X(k)$ as

$$X(k) = (\mathbf{ant}(X) \cap \mathbf{BG}(k) \to \mathbf{suc}(X) \cap \mathbf{BG}(k)).$$

Here, we note that $X(k) = X$ if $k \geq n$; and that $X(n-1) = X_\ominus$ if $n \neq 0$.

Definition 3.6 For any $X \in \mathbf{G}(n)$, we define the sets $X\Downarrow$ inductively as follows:

(1) $X \in X\Downarrow$,

(2) if $Z \in X\Downarrow - \bigcup_{i=1}^{\infty} \mathbf{G}^*(i)$, then $Y \in X\Downarrow$ for any $Y \in \mathbf{next}(Z)$.

Example 3.7

(1) $T2.3(3) = T2.3, \quad T2.3(2) = T2, \quad T2.3(1) = T$,

(2) $T\Downarrow = \{T1^\bullet\} \cup T2\Downarrow \cup T3\Downarrow \cup T4\Downarrow$,

(3) $T2.3 \in T2.3\Downarrow \subseteq T2\Downarrow \subseteq T\Downarrow$.

Lemma 3.8 *Let X and Y be sequents in $\mathbf{G}^+(n)$ and $\mathbf{G}(k)$, respectively. Then*

(1) *$n > k$ implies $X(k) \in \mathbf{G}(k) - \mathbf{G}^*(k)$ and $X \in X(k)\Downarrow \cup \mathbf{prov}(X_\ominus)$,*

(2) *if $X \in \mathbf{G}(n)$, then the following three conditions are equivalent:*

(2.1) *$\mathbf{ant}(Y) \subseteq \mathbf{ant}(X)$ and $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$,*

(2.2) *$n \geq k$ and $Y = X(k)$,*

(2.3) *$X \in Y\Downarrow$.*

Proof. We obtain (1) by an induction on n. By (1), we have that (2.2) implies (2.3). By an induction on $X \in Y\Downarrow$, we can show that (2.3) implies (2.1). By Lemma 2.5(3), (1), and (2.1), we have $n \geq k$, $X(k) \in \mathbf{G}(k)$, and $\mathbf{ant}(Y) = \mathbf{ant}(X(k))$; and hence, we obtain that (2.1) implies (2.2). ⊣

Lemma 3.9 *Let X_n and Y_k be sequents in $\mathbf{G}(n)$ and $\mathbf{G}^*(k)$, respectively. If $n \geq k$ and $(\mathbf{ant}(Y_k))^\Box \neq (\mathbf{ant}(X_n(k)))^\Box$, then $(\Box\mathbf{for}(X_n) \supset \mathbf{for}(Y_k)) \equiv \mathbf{for}(Y_k)$.*

Proof. We show

$$(\mathbf{ant}(Y_k))^\Box \neq (\mathbf{ant}(X_n(k)))^\Box \text{ implies } (\to \mathbf{for}(Y_k), \Box\mathbf{for}(X_n)) \in \mathbf{K4}$$

by an induction on n. Basis ($n = k$) can be shown by Lemma 2.10. Induction step ($n > k$) can be shown by $\Box\mathbf{for}(X_n(n-1)) \to \Box\mathbf{for}(X) \in \mathbf{K4}$. ⊣

Lemma 3.10 *Let X_n be a sequent in $\mathbf{G}(n) - \mathbf{G}^*(n)$. Let X_{n+1} and Y_k be sequents in $\mathbf{next}^+(X_n) - (\mathbf{pr}_0(X_n) \cup \mathbf{pr}_1(X_n) \cup \mathbf{pr}_2(X_n) \cup \mathbf{pr}_3(X_n) \cup \mathbf{pr}_4(X_n))$ and $\mathbf{G}(k)$, respectively. If $n \geq k$ and $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_{n+1})$, then*

(1) *$Y_k \in \mathbf{G}^*(k)$ implies $(\Box\mathbf{for}(Z) \supset \mathbf{for}(Y_k)) \equiv \mathbf{for}(Y_k)$, for any $Z \in \{Z' \mid \Box\mathbf{for}(Z') \in \mathbf{ant}(X_{n+1})\}$,*

(2) *$Y_k \in \mathbf{G}^*(k)$ implies $(\mathbf{for}(Z) \supset \mathbf{for}(Y_k)) \equiv \mathbf{for}(Y_k)$, for any $Z \in \{Z' \mid \Box\mathbf{for}(Z') \in \mathbf{ant}(X_{n+1})\}$,*

(3) *there exists a sequent $Y \in \mathbf{ED}(n)$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_{n+1})$ and $Y \in Y_k\Downarrow$.*

Proof. First, we note that $X_n \notin \mathbf{K4}$ by Lemma 2.5(1).

For (1). Suppose that $\Box\mathbf{for}(Z) \in (\mathbf{ant}(X_{n+1}))^\Box \cap \Box\mathbf{for}(\mathbf{G}(i))$ for some $i \leq n$. We divide the cases.

The case that $i \geq k$. By Lemma 3.9, we can assume that $(\mathbf{ant}(Y_k))^\Box = (\mathbf{ant}(Z(k)))^\Box$. Using $Y_k \in \mathbf{G}^*(k)$ and Lemma 2.11(1), we have $Z(k) \in \mathbf{G}^*(k)$. Using Lemma 3.8(1), we have $i = k$ and $Z(k) = Z$. Therefore, if $Z \in \mathbf{G}^\circ(k)$, then $k = n$ is in contradiction with $X_{n+1} \notin \mathbf{pr}_4(X_n)$; and $k < n$

is in contradiction with Lemma 2.11(2) and $X_n \notin \mathbf{K4}$. If $Z \in \mathbf{G}^\bullet(n)$, then we have $\Box\mathbf{for}(Z(k-1)) \in (\mathbf{ant}(Z))^\Box = (\mathbf{ant}(Y_k))^\Box$, and clearly, $(\Box\mathbf{for}(Z) \supset \mathbf{for}(Y_k)) \equiv \mathbf{for}(Y_k)$.

The case that $i < k$ can be shown similarly to Lemma 3.10 in [Sas10].

For (2). Suppose that $\Box\mathbf{for}(Z) \in (\mathbf{ant}(X_{n+1}))^\Box \cap \Box\mathbf{for}(\mathbf{G}(i))$ for some $i \leq n$. We divide the cases.

The case that $i \geq k$. If $Y_k \neq Z(k)$, then by Lemma 2.5(2), we have

$$\mathbf{for}(Y_k) \vee \mathbf{for}(Z(k)) \in \mathbf{K4},$$

and therefore,

$$\mathbf{for}(Y_k) \vee \mathbf{for}(Z) \in \mathbf{K4},$$

and hence, we obtain (2). If $Y_k = Z(k)$, then by Lemma 3.8(1), we have $i = k$ and $Z = Z(k) = Y_k$, which is in contradiction with $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_{n+1})$, $\Box\mathbf{for}(Z) \in \mathbf{ant}(X_{n+1})$, and Lemma 2.5(3).

The case that $i < k$. If $Y_k(i) \neq Z$, then by Lemma 2.5(2), we have

$$\mathbf{for}(Y_k(i)) \vee \mathbf{for}(Z) \in \mathbf{K4},$$

and therefore,

$$\mathbf{for}(Y_k) \vee \mathbf{for}(Z) \in \mathbf{K4},$$

and hence, we obtain (2). We assume that $Y_k(i) = Z$. By $i < k \leq n$, we have

$$\Box\mathbf{for}(Y_k(i)) = \Box\mathbf{for}(Z) \in \mathbf{ant}(X_n).$$

Using $X_n \notin \mathbf{K4}$ and $\Box\mathbf{for}(Y_k(i)) \to \Box\mathbf{for}(Y_k(k-1)) \in \mathbf{K4}$, we have

$$\Box\mathbf{for}(Y_k(k-1)) \in \mathbf{ant}(X_n) \subseteq \mathbf{ant}(X_{n+1}),$$

which is in contradiction with $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_{n+1})$ and $X_{n+1} \notin \mathbf{pr}_0(X_n)$.

For (3). We use an induction on n. Basis ($n = k$) is clear since Y_k satisfies the conditions.

Induction step ($n > k$). By $n > k$, Lemma 3.8, and Lemma 3.3, we have $X_n \in \mathbf{next}(X_n(n-1)) \subseteq \mathbf{next}^+(X_n(n-1)) - (\mathbf{pr}_0(X_n(n-1)) \cup \mathbf{pr}_1(X_n(n-1)) \cup \mathbf{pr}_2(X_n(n-1)) \cup \mathbf{pr}_3(X_n(n-1)) \cup \mathbf{pr}_4(X_n(n-1)))$ and $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_n)$. Using the induction hypothesis, there exists a sequent $Y' \in \mathbf{ED}(n-1)$ such that $\Box\mathbf{for}(Y') \in \mathbf{suc}(X_{n+1}(n))$ and $Y' \in Y_k\Downarrow$. If $Y' \in \bigcup_{i=0}^{n-1} \mathbf{G}^*(i)$, then Y' satisfies the conditions. We assume that $Y' \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$. Then by $\Box\mathbf{for}(Y') \in \mathbf{suc}(X_{n+1}(n))$ and $X_{n+1} \notin \mathbf{pr}_2(X_n)$,

$$\Box\mathbf{for}(\mathbf{next}(Y', X_{n+1}(n))) \not\subseteq \mathbf{ant}(X_{n+1}).$$

Therefore, there exists a sequent $Y_n \in \mathbf{next}(Y')$ such that $\Box\mathbf{for}(Y_n) \notin \mathbf{ant}(X_{n+1})$. Hence, Y_n satisfies the conditions, and hence we obtain (2). ⊣

Definition 3.11 Let X be a sequent in $\mathbf{G}^+(n+1)$ and Y be a sequent satisfying $\Box\mathbf{for}(Y) \in \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n))$. We define the sequents $\mathbf{sat}^0(X, Y)$ and $\mathbf{sat}(X, Y)$, and the set $\mathbf{SAT}(X, Y)$ of a sequent as follows:

$$\mathbf{sat}^0(X, Y) = (\{A \mid \Box A \in \mathbf{ant}(X)\}, (\mathbf{ant}(X))^\Box, \mathbf{ant}(Y) \to \mathbf{suc}(Y)),$$

$$\mathbf{sat}(X, Y) = (\Gamma_d, \Gamma_c, \mathbf{ant}(\mathbf{sat}^0(X, Y)) \to \mathbf{suc}(\mathbf{sat}^0(X, Y)), \Delta_c, \Delta_d, \Delta_f),$$

$$\mathbf{SAT}(X, Y) = \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X, Y)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X, Y))\},$$

where

$\Gamma_c = \{\bigwedge\Sigma \mid \Sigma \subseteq \mathbf{BG}(n), \Sigma \subseteq \mathbf{ant}(Y), \#(\Sigma) > 1\}$,

$\Gamma_d = \{\bigvee\Sigma \mid \Sigma \subseteq \mathbf{BG}(n), \Sigma \not\subseteq \mathbf{suc}(Y), \#(\Sigma) > 1\}$,

$\Delta_c = \{\bigwedge\Sigma \mid \Sigma \subseteq \mathbf{BG}(n), \Sigma \not\subseteq \mathbf{ant}(Y), \#(\Sigma) > 1\}$.

$\Delta_d = \{\bigvee\Sigma \mid \Sigma \subseteq \mathbf{BG}(n), \Sigma \subseteq \mathbf{suc}(Y), \#(\Sigma) > 1\}$.

$\Delta_f = \{\mathbf{for}(Y(k)) \mid 0 \leq k \leq n\}$.

Lemma 3.12 *Let $\mathcal{P}$ be a cut-free proof figure in* **K4** *whose end sequent is $\Phi \to \Psi$. Then for any $X_n \in \mathbf{G}(n) - \mathbf{G}^*(n)$, for any $X_{n+1} \in \mathbf{next}^+(X_n) - (\mathbf{pr}_0(X_n) \cup \mathbf{pr}_1(X_n) \cup \mathbf{pr}_2(X_n) \cup \mathbf{pr}_3(X_n) \cup \mathbf{pr}_4(X_n))$, and for any Y satisfying $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_{n+1})$,*

$$(\Phi \to \Psi) \notin \mathbf{SAT}(X_{n+1}, Y).$$

Proof. First, we show the following three conditions:

(1) for any $k \in \{0, \cdots, n\}$, $\Box\mathbf{for}(Y(k)) \notin \mathbf{ant}(X_{n+1})$,

(2) if $A_1 \vee A_2 \in \mathbf{suc}(\mathbf{sat}(X_{n+1}, Y))$, then $A_1 \vee A_2 \in \Delta_d$,

(3) if $A_1 \vee A_2 \in \mathbf{ant}(\mathbf{sat}(X_{n+1}, Y))$, then $A_1 \vee A_2 \in \Gamma_d$.

For (1). Suppose that $\Box\mathbf{for}(Y(k)) \in \mathbf{ant}(X_{n+1})$. If $k = n$, then $\Box\mathbf{for}(Y) \in \mathbf{ant}(X_{n+1}) \cap \mathbf{suc}(X_{n+1})$, which is in contradiction with Lemma 2.5(3). If $k < n$, then we have $\Box\mathbf{for}(Y(k)) \in \mathbf{ant}(X_n)$. Using $\Box\mathbf{for}(Y(k)) \to \Box\mathbf{for}(Y(n-1)) \in \mathbf{K4}$, we have $\Box\mathbf{for}(Y(n-1)) \in \mathbf{ant}(X_n) \subseteq \mathbf{ant}(X_{n+1})$, which is in contradiction with $X_{n+1} \notin \mathbf{pr}_0(X_n)$.

For (2). Suppose that $A_1 \vee A_2 \in \mathbf{suc}(\mathbf{sat}(X_{n+1}, Y))$. Then we have $A_1 \vee A_2 \in \Delta_d \cup \Delta_f$. If $A_1 \vee A_2 \in \Delta_d$, then (2) is clear. We assume that $A_1 \vee A_2 \in \Delta_f$. Then we have

$$A_1 \vee A_2 = \mathbf{for}(Y(k)) = \bigvee \mathbf{suc}(Y(k)) \text{ and } \mathbf{suc}(Y(k)) = \mathbf{BG}(k).$$

Using $\mathbf{suc}(Y(k)) \subseteq \mathbf{suc}(Y)$, we obtain (2).

For (3). Suppose that $A_1 \vee A_2 \in \mathbf{ant}(\mathbf{sat}(X_{n+1}, Y))$. Then we have $A_1 \vee A_2 \in \Gamma_d \cup \{A \mid \Box A \in \mathbf{ant}(X_{n+1})\}$. If $A_1 \vee A_2 \in \Gamma_d$, then (3) is clear. We assume that $\Box(A_1 \vee A_2) \in \mathbf{ant}(X_{n+1})$. Then there exists $Z \in \mathbf{G}(k)$ such that

$$A_1 \vee A_2 = \mathbf{for}(Z(k)) = \bigvee \mathbf{suc}(Z(k)) \text{ and } \mathbf{suc}(Z(k)) = \mathbf{BG}(k).$$

Also, by (1), we have $Z(k) \neq Y(k)$. Therefore,

$$\mathbf{suc}(Z(k)) = \mathbf{BG}(k) \not\subseteq Y(k) \subseteq \mathbf{suc}(Y).$$

Hence, we obtain (3).

In order to prove the lemma, we use an induction on $\mathcal{P}$.

Basis ($\mathcal{P}$ consists of an axiom). By Lemma 2.5(3), we can show $\perp \notin \mathbf{ant}(\mathbf{sat}(X_{n+1}, Y))$. We show $\mathbf{ant}(\mathbf{sat}(X_{n+1}, Y)) \cap \mathbf{suc}(\mathbf{sat}(X_{n+1}, Y)) = \emptyset$.

Suppose that $A \in \mathbf{ant}(\mathbf{sat}(X_{n+1}, Y)) \cap \mathbf{suc}(\mathbf{sat}(X_{n+1}, Y))$. We divide the cases.

The case that $A = p_i$. By $p_i \in \mathbf{suc}(\mathbf{sat}(X_{n+1}, Y))$, we have $p_i \in \mathbf{suc}(Y)$. Using Lemma 2.5, we have $p_i \notin \mathbf{ant}(Y)$. Using $p_i \in \mathbf{ant}(\mathbf{sat}(X_{n+1}, Y))$, we have $\Box p_i \in \mathbf{ant}(X_{n+1})$, and thus, $i = m = 1$. In other words, $\Box\mathbf{for}(Y(0)) = \Box p_1 \in \mathbf{ant}(X_{n+1})$, which is in contradiction with (1).

The case that $A = A_1 \wedge A_2$ is shown from $\Gamma_c \cap \Delta_c = \emptyset$.

The case that $A = A_1 \vee A_2$ is shown from $\Gamma_d \cap \Delta_d = \emptyset$, (2), and (3).

The case that $A = A_1 \supset A_2$. By $A \in \mathbf{suc}(\mathbf{sat}(X_{n+1}, Y))$, we have $A = \mathbf{for}(Y(k)) \in \Delta_f$. Using $A \in \mathbf{ant}(\mathbf{sat}(X_{n+1}, Y))$, we have $\Box A \in \mathbf{ant}(X_{n+1})$, which is in contradiction with (1).

The case that $A = \Box A_1$. By $X_{n+1} \notin \mathbf{pr}_1(X_n)$, we have $(\mathbf{ant}(X_n))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box}$. Therefore, $A \in \mathbf{ant}(Y) \cap \mathbf{suc}(Y)$, which is in contradiction with Lemma 2.5(3).

Induction step ($\mathcal{P}$ has the inference rule I introducing the end sequent $\Phi \to \Psi$). Suppose that $(\Phi \to \Psi) \in \mathbf{SAT}(X_{n+1}, Y)$. We divide the cases.

The case that I is either a weakening rule or $(\to\supset)$ is clear.

The case that I is $(\wedge \to)$. I is of the form of

$$\frac{A_i, \Phi' \to \Psi}{A_1 \wedge A_2, \Phi' \to \Psi},$$

where $\{A_1 \wedge A_2\} \cup \Phi' = \Phi$. We note that $A_1 \wedge A_2 \in \Gamma_c$. Therefore, $A_1 \in \Gamma_c \cup \mathbf{ant}(Y) \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1}, Y))$ and $A_2 \in \mathbf{ant}(Y) \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1}, Y))$. Hence, the upper sequent of I belongs to $\mathbf{SAT}(X_{n+1}, Y)$, which is in contradiction with the induction hypothesis.

The case that I is $(\to \wedge)$. I is of the form of

$$\frac{\Phi \to \Psi', A_1 \quad \Phi \to \Psi', A_2}{\Phi \to \Psi', A_1 \wedge A_2},$$

where $\{A_1 \wedge A_2\} \cup \Psi' = \Psi$. We note that $A_1 \wedge A_2 \in \Delta_c$. If $A_2 \in \mathbf{ant}(Y)$, then either $A_1 \in \Delta_c$ or $A_1 \in \mathbf{BG}(n) - \mathbf{ant}(Y) = \mathbf{suc}(Y)$, and hence, the left upper sequent of I belongs to $\mathbf{SAT}(X_{n+1}, Y)$, which is in contradiction with the induction hypothesis. If $A_2 \notin \mathbf{ant}(Y)$, then similarly, $A_2 \in \mathbf{BG}(n) - \mathbf{ant}(Y) = \mathbf{suc}(Y)$, and hence, the right upper sequent of I belongs to $\mathbf{SAT}(X_{n+1}, Y)$, which is in contradiction with the induction hypothesis.

The case that I is either $(\to \vee)$ or $(\vee \to)$. By (2) and (3), we can show the lemma similarly to the above two cases, respectively.

The case that I is $(\supset\to)$. I is of the form of

$$\frac{\Phi' \to \Psi, A_1 \quad A_2, \Phi' \to \Psi}{A_1 \supset A_2, \Phi' \to \Psi},$$

where $\{A_1 \supset A_2\} \cup \Phi' = \Phi$. We note that $\Box(A_1 \supset A_2) = \Box\mathbf{for}(Z) \in \mathbf{ant}(X_{n+1})$ for some $Z \in \mathbf{G}(k)$. By (1), we have $Z \neq Y(k)$. If $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y(k))$, then there exists a formula $B \in \mathbf{ant}(Z) \cap \mathbf{suc}(Y(k))$, and thus, $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y)$. Therefore, we have either $A_1 = \bigwedge \mathbf{ant}(Z) \in \Delta_c$ or $A_1 = \bigwedge \mathbf{ant}(Z) = B \in \mathbf{suc}(Y)$. Hence, the left upper sequent of I belongs to $\mathbf{SAT}(X_{n+1}, Y)$, which is in contradiction with the induction hypothesis. If $\mathbf{ant}(Y(k)) \not\subseteq \mathbf{ant}(Z)$, then we have $\mathbf{suc}(Z) \not\subseteq \mathbf{suc}(Y(k))$, and similarly, we have either $A_2 = \bigvee \mathbf{suc}(Z) \in \Gamma_d$ or $A_2 = \bigvee \mathbf{suc}(Z) \in \mathbf{ant}(Y)$. Hence, the right upper sequent of I belongs to $\mathbf{SAT}(X_{n+1}, Y)$, which is in contradiction with the induction hypothesis.

The case that I is $(\Box)$. There exists a sequent $Z_k \in \mathbf{G}(k)$ $(k \leq n)$ such that

$$I \text{ is } \frac{\Phi, \Box\Phi \to \mathbf{for}(Z_k)}{\Box\Phi \to \Box\mathbf{for}(Z_k)}, \tag{5}$$

$$\{\Box\mathbf{for}(Z_k)\} = \Psi \subseteq (\mathbf{suc}(Y))^{\Box}, \tag{6}$$

and

$$\Box\Phi \subseteq (\mathbf{ant}(X_{n+1}))^{\Box} \cup (\mathbf{ant}(Y))^{\Box}. \tag{7}$$

We define the sequent Y_{n+1} as the sequent $Y_{\oplus}$ in Lemma 3.4. Then by Lemma 3.4, we have

$$Y_{n+1} \in \mathbf{next}^{+}(Y) - (\mathbf{pr}_0(Y) \cup \mathbf{pr}_1(Y) \cup \mathbf{pr}_2(Y) \cup \mathbf{pr}_3(Y) \cup \mathbf{pr}_4(Y)).$$

Also, by (6) and (7), we have

$$\Box\mathbf{for}(Z_k) \in \mathbf{suc}(Y_{n+1}), \tag{8}$$

$$\Box\Phi \subseteq (\mathbf{ant}(Y_{n+1}))^{\Box}. \tag{9}$$

By (8) and Lemma 3.10(3), there exists a sequent $Z \in \mathbf{ED}(n)$ such that

$$\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_{n+1}), \tag{10}$$

$$Z \in Z_k{\Downarrow}. \tag{11}$$

We divide the cases.

The case that $Z \in \mathbf{G}^*(i)$ for some $i \leq n$. By (5) and (11), we have $\Phi, \Box\Phi \to \mathbf{for}(Z) \in \mathbf{K4}$. Also, by (9), (10), Lemma 3.10(1), and Lemma 3.10(2), we have $\mathbf{for}(\Phi, \Box\Phi \to \mathbf{for}(Z)) \equiv \mathbf{for}(Z)$. Therefore, we have $Z \in \mathbf{K4}$, which is in contradiction with $Z \in \mathbf{ED}(n)$ and Lemma 2.5(1).

The case that $Z \in \mathbf{G}(n) - \mathbf{G}^*(n)$. By (9), (10), and (11), the upper sequent of I belongs to $\mathbf{SAT}(Y_{n+1}, Z)$. This is in contradiction with the induction hypothesis. $\dashv$

Corollary 3.13

(1) *Let $\mathcal{P}$ be a cut-free proof figure in* **K4** *whose end sequent is $\Phi \to \Psi$. Then for any $X_n \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and for any $X_{n+1} \in \mathbf{next}^+(X_n) - (\mathbf{pr}_0(X_n) \cup \mathbf{pr}_1(X_n) \cup \mathbf{pr}_2(X_n) \cup \mathbf{pr}_3(X_n) \cup \mathbf{pr}_4(X_n))$,*

$$(\Phi \to \Psi) \notin \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(X_{n+1}), \Psi^* \subseteq \mathbf{suc}(X_{n+1})\}.$$

(2) *Let X be a sequent in $\mathbf{G}(n) - \mathbf{G}^*(n)$. Then*

$$\mathbf{pr}_0(X) \cup \mathbf{pr}_1(X) \cup \mathbf{pr}_2(X) \cup \mathbf{pr}_3(X_n) \cup \mathbf{pr}_4(X) \supseteq \mathbf{prov}(X).$$

Proof. For (1). We use an induction on $\mathcal{P}$.

Basis ($\mathcal{P}$ consists of an axiom) can be shown from $\bot \notin \mathbf{ant}(X_{n+1})$ and Lemma 2.5.

Induction step ($\mathcal{P}$ has the inference rule I introducing the end sequent $\Phi \to \Psi$). Suppose that

$$(\Phi \to \Psi) \in \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1})), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_{n+1}))\}.$$

If I is a weakening rule, then the upper sequent also belongs to the above set, which is in contradiction with the induction hypothesis. We assume that I is $(\Box)$. Then the upper sequent of I is also provable in **K4**, which is in contradiction with Lemma 3.12.

For(2). By (1) and Lemma 1.1. $\dashv$

By Lemma 3.3 and Corollary 3.13(2), we obtain Theorem 3.2.

References

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