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Abstract

The polyhedral split decomposition, introduced by Hirai, is a decomposition of a polyhedral convex function into a sum of split functions and the residue, where a split function is a polyhedral convex function whose value at a point is defined to be the distance between the point and a hyperplane. The polyhedral split decomposition, applied to the convex extension of a metric, is known to obtain Buneman's decomposition of a metric geometrically. In this paper we shed a light on this fact from discrete convex analysis, by observing that this is a decomposition of an M-convex function into a sum of M-convex functions, whereas a sum of M-convex functions is not necessarily M-convex in general. We discuss the reason why the M-convexity is preserved in this polyhedral split decomposition. By applying the polyhedral split decomposition to a quadratic M-convex function, we also observe that a quadratic M-convex function induces a lattice dicing or, equivalently, a zonotope which fills the space facet-to-facet by its translations. Inspired by a result on lattice dicings, we propose another canonical representation of a quadratic M-convex function as a positive combination of rank one forms.

Keywords: M-convex function, polyhedral split decomposition, tree metric, split-decomposability, lattice dicing.

1 Introduction

A tree metric is known to be representable as a sum of split metrics. In [7], this classical result is reobtained geometrically via the *polyhedral split decomposition* of a metric. In this paper we intend to shed light on this decomposition from *discrete convex analysis* advocated by Murota [9].

To review the previous results, we begin by classifying a *distance*, *metric*, *tree metric* and *split metric* on a finite set X. A distance is defined as a function $d : X \times X \to \mathbf{R}$ such that $d(i,i) = 0$ for all $i \in X$ and $d(i,j) \leq d(i,k) + d(k,j)$ for all $i,j,k \in X$. A metric d is a symmetric and nonnegative distance, that is, a distance d such that $d(i,j) = d(j,i) \geq 0$ for all $i,j \in X$. A metric d is called a tree metric if there exits a tree with nonnegative edge lengths such that $d(i,j)$ is equal to the length of the path in the tree between the vertices indexed by i and j for all $i,j \in X$. An X-*split* is a partition

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of X into two nonempty sets, i.e., a pair $\{A, B\}$ of A and B such that $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq X$, $A \cap B = \emptyset$ and $A \cup B = X$. The split metric $\xi_{\{A,B\}} : X \times X \to \{0, 1\}$ associated with an X-split $\{A, B\}$ is defined by

$$\xi_{\{A,B\}}(i, j) = \begin{cases} 0 & \text{if } \ i, j \in A \text{ or } i, j \in B, \\ 1 & \text{otherwise,} \end{cases}$$

for all $i, j \in X$.

The polyhedral split decomposition of polyhedral convex functions is introduced by Hirai [5]. By the polyhedral split decomposition, a polyhedral convex function is decomposed into a sum of split functions and the residue, where a split function is a polyhedral convex function whose value at a point is defined to be the distance between the point and a hyperplane. Furthermore, the polyhedral split decomposition is applicable to a discrete function thorough its convex extension.

In [1], Buneman introduced an index of fundamental importance, called the *Buneman index*, to extract a split metric from a metric d, and showed that, if d is a tree metric, d can be represented as the sum of the extracted split metrics. A distance d on X can be regarded as a discrete function on $\{\chi_i - \chi_j \mid i, j \in X\}$ by setting $d(\chi_i - \chi_j) = d(i, j)$. In [7], the polyhedral split decomposition is applied to the convex extension of d and the result by Buneman is recovered in a purely geometric way.

In discrete convex analysis, the convex extension of a distance is known to be a *positively homogeneous M-convex function*. The notion of *M-convex functions* is introduced by Murota [8] as a generalization of *valuated matroids* by Dress and Wenzel [2]. Positively homogeneous M-convex functions form a most fundamental subclass of M-convex functions.

An interesting fact about the polyhedral split decomposition of a distance is that it gives a decomposition of a polyhedral M-convex function into a sum of polyhedral M-convex functions. This is a remarkable fact indeed because, in general, a sum of M-convex functions is not necessarily M-convex. As the polyhedral split decomposition is a geometric notion, we aim at giving a geometric explanation to the reason why the M-convexity is preserved in this decomposition, in particular, in terms of *polyhedral subdivisions* induced by M-convex functions. Those polyhedral subdivisions must consist of M-convex polyhedra.

This paper also deals with *quadratic M-convex functions*. It is known that there is a one-to-one correspondence between tree metrics and quadratic M-convex functions [6]. For a quadratic M-convex function, a positively homogeneous M-convex function can be defined at each point of its domain. We show that such a positively homogeneous M-convex function is representable as the sum of a tree metric and a linear function; namely, a quadratic M-convex function is split-decomposable at every point, where a discrete function is said to be split-decomposable if its convex extension is decomposed into a sum of split functions and a linear function by the polyhedral split decomposition. Moreover, this fact indicates that a quadratic M-convex function induces a lattice dicing, or equivalently, a zonotope which fills the space facet-to-facet by its translations. As a well-known result on lattice dicings, there is a routine for constructing a quadratic function that induces a lattice dicing [3, 4]. Inspired by this, we propose another canonical form of a quadratic M-convex function. The form is written as a positive combination of rank one forms.

This paper is organized as follows. Section 2 briefly describes the polyhedral and discrete split decompositions on the basis of [5]. In Section 3, we introduce basic terminology in discrete convex analysis and apply the polyhedral split decomposition to the convex extension of a distance according to [7]. We give in subsection 3.4 a geometric explanation for the M-convexity in the decomposition. In Section 4, we apply the polyhedral split decomposition to a quadratic M-convex function and discuss the induced lattice dicing.

2 Polyhedral and discrete split decomposition

This section describes the polyhedral split decomposition of polyhedral convex functions and the discrete split decomposition of discrete functions on the basis of [5]. The discrete split decomposition of a discrete function is nothing but the polyhedral split decomposition of the convex extension of the discrete function.

2.1 Preliminaries

Let $\mathbf{R}, \mathbf{R}_+$, and $\mathbf{R}_{++}$ be the set of real numbers, nonnegative real numbers, and positive real numbers, respectively. We denote by $\mathbf{R}^n$ the n dimensional Euclidean space with the standard inner product $\langle \cdot, \cdot \rangle$. Let $\mathbf{0}$ and $\mathbf{1}$ be the all-zero and all-one vectors in $\mathbf{R}^n$, respectively.

For $x, y \in \mathbf{R}^n$, let $[x, y]$ denote the closed line segment between x and y. We refer to an $(n-1)$ dimensional affine subspace of $\mathbf{R}^n$ as a hyperplane. In particular, for $(a, r) \in \mathbf{R}^n \times \mathbf{R}$, we define a hyperplane $H_{a,r} = \{x \in \mathbf{R}^n \mid \langle a, x \rangle = r\}$, closed half spaces $H^-_{a,r} = \{x \in \mathbf{R}^n \mid \langle a, x \rangle \leq r\}$ and $H^+_{a,r} = \{x \in \mathbf{R}^n \mid \langle a, x \rangle \geq r\}$.

For a set $S \subseteq \mathbf{R}^n$, we denote by $\operatorname{cone} S$ the conical hull of S, i.e.,

$$\operatorname{cone} S = \left\{ \sum_{t \in T} \lambda_t t \mid T \subseteq S : \text{a finite set}, \lambda = (\lambda_t) \in \mathbf{R}^T_+ \right\}.$$

The *indicator function* of a set $S \subseteq \mathbf{R}^n$ is the function $\delta_S : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by $\delta_S(x) = 1$ if $x \in S$ and $\delta_S(x) = 0$ otherwise.

For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, the *effective domain* of f, denoted by $\operatorname{dom} f$, is the set defined by $\operatorname{dom} f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\}$, and the *epigraph* of f, denoted by $\operatorname{epi} f$, is the set given by $\operatorname{epi} f = \{(x, \beta) \in \mathbf{R}^n \times \mathbf{R} \mid \beta \geq f(x)\}$.

A *polyhedral complex* $\mathcal{C}$ is a finite collection of polyhedra such that

(1) if $P \in \mathcal{C}$, all the faces of P are also in $\mathcal{C}$, and

(2) the nonempty intersection $P \cap Q$ of two polyhedra $P, Q \in \mathcal{C}$ is a face of P and Q.

The underlying set of $\mathcal{C}$ is the set $|\mathcal{C}| = \bigcup_{P \in \mathcal{C}} P$. A *polyhedral subdivision* of a polyhedron P is a polyhedral complex $\mathcal{C}$ with $|\mathcal{C}| = P$.

A convex function f is said to be *polyhedral* if its epigraph $\operatorname{epi} f$ is a polyhedron. Let f be a polyhedral convex function. By defining, for each proper face F of $\operatorname{epi} f$, a set F' of points in $\operatorname{dom} f$ as $F' = \{x \in \operatorname{dom} f \mid (x, f(x)) \in F\}$, we obtain a collection of subsets of $\operatorname{dom} f$. One can easily see that each set F' in this collection is a polyhedron, that is, the collection is a polyhedral subdivision of $\operatorname{dom} f$, which we denote by $\mathcal{T}(f)$. The following is a well-known fact.

Lemma 2.1. *For a polyhedral convex function f, the polyhedral subdivision $\mathcal{T}(f)$ is given by*

$$\mathcal{T}(f) = \{F \subseteq \mathbf{R}^n \mid F = \operatorname{argmin}(f - \langle p, \cdot \rangle) \text{ for some } p \in \mathbf{R}^n\}.$$

For two polyhedral subdivisions $\mathcal{C}_1$ and $\mathcal{C}_2$, the *common refinement* $\mathcal{C}_1 \wedge \mathcal{C}_2$ is defined by $\mathcal{C}_1 \wedge \mathcal{C}_2 = \{F \cap G \mid F \in \mathcal{C}_1, G \in \mathcal{C}_2, F \cap G \neq \emptyset\}$. Note that $\mathcal{C}_1 \wedge \mathcal{C}_2$ is a polyhedral subdivision of $|\mathcal{C}_1| \cap |\mathcal{C}_2|$. In particular, for a finite set of hyperplanes $\mathcal{H}$, we define the polyhedral subdivision $\mathcal{A}(\mathcal{H})$ of $\mathbf{R}^n$ as

$$\mathcal{A}(\mathcal{H}) = \bigwedge_{H \in \mathcal{H}} \{H, H^+, H^-\}.$$

That is, $\mathcal{A}(\mathcal{H})$ is the partition of $\mathbf{R}^n$ by hyperplanes in $\mathcal{H}$.

The polyhedral subdivision by the sum of two polyhedral convex functions amounts to the common refinement of the two polyhedral subdivisions associated with the given polyhedral convex functions.

Lemma 2.2. *For two polyhedral convex functions f, g with* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$*, we have*

$$\mathcal{T}(f+g) = \mathcal{T}(f) \wedge \mathcal{T}(g).$$

2.2 Polyhedral split decomposition

For a hyperplane $H_{a,r}$, the *split function* $l_{H_{a,r}} : \mathbf{R}^n \to \mathbf{R}$ associated with $H_{a,r}$ is defined to be the function such that the value $l_{H_{a,r}}(x)$ of each point x in $\mathbf{R}^n$ is $\|a\|$ times the distance between the point x and the hyperplane $H_{a,r}$, i.e., $l_{H_{a,r}}$ is given by

$$l_{H_{a,r}}(x) = |\langle a, x \rangle - r| \quad (x \in \mathbf{R}^n).$$

For a polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R}$ and a hyperplane $H_{a,r}$, we define the *quotient* $c_{H_{a,r}}(f)$ of f by $l_{H_{a,r}}$ as

$$c_{H_{a,r}}(f) = \sup\{t \in \mathbf{R}_+ \mid f - t l_{H_{a,r}} \text{ is convex}\}.$$

Let (a, r) and (a', r') be vectors in $\mathbf{R}^n \times \mathbf{R}$ such that $(a, r) = k(a', r')$ for some $k \in \mathbf{R}$ with $k \neq 0$. Since, for $t \in \mathbf{R}_+$, $f - t l_{H_{a,r}}$ is convex if and only if $f - |k| t l_{H_{a',r'}}$ is convex, we have $c_{H_{a,r}}(f) = |k| c_{H_{a',r'}}(f)$. Hence, $c_H(f) l_H$ is independent of the equation of a hyperplane H. From now on, unless otherwise stated, a hyperplane H is assumed to be represented as $H = H_{a,r}$ for a normal vector a with $\|a\| = 1$, and so $l_H(x)$ amounts to the distance between x and H.

We define the set of hyperplanes $\mathcal{H}(f)$ as

$$\mathcal{H}(f) = \{H : \text{hyperplane} \mid 0 < c_H(f) < +\infty\}.$$

The basic idea for the polyhedral split decomposition of a polyhedral convex function f is to subtract split functions associated with hyperplanes in $\mathcal{H}(f)$ from f successively. In fact, if the effective domain of f is full-dimensional, this idea directly applies to f thanks to the following fact.

Lemma 2.3 ([5, Lemma 2.5])**.** *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\dim \operatorname{dom} f = n$*. Suppose that $H, H' \in \mathcal{H}(f)$ and $t \in [0, c_H(f)]$. Then, we have*

$$c_{H'}(f - t l_H) = \begin{cases} c_{H'}(f) - t & \text{if } H = H' \\ c_{H'}(f) & \text{otherwise.} \end{cases}$$

If $\operatorname{dom} f$ is not full-dimensional, there exist infinitely many hyperplanes having the same intersection with $\operatorname{dom} f$. For this, and only for this, $\operatorname{dom} f$ is assumed to be full-dimensional.

Theorem 2.4 ([5, Theorem 2.2])**.** *A polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ with $\dim \operatorname{dom} f = n$ is uniquely decomposable as*

$$f = \sum_{H \in \mathcal{H}(f)} c_H(f) l_H + f', \tag{2.1}$$

where $f' : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is a polyhedral convex function with $c_{H'}(f') \in \{0, +\infty\}$ for any hyperplane H'.

Obviously, we have $\mathcal{T}(\alpha l_H) = \{H, H^+, H^-\}$ for any $\alpha \in \mathbf{R}_{++}$. Hence, by Lemma 2.2, the decomposition (2.1) produces a subdivision as in (2.2) below.

Lemma 2.5. *For a polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R}$, the polyhedral subdivision $\mathcal{T}(f)$ is represented as*

$$\mathcal{T}(f) = \mathcal{A}(\mathcal{H}(f)) \wedge \mathcal{T}(f'). \tag{2.2}$$

2.3 Discrete split decomposition

In this paper, a *discrete function* means a function defined on a finite set of points in $\mathbf{R}^n$. Let K be a finite set of points in $\mathbf{R}^n$. If K contains the origin $\mathbf{0}$, we assume that $f(\mathbf{0}) = 0$.

For a discrete function $f : K \to \mathbf{R}$, the *homogeneous convex closure* of f is defined by

$$\overline{f}(x) = \inf \left\{ \sum_{y \in K} \lambda_y f(y) \mid \sum_{y \in K} \lambda_y y = x, \lambda_y \geq 0 \ (y \in K) \right\} + \delta_{\operatorname{cone} K}(x) \quad (x \in \mathbf{R}^n). \tag{2.3}$$

Since K is a finite set, $\overline{f}$ is a polyhedral convex function with $\operatorname{dom} f = \operatorname{cone} K$. Furthermore, by definition, $\overline{f}$ is positively homogeneous, i.e., $\overline{f}(\alpha x) = \alpha \overline{f}(x)$ holds for $\alpha \geq 0$ and $x \in \mathbf{R}^n$.

For a function $f : \mathbf{R}^n \to \mathbf{R}$, we denote the restriction of f to K by f^K. A discrete function $g : K \to \mathbf{R}$ is said to be *convex-extensible* if it satisfies $\overline{g}^K = g$. If f is convex-extensible, we call $\overline{f}$ the *homogeneous convex extension* of f (the *extension* of f for short).

The discrete split decomposition is based on the next lemma; see the proof of Theorem 3.2 in [5].

Lemma 2.6. *Let $f : K \to \mathbf{R}$ be a discrete function on K with $\dim \operatorname{cone} K = n$. For the extension $\overline{f}$ of f, suppose that $H \in \mathcal{H}(\overline{f})$ and $t \in [0, c_H(\overline{f})]$. Then, we have $\overline{f} - t l_H = \overline{f - t l_H^K}$.*

Applying the polyhedral split decomposition to the extension of a discrete function, we obtain the discrete split decomposition.

Theorem 2.7 ([5, Theorem 3.2])**.** *Let $f : K \to \mathbf{R}$ be a discrete function on K with* $\dim \operatorname{cone} K = n$. *Then, f is uniquely decomposable as*

$$f = \sum_{H \in \mathcal{H}(\overline{f})} c_H(\overline{f}) l_H^K + f',$$

where $f' : K \to \mathbf{R}$ satisfies $c_{H'}(\overline{f'}) \in \{0, +\infty\}$ for any linear hyperplane H'. Furthermore, we have

$$\overline{f} = \sum_{H \in \mathcal{H}(\overline{f})} c_H(\overline{f}) l_H + \overline{f'}.$$

If, in addition, f is convex-extensible, then f' is also convex-extensible.

What is worth discussing is a relation between K and $\mathcal{H}(\overline{f})$ for a convex-extensible discrete function f on K. Since $\overline{f}(\mathbf{0}) = 0$, each hyperplane $H \in \mathcal{H}(\overline{f})$ is linear, i.e., $H = H_{a,0}$ for some $a \in \mathbf{R}^n$. By the definition of $\overline{f}$, an extremal ray of a cone in $\mathcal{T}(\overline{f})$ is written as αv for some vector v in K and $\alpha \in \mathbf{R}_+$. Since $\mathcal{T}(\overline{f}) = \mathcal{A}(\mathcal{H}(\overline{f})) \wedge \mathcal{T}(\overline{f'})$, the hyperplanes in $\mathcal{H}(\overline{f})$ are limited by the point set K. In fact, $\mathcal{H}(\overline{f})$ must satisfy the K-admissibility defined as follows. A set of linear hyperplanes $\mathcal{H}$ is called *K-admissible* if $\mathcal{H}$ satisfies the following.

(A1) For each $H \in \mathcal{H}$, H intersects with the relative interior of $\operatorname{cone} K$.

(A2) For each $F \in \mathcal{A}(\mathcal{H})$, $\operatorname{cone}(F \cap K) = F \cap \operatorname{cone} K$.

For simplicity, a hyperplane H is called *K-admissible* if the set $\{H\}$ is K-admissible. We define the set of linear hyperplanes $\mathcal{H}_K$ as

$$\mathcal{H}_K = \{H \mid H : \text{a } K\text{-admissible linear hyperplane}\}.$$

Since $\mathcal{H}(\overline{f})$ is K-admissible, we have $\mathcal{H}(\overline{f}) \subseteq \mathcal{H}_K$ for any discrete function $f : K \to \mathbf{R}$.

The K-admissibility also relates to the *split-decomposability.* A function f on a finite set K with $\dim \operatorname{cone} K = n$ is said to be *split-decomposable* if the *residue* f' given by $f' = f - \sum_{H \in \mathcal{H}(\overline{f})} c_H(\overline{f}) l_H^K$ is equal to the restriction of a linear function.

An addition of the restriction of a linear function to a discrete function f does not change the quotient for $\overline{f}$ and hence does not affect the split-decomposability of f, which follows from the next lemma. Note that the quotient $c_H(\langle q, \cdot \rangle)$ equals zero for any hyperplane H.

Lemma 2.8. *Let f be a discrete function on K with* $\dim \operatorname{cone} K = n$. *For any factor $\alpha \in \mathbf{R}_{++}$ and any vector $q \in \mathbf{R}^n$, we have $\overline{\alpha f + (\langle q, \cdot \rangle)^K} = \alpha \overline{f} + \langle q, \cdot \rangle$*

Moreover, according to the following proposition, every split-decomposable function is constructed from a K-admissible set of hyperplanes. Thus, split-decomposable functions are also determined by K through the K-admissible sets of hyperplanes since the K-admissibility depends on K.

Proposition 2.9 ([5, Proposition 3.5])**.** *Let K be a finite set such that* $\dim \operatorname{cone} K = n$. *For a subset $\mathcal{H}$ of hyperplanes $\mathcal{H}_K$ and positive weights $\{\alpha_H \in \mathbf{R}_{++} \mid H \in \mathcal{H}\}$ on the hyperplanes in $\mathcal{H}$, let $f = \sum_{H \in \mathcal{H}} \alpha_H l_H^K$. Then the following conditions* (a), (b) *and* (c) *are equivalent.*

(a) $\overline{f} = \sum_{H \in \mathcal{H}} \alpha_H l_H + \delta_{\operatorname{cone} K}$.

(b) *$\mathcal{H} = \mathcal{H}(\overline{f})$ and $\alpha_H = c_H(\overline{f})$ for every $H \in \mathcal{H}$.*

(c) *$\mathcal{H}$ is K-admissible.*

3 M-convex function and polyhedral split decomposition

3.1 M-convex functions

The aim of this subsection is to introduce M-convex functions. For a point $x \in \mathbf{R}^n$, we define $\mathrm{supp}^+ x = \{i \mid x(i) > 0, i \in X\}$ and $\mathrm{supp}^- x = \{i \mid x(i) < 0, i \in X\}$. A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is said to be *M-convex* if it satisfies the following exchange property:

(**M-EXC**[**Z**]) For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

The M-convexity is generalized to a polyhedral convex function as follows. A polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is said to be *M-convex* if it satisfies the following exchange property:

(**M-EXC**[**R**]) For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$, there exist $v \in \mathrm{supp}^-(x - y)$ and a positive number $\alpha_0 \in \mathbf{R}_{++}$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v))$$

for all $\alpha \in [0, \alpha_0]$.

The effective domain of a polyhedral M-convex function is an M-convex polyhedron, which is defined as follows. A nonempty polyhedron $B \subseteq \mathbf{R}^n$ is defined to be an *M-convex polyhedron* if it satisfies the following:

(**B-EXC**[**R**]) For $x, y \in B$ and $u \in \mathrm{supp}^+(x - y)$, there exist $v \in \mathrm{supp}^-(x - y)$ and a positive number $\alpha_0 \in \mathbf{R}_{++}$ such that $x - \alpha(\chi_u - \chi_v) \in B$ and $y + \alpha(\chi_u - \chi_v) \in B$ for all $\alpha \in [0, \alpha_0]$.

In fact, a polyhedral convex function f whose subdivision $\mathcal{T}(f)$ consists of M-convex polyhedra is M-convex, which is a consequence of Lemma 2.1 and the following theorem.

Theorem 3.1 ([10, Theorem 5.2]). *For a polyhedral convex function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$*, the following two conditions* (1) *and* (2) *are equivalent.*

(1) f *is a polyhedral M-convex function.*

(2) $\mathrm{argmin}(f - \langle p, \cdot \rangle)$ *is an M-convex polyhedron for every* $p \in \mathbf{R}^n$ *with* $\inf(f - \langle p, \cdot \rangle) > -\infty$.

In general, a sum of polyhedral M-convex functions is not necessarily M-convex. M-convexity of the sum depends on whether the polyhedral subdivision induced by the sum consists of M-convex polyhedra.

3.2 Homogeneous convex extension of a distance

Let $X = \{1, 2, \ldots, n\}$, and let Ω be the finite set defined by

$$\Omega = \{\chi_i - \chi_j \mid i, j \in X\}.$$

A distance d on X can be regarded as a discrete function defined on the set Ω by setting

$$d(\chi_i - \chi_j) = d(i, j) \quad (i, j \in X).$$

It is important that d is a distance since the convex-extensibility on Ω is guaranteed by the triangle inequality.

Lemma 3.2. *A discrete function $f : \Omega \to \mathbf{R}$ with $f(\mathbf{0}) = 0$ is convex-extensible if and only if f satisfies $f(\chi_i - \chi_j) \leq f(\chi_i - \chi_k) + f(\chi_k - \chi_j)$ for all $i, j, k \in X$.*

In discrete convex analysis, it is known that a positively homogeneous M-convex function is given by the homogeneous convex extension of a distance on Ω. However, since $\dim \operatorname{cone} \Omega = n - 1$, we cannot directly apply the polyhedral split decomposition to the homogeneous convex extension. To cope with this difficulty, we extend Ω to $\Omega_{\mathbf{1}} = \Omega \cup \{\mathbf{1}, -\mathbf{1}\}$, and define $d(\mathbf{1}) = d(-\mathbf{1}) = 0$. Then, $\operatorname{cone} \Omega_{\mathbf{1}} = \mathbf{R}^n$. The homogeneous convex extension of d on $\Omega_{\mathbf{1}}$ is given, for each x in $\mathbf{R}^n$, by

$$\overline{d}(x) = \inf \left\{ \sum_{i,j \in X} \lambda_{ij} d(\chi_i - \chi_j) \;\middle|\; \begin{array}{c} \sum_{i,j \in X} \lambda_{ij}(\chi_i - \chi_j) + \lambda_{\mathbf{1}}\mathbf{1} + \lambda_{-\mathbf{1}}(-\mathbf{1}) = x, \\ \lambda_{ij} \geq 0 \ (i, j \in X), \ \lambda_{\mathbf{1}} \geq 0, \ \lambda_{-\mathbf{1}} \geq 0 \end{array} \right\}.$$

For any point $x \in \mathbf{R}^n$, x is represented as $x = x' + \lambda \mathbf{1}$ with $\sum_{i=1}^n x'(i) = 0$. In fact, since $d(\mathbf{1}) = d(-\mathbf{1}) = 0$, we have $\overline{d}(x) = \overline{d}(x')$, which allows us to identify the extension of d on Ω with that on $\Omega_{\mathbf{1}}$.

Figure 1 (c) illustrates the homogeneous convex extension on Ω of a metric d on $X = \{i, j, k\}$. Since a point in Ω is on a linear hyperplane as in Figure 1 (a), we can project $\{(\chi_i - \chi_j, d(i, j)) \mid i, j \in X\}$ to a 3-dimensional space as shown in Figure 1 (b).

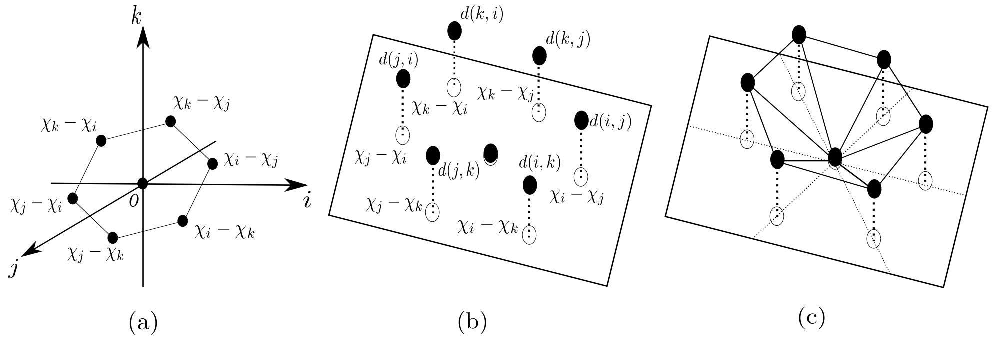


Figure 1: The homogeneous convex extension on Ω of a metric d on $X = \{i, j, k\}$.

3.3 Discrete split decomposition of a distance

To apply the the polyhedral split decomposition, we need the set $\mathcal{H}_{\Omega_{\mathbf{1}}}$ of $\Omega_{\mathbf{1}}$-admissible hyperplanes. For an X-split $\{A, B\}$, we denote by $H_{\{A,B\}}$ the hyperplane $H_{a_{\{A,B\}},0}$ with

$$a_{\{A,B\}} = \frac{|A|\,|B|}{|A| + |B|} \left(\frac{\chi_A}{|A|} - \frac{\chi_B}{|B|} \right).$$

For the simplicity of the quotients for $\overline{d}$, we use $a_{\{A,B\}}$ though $\|a_{\{A,B\}}\| \neq 1$.

Proposition 3.3 ([7, Proposition 9.3])**.** *The set of $\Omega_{\mathbf{1}}$-admissible hyperplanes is given by*

$$\mathcal{H}_{\Omega_{\mathbf{1}}} = \{H_{\{A,B\}} \mid \{A, B\} : \text{an } X\text{-split}\} \cup \{H_{\mathbf{1},0}\}.$$

Furthermore, for each $H_{\{A,B\}}$, the quotient $c_{H_{\{A,B\}}}(\overline{d})$ for a split function $l_{H_{\{A,B\}}}$ is given by $c_{H_{\{A,B\}}}(\overline{d}) = \max\{0, \tilde{c}_{H_{\{A,B\}}}(d)\}$, where $\tilde{c}_{H_{\{A,B\}}}$ is represented as

$$\tilde{c}_{H_{\{A,B\}}}(d) = \frac{1}{2} \min_{i,j \in A,\ k,l \in B} \left\{ \begin{array}{l} d(i,k) + d(l,j) - d(i,j) - d(l,k), \\ d(i,l) + d(k,j) - d(i,j) - d(k,l) \end{array} \right\}.$$

If, in particular, d is a metric, we have $\tilde{c}_{H_{\{A,B\}}}(d) = b^d_{\{A,B\}}$, where $b^d_{\{A,B\}}$ is called the *Buneman index* defined by

$$b^d_{\{A,B\}} = \frac{1}{2} \min_{i,j \in A,\ k,l \in B} \{d(i,k) + d(j,l) - d(i,j) - d(k,l)\}\,.$$

Theorem 3.4 ([7, Theorem 9.6])**.** *Let $d : X \times X \to \mathbf{R}$ be a distance. Then d can be decomposed as*

$$d = \sum_{\{A,B\} \in \Sigma(d)} c_{H_{\{A,B\}}}(\overline{d}) l^{\Omega_\mathbf{1}}_{H_{\{A,B\}}} + d', \tag{3.1}$$

where $\Sigma(d) = \{\{A,B\} \mid \{A,B\} : \text{an } X\text{-split with } c_{H_{\{A,B\}}}(\overline{d}) > 0\}$ and $d' : X \times X \to \mathbf{R}$ is a distance with $c_{H_{\{A,B\}}}(\overline{d'}) = 0$ for any X-split $\{A,B\}$. Furthermore, we have

$$\overline{d} = \sum_{\{A,B\} \in \Sigma(d)} c_{H_{\{A,B\}}}(\overline{d})\, l_{H_{\{A,B\}}} + \overline{d'}. \tag{3.2}$$

The $\Omega_\mathbf{1}$-admissibility of a set of hyperplanes can be translated to the compatibility of X-splits. Two X-splits $\{A,B\}$ and $\{C,D\}$ are called *compatible* if at least one of the sets $A \cap C, A \cap D, B \cap C$ and $B \cap D$ is the empty set. A collection of X-splits is called *pairwise compatible* if any two X-splits in the collection are compatible. For a subset $\mathcal{H}$ of $\mathcal{H}_{\Omega_\mathbf{1}}$, we denote

$$\Sigma_\mathcal{H} = \{\{A,B\} \mid \{A,B\} : \text{an } X\text{-split with } H_{\{A,B\}} \in \mathcal{H}\}.$$

Proposition 3.5 ([7, Proposition 9.8])**.** *A set of hyperplanes $\mathcal{H} \subseteq \mathcal{H}_{\Omega_\mathbf{1}}$ is $\Omega_\mathbf{1}$-admissible if and only if $\Sigma_\mathcal{H}$ is pairwise compatible.*

Furthermore, the decomposition (3.1) can be interpreted as a decomposition of d into a sum of split metrics and some distance d'. It is easily seen that, for a split function $l_{H_{\{A,B\}}}$, we have $l_{H_{\{A,B\}}}(\chi_i - \chi_j) = 1$ if $|\{i,j\} \cap A| = 1$ and $l_{H_{\{A,B\}}}(\chi_i - \chi_j) = 0$ otherwise. Hence, $l^{\Omega_\mathbf{1}}_{H_{\{A,B\}}}$ can be identified with the split metric $\xi_{\{A,B\}}$. Since $H_{\{A,B\}}$ is $\Omega_\mathbf{1}$-admissible, the extension of $l^{\Omega_\mathbf{1}}_{H_{\{A,B\}}}$ coincides with $l_{H_{\{A,B\}}}$ by Proposition 2.9.

Proposition 3.6. *For an X-split $\{A,B\}$, the split function $l_{H_{\{A,B\}}}$ is the homogeneous convex extension of the split metric $\xi_{\{A,B\}}$ with respect to $\{A,B\}$.*

It is known that a metric d is a tree metric if and only if d is represented as a sum of split metrics for pairwise compatible X-splits.

Proposition 3.7 ([7, Proposition 9.7])**.** *A metric d is a tree metric if and only if the extension of d is decomposed as*

$$\overline{d} = \sum_{\{A,B\} \in \Sigma(d)} c_{H_{\{A,B\}}}(\overline{d}) l_{H_{\{A,B\}}}.$$

By Proposition 3.7, a split-decomposable function f on $\Omega_\mathbf{1}$ (with $f(\mathbf{1}) = f(-\mathbf{1}) = 0$) corresponds to a sum of a tree metric and a linear function.

Figure 2 illustrates the polyhedral split decomposition of a metric on X with $|X| = 3$. It is known that every 3-point metric can be represented as a sum of split metrics, i.e., $d' = 0$ in the decomposition (3.1).

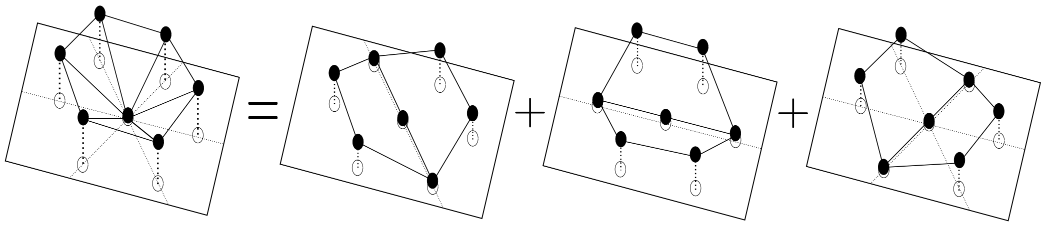


Figure 2: The polyhedral split decomposition of a metric on $X = \{i, j, k\}$.

3.4 M-convexity of split functions

As in Proposition 3.6, the split function $l_{H_{\{A,B\}}}$ is the homogeneous convex extension of the split metric $\xi_{\{A,B\}}$, which immediately implies that $l_{H_{\{A,B\}}}$ is a positively homogeneous M-convex function (if $l_{H_{\{A,B\}}}$ is restricted on the hyperplane $H_{\mathbf{1},0}$). Then, we are led to the following statement that reveals a remarkable phenomenon for M-convex functions. Recall that, in general, the sum of M-convex functions is not necessarily M-convex.

Theorem 3.8. *The polyhedral split decomposition of a positively homogeneous M-convex function as in* (3.2) *is a decomposition of a polyhedral M-convex function into a sum of polyhedral M-convex functions.*

The rest of this subsection is devoted to unraveling why the M-convexity is preserved in the polyhedral split decomposition of the extension of a distance. Since the polyhedral split decomposition is a geometric notion, we explain it especially in terms of geometry. To this end, we study the polyhedral subdivision induced by split functions since, by Theorem 3.1 and Lemma 2.1, the M-convexity of $\overline{d}$ is equivalent to the M-convexity of all cones in $\mathcal{T}(\overline{d})$.

Consider that we add a split function $l_{H_{\{A,B\}}}$ to the extension $\overline{d'}$ of d' in (3.1). Since the polyhedral split decomposition is designed on the basis of Lemma 2.6, $l_{H_{\{A,B\}}} + \overline{d'}$ coincides with the extension of $l^{\Omega_{\mathbf{1}}}_{H_{\{A,B\}}} + d'$, that is,

$$l_{H_{\{A,B\}}} + \overline{d'} = \overline{l^{\Omega_{\mathbf{1}}}_{H_{\{A,B\}}} + d'}. \tag{3.3}$$

Since the right-hand side of (3.3) is the extension of a discrete function on $\Omega_{\mathbf{1}}$, the equality requires that

$$\mathrm{cone}(F \cap \Omega) = F \cap \mathrm{cone}\, \Omega \quad \text{for each cone } F \in \mathcal{T}(l_{H_{\{A,B\}}} + \overline{d'}). \tag{3.4}$$

By interpreting the decomposition (3.2) as successive additions of split functions to $\overline{d'}$, such a property must hold at each of the additions. Since each addition of a split function produces a common refinement of $\mathcal{T}(l_{H_{\{A,B\}}}) = \{H_{\{A,B\}}, H^+_{\{A,B\}}, H^-_{\{A,B\}}\}$ and the present polyhedral subdivision, a cone appearing in the addition has a form as $H_{\{A,B\}} \cap F$, $H^+_{\{A,B\}} \cap F$, or $H^-_{\{A,B\}} \cap F$ for a cone F. As $\overline{d'}$ is M-convex, (the intersections with cone Ω of) cones in $\mathcal{T}(\overline{d'})$ is M-convex. Hence, according to the interpretation above, the following observation points out that the preservation of M-convexity in the additions of split functions is due to a property as in (3.4).

Proposition 3.9. *Let F be an M-convex cone, and let $\{A, B\}$ be an X-split. Then the following* (1), (2) *and* (3) *hold.*

(1) $\operatorname{cone}((H_{\{A,B\}} \cap F) \cap \Omega) = (H_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$ *if and only if* $(H_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$ *is an M-convex cone.*

(2) $\operatorname{cone}((H^+_{\{A,B\}} \cap F) \cap \Omega) = (H^+_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$ *if and only if* $(H^+_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$ *is an M-convex cone.*

(3) $\operatorname{cone}((H^-_{\{A,B\}} \cap F) \cap \Omega) = (H^-_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$ *if and only if* $(H^-_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$ *is an M-convex cone.*

Proof. In this proof, we use the fact that a cone is M-convex if and only if every ray of the cone has the direction $\chi_i - \chi_j$ for some $i, j \in X$.

We show (1). The only-if part is obvious since the set $(H_{\{A,B\}} \cap F) \cap \Omega$ consists of vectors $\chi_i - \chi_j$ for $i, j \in X$. By the characterization of M-convex cones, $(H_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$ is M-convex.

Next, we show the if part. Clearly, $\operatorname{cone}((H_{\{A,B\}} \cap F) \cap \Omega) \subseteq (H_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$. Then, we show the reverse inclusion. We need to consider only the rays of $(H_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$. Since $(H_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$ is M-convex, its ray has the direction $\chi_i - \chi_j$ for $i, j \in X$. Obviously, $\chi_i - \chi_j \in \Omega$. Hence, $\chi_i - \chi_j \in (H_{\{A,B\}} \cap F) \cap \Omega$, and thus $\operatorname{cone}((H_{\{A,B\}} \cap F) \cap \Omega) = (H_{\{A,B\}} \cap F) \cap \operatorname{cone} \Omega$.

The assertions (2) and (3) are shown similarly. □

4 Quadratic M-convex functions and lattice dicings

In this section, we apply the polyhedral split decomposition to a quadratic M-convex function. Then, quadratic M-convex function turns out to be a function that is split-decomposable around each point in its domain. Furthermore, this result indicates that there is a lattice dicing or, equivalently, a zonotope which fills the space facet-to-facet by its translation copies [3, 4]. Inspired by a result for lattice dicings, we obtain another canonical representation of quadratic M-convex functions.

We start with the property of M-convex functions as described in the next theorem; see also [9, Theorem 6.61].

Theorem 4.1 ([10, Theorem 4.15])**.** *For an M-convex function* $f : \mathbf{Z}^n \to \mathbf{R}$ *and* $x \in \operatorname{dom} f$*, define* $\gamma_{f,x}(u,v) = f(x + \chi_u - \chi_v) - f(x)$ $(u, v \in X)$*. Then* $\gamma_{f,x}$ *is a distance.*

For each $x \in \operatorname{dom} f$, we regard $\gamma_{f,x}$ as a discrete function on $\Omega_{\mathbf{1}}$ by setting

$$\gamma_{f,x}(\chi_i - \chi_j) = \gamma_{f,x}(i,j) \ (i,j \in X), \quad \gamma_{f,x}(\mathbf{1}) = \gamma_{f,x}(-\mathbf{1}) = 0.$$

Then, by Lemma 3.2 and Theorem 4.1, $\gamma_{f,x}$ is convex-extensible on $\Omega_{\mathbf{1}}$. In addition, the discrete split decomposition is applicable to $\gamma_{f,x}$.

We are particularly interested in the case that f is a quadratic M-convex function on $\mathbf{Z}^n \cap \operatorname{cone} \Omega$, i.e., f can be represented as $f(x) = \frac{1}{2} x^\top A x$ for all $x \in \mathbf{Z}^n \cap \operatorname{cone} \Omega$ with some coefficient matrix A. The interest is because a quadratic M-convex function is available from a tree metric, i.e., a split-decomposable function on $\Omega_{\mathbf{1}}$. For a distance $d : X \times X \to \mathbf{R}_+$, a matrix $D = (d_{ij})$ is defined by $d_{ij} = d(i,j)$ for all $i, j \in X$ and called the *distance matrix* of d.

Theorem 4.2 ([6, Theorem 3.1]). *A quadratic form $f(x)$ defined on $\mathbf{Z}^n \cap \mathrm{cone}\,\Omega$ is M-convex if and only if there exists a tree metric $d : X \times X \to \mathbf{R}_+$ such that*

$$f(x) = -\frac{1}{2}x^\top Dx \quad (x \in \mathbf{Z}^n \cap \mathrm{cone}\,\Omega),$$

where D is the distance matrix of d.

Let f be a quadratic M-convex function on $\mathbf{Z}^n \cap \mathrm{cone}\,\Omega$. Then, by Theorem 4.2, f is represented as $f(x) = -\frac{1}{2}x^\top Dx$ with the distance matrix D of a tree metric d. Moreover, $\gamma_{f,x}$ for f and $x \in \mathbf{Z}^n \cap \mathrm{cone}\,\Omega$ is given by

$$\begin{aligned}
\gamma_{f,x}(u,v) &= f(x + \chi_u - \chi_v) - f(x) \\
&= -\frac{1}{2}(x + \chi_u - \chi_v)^\top D(x + \chi_u - \chi_v) + \frac{1}{2}x^\top Dx \\
&= -x^\top D\chi_u + x^\top D\chi_v - \frac{1}{2}\chi_u^\top D\chi_u + \chi_v^\top D\chi_u + \frac{1}{2}\chi_v^\top D\chi_v \\
&= \langle -x^\top D, \chi_u - \chi_v\rangle + d(u,v) \quad (u, v \in X).
\end{aligned}$$

Therefore, $\gamma_{f,x}$ can be regarded as a discrete function on $\Omega_{\mathbf{1}}$ as follows:

$$\gamma_{f,x}(\cdot) = d(\cdot) + (\langle -x^\top D, \cdot\rangle)^{\Omega_{\mathbf{1}}}.$$

By Lemma 2.8, we have $\overline{\gamma_{f,x}} = \overline{d} + \langle -x^\top D, \cdot\rangle$. Since $c_{H_{A,B}}(\overline{\gamma_{f,x}})$ for $H_{\{A,B\}}$ depends only on d, the quotient $c_{H_{A,B}}(\overline{\gamma_{f,x}})$ coincides with $c_{H_{A,B}}(\overline{d}) = \max\{0, b^d_{\{A,B\}}\}$. Furthermore, since d is a tree metric, we have the following by Proposition 3.7.

Theorem 4.3. *For a quadratic M-convex function f on $\mathbf{Z}^n \cap \mathrm{cone}\,\Omega$ and any point $x \in \mathbf{Z}^n \cap \mathrm{cone}\,\Omega$, the function $\gamma_{f,x}(\cdot)$ is split-decomposable.*

As the set $\mathcal{H}(\overline{\gamma_{f,x}})$ is independent of a point x, by abuse of notation, we use $\mathcal{H}(f)$ for $\mathcal{H}(\overline{\gamma_{f,x}})$.

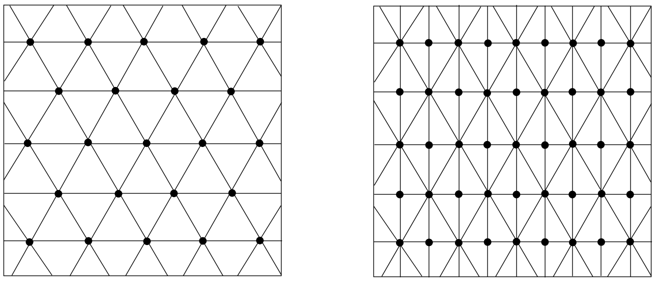
Let x be a point in $\mathbf{Z}^n \cap \mathrm{cone}\,\Omega$. Since $\gamma_{f,x}$ is split-decomposable, the polyhedral subdivision $\mathcal{T}(\overline{\gamma_{f,x}})$ induced by $\overline{\gamma_{f,x}}$ coincides with the polyhedral subdivision $\mathcal{A}(\mathcal{H}(f))$ given by the hyperplane arrangement $\mathcal{H}(f)$. Hence, if $\mathcal{H}(f)$ contains n hyperplanes with linearly independent normal vectors, the point x amounts to the intersection point of $H_{\mathbf{1},0}$ and translations of linear hyperplanes in $\mathcal{H}(f)$. Note that x is a point in the lattice generated by Ω. Hence, by appropriately translating $H_{\mathbf{1},0}$ too, every point of the lattice generated by $\Omega_{\mathbf{1}}$ appears as the intersection point of translations of linear hyperplanes in $\mathcal{H}(f) \cup \{H_{\mathbf{1},0}\}$, which means that there is a lattice dicing, described below, with respect to the lattice generated by $\Omega_{\mathbf{1}}$.

We here introduce the *lattice dicing* in $\mathbf{R}^n$. Let $\mathcal{D}$ be a finite set $\{D_1, D_2, \ldots, D_m\}$ of families of equispaced parallel hyperplanes in $\mathbf{R}^n$. A *lattice dicing* formed by $\mathcal{D}$ is an arrangement of hyperplanes in the families in $\mathcal{D}$ that satisfies both the following two properties:

(D1) Among hyperplanes in the families of $\mathcal{D}$, there are n hyperplanes with linearly independent normal vectors.

(D2) For each vertex of the arrangement, there is one hyperplane from each family.

By the condition (D2), the vertex set of a lattice dicing forms a lattice. Note that not every central hyperplane arrangement provides a lattice dicing. For example, we can choose at most three lines to make a lattice dicing in $\mathbf{R}^2$. Figure 3 (a) is a lattice dicing in $\mathbf{R}^2$, and (b) is not a lattice dicing. Figure 3 (a) also illustrates the subdivision of $\mathrm{cone}\,\Omega$ for X with $|X| = 3$ induced by a quadratic M-convex function. In addition, it is known that a central hyperplane arrangement produces a zonotope that is the Minkowski sum of the normal vectors of the hyperplanes in the arrangement. For a lattice dicing of a space, by constructing a zonotope around each of the vertices of the lattice dicing, the space is filled with facet-to-facet zonotopes.



(a) (b)

Figure 3: (a) a lattice dicing, (b) not a lattice dicing.

We now turn to the relation between lattice dicings and $\Omega_{\mathbf{1}}$-admissible sets of hyperplanes. Let $\mathcal{H} = \{H_{a_1,0}, H_{a_2,0}, \ldots, H_{a_{m-1},0}, H_{a_m,0}\}$ be an $\Omega_{\mathbf{1}}$-admissible set of hyperplanes where $a_i = a_{\{A_i,B_i\}}$ for each $i = 1, \ldots, m-1$ and $a_m = \mathbf{1}$. For each $H_{a_i,0} \in \mathcal{H}$, let D_i be the set of equispaced parallel hyperplanes $H_{a_i,k}$ for each $k \in \mathbf{Z}$. Suppose that $\mathcal{H}$ contains n hyperplanes with linearly independent normal vectors. Then, by the argument above, the set $\mathcal{D}(\mathcal{H})$ defined to be $\{D_1, D_2, \ldots, D_m\}$ forms a lattice dicing with respect to the lattice generated by $\Omega_{\mathbf{1}}$.

Corollary 4.4. *Let $\mathcal{H}$ be a subset of $\mathcal{H}_{\Omega_{\mathbf{1}}}$ containing $H_{\mathbf{1},0}$ and n hyperplanes with linearly independent normal vectors. If $\mathcal{H}$ is $\Omega_{\mathbf{1}}$-admissible, then the set $\mathcal{D}(\mathcal{H})$ forms a lattice dicing with respect to the lattice generated by $\Omega_{\mathbf{1}}$.*

Furthermore, as a result on the lattice dicings, it is known that, for a set $\mathcal{H}$ of hyperplanes which provides a lattice dicing, there is a routine for obtaining a quadratic function that induces the lattice dicing. Inspired by this, we propose another canonical form for a quadratic M-convex function.

Let f be a quadratic M-convex function, i.e., $f(x) = -\frac{1}{2}x^\top Dx$ for a tree metric d. Then, the set $\mathcal{H}(f) = \{H_{a_1,0}, H_{a_2,0}, \ldots, H_{a_m,0}\}$ of hyperplanes is $\Omega_{\mathbf{1}}$-admissible, where $a_i = a_{\{A_i,B_i\}}$ for each $i = 1, \ldots, m$. We define a quadratic function g by

$$\begin{aligned} g(x) &= \sum_{i=1}^{m} b^d_{\{A_i,B_i\}}(\langle a_i, x\rangle)^2 \\ &= x^\top \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} \mathrm{diag}[b^d_{\{A_1,B_1\}}, b^d_{\{A_2,B_2\}}, \ldots, b^d_{\{A_m,B_m\}}] \begin{bmatrix} a_1^\top \\ a_2^\top \\ \vdots \\ a_m^\top \end{bmatrix} x, \end{aligned}$$

where $\mathrm{diag}[b^d_{\{A_1,B_1\}}, b^d_{\{A_2,B_2\}}, \ldots, b^d_{\{A_m,B_m\}}]$ is a diagonal matrix whose diagonal entries are $b^d_{\{A_1,B_1\}}, b^d_{\{A_2,B_2\}}, \ldots, b^d_{\{A_m,B_m\}}$. Writing $g(x) = x^\top Qx$, we have $-\frac{1}{2}D \neq Q$. Nevertheless, a computation shows that $f(x) = g(x)$ for each point $x \in \mathbf{Z}^n \cap \mathrm{cone}\,\Omega$.

Theorem 4.5. *A quadratic form $g(x)$ defined on $\mathbf{Z}^n \cap \mathrm{cone}\,\Omega$ is M-convex if and only if there exist a pairwise compatible set of X-splits $\{\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}\}$ and positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $g(x) = \sum_{i=1}^m \lambda_i(\langle a_i, x\rangle)^2$ where a_i is the vector given by*

$$a_i = \frac{|A_i|\,|B_i|}{|A_i| + |B_i|}\left(\frac{\chi_{A_i}}{|A_i|} - \frac{\chi_{B_i}}{|B_i|}\right).$$

Obviously, the term $(\langle a, x\rangle)^2$ is a rank one form. Hence, $g(x) = \sum_{i=1}^m \lambda_i(\langle a_i, x\rangle)^2$ is a positive combination of rank one forms.

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