NANZAN-TR-2009-02
(revised version of NANZAN-TR-2008-03)

Hub Arc Location with Competition

Mihiro Sasaki, James F. Campbell, Andreas T. Ernst,
Mohan Krishnamoorthy

July 2009

Technical Report of the Nanzan Academic Society
Information Sciences and Engineering

Hub Arc Location with Competition

Mihiro Sasaki

Department of Information Systems and Mathematical Sciences, Nanzan University, Japan

James F. Campbell

College of Business Administration, University of Missouri - St. Louis, USA

Andreas T. Ernst

CSIRO Mathematical and Information Sciences, Australia

Mohan Krishnamoorthy

Department of Mechanical Engineering, Monash University, Australia

July 3, 2009

Abstract

In this paper, we consider the design of large-scale hub-and-spoke transportation networks in a competitive environment. We adopt the hub arc location model that locates arcs with discounted transport costs connecting pairs of hub facilities. Two firms compete for customers in a Stackelberg framework where the leader firm locates hub arcs to maximize its revenue, given that the follower firm will subsequently locate its own hub arcs to maximize its own revenue. Several mechanisms are presented to allocate traffic between the two firms based on the relative utility of travel via the competing hub networks. Results with up to three hub arcs for each competing firm show the role of a competitive environment in designing transportation systems.

key words: Hub location, competition, transportation, Stackelberg

1 Introduction

Hub-and-spoke networks play an important role in many transportation systems. These networks provide efficient transportation between many origins and destinations (e.g., cities) via a set of hubs that serve as switching and flow consolidation points, hub arcs that connect two hubs with a discounted travel cost, and access arcs that connect the non-hub nodes and hubs. Hub networks use fewer arcs than in a point-to-point network and thus can reduce transportation costs by exploiting the economies of scale from consolidated flows. Because of the rich applications in the real world, studies on various hub location models have attracted much attention since O'Kelly [15]. The large, and growing, literature on hub location research is summarized in Alumur and Kara [2] and Campbell et al. [4]. Nearly all hub location research has been directed at finding an optimal (or near-optimal) hub network for a single firm to serve a given set of demand specified as flows between many origins and destinations.

However, real-world hub-based transportation systems typically operate in a competitive environment where several carriers (e.g., airlines or motor carriers) exist in a market and compete in transporting freight or passengers throughout a geographic region. In this case, the customers (passengers or freight shippers) must decide which competing carrier(s) to use, and this is typically based on the relative level of service provided and the costs (or fare) charged. Thus, competitive hub models require designing hub networks for each competitor and allocating the demand among the competitors. The objective is usually to maximize the market share captured, where market share may be measured in terms of the percentage of the passengers, freight, revenue, or profit captured. This competition to capture market share is likely to influence the optimal hub locations and hub network design. Reviews of competitive location research for general (non-hub) networks include [8, 9, 17, 22].

Although a variety of hub location models have been studied in the last two decades, studies on competitive hub location problems are scarce. The earliest work is Marianov et al. [14], which formulated a competitive hub median problem on a network. This model assumes that one firm locates p hubs optimally (as in a multiple allocation p-hub median problem [4]) and then the second firm locates p hubs, given the locations of the first firm's hubs, to maximize the flow captured. Wagner [23] provides improved formulations and results for the problems presented in [14] with optimal solutions for up to 50 nodes and 5 hubs. These works allocate customers between the two firms based on the relative *costs* of the OD paths and include both a binary "all-or-nothing" allocation, where all passengers for each OD pair are allocated to whichever firm provides the lowest cost OD path (with ties being allocated to the first firm), and a five-level fractional allocation, where each firm captures 0%, 25%, 50%, 75% or 100% of the passengers, depending on the relative OD path costs for the two competitors. Eiselt and Marianov [10] extend this line of research by replacing the discrete passenger allocation mechanism with a continuous proportional allocation based on the relative travel time and travel cost of the OD paths of the competitors. They provide solutions with up to five hubs for each competitor using heuristic procedures where the first of the two competing firms locates its hubs either at random or to provide the optimal p-hub median network. The models in [10, 14, 23] assume the hub-level network is fully connected (i.e., there is a hub arc between every pair of hub nodes) and OD paths are limited to include at most three arcs and one hub arc. These are also sequential competitive location models in that the first firm locates its hubs without anticipation of a future competitor.

In practice, competitors are often aware of each other and one firm (the leader) will locate its hubs in anticipation of another firm (the follower) optimally locating its hubs based on the known locations of the leader's hubs. In this case, the leader seeks to locate its hubs so that its objective is optimized *after* the follower best locates its hubs. This is a Stackelberg hub location problem, analogous to the Stackelberg location problem on a network introduced by Hakimi [11] and used in several other studies of non-hub facility location (e.g., [18, 21]). Sasaki and Fukushima [20] presented a continuous Stackelberg hub location model where passenger allocations are determined by a logit function. The results showed that the leader firm may suffer heavy losses if it neglects to consider the competitor's strategies. Sasaki [19] considers a discrete Stackelberg hub location model with flow threshold constraints to ensure that a firm does not carry an unrealistically low level of flow for any OD pair. The model is formulated as a bilevel programming problem where the upper (leader) and lower (follower) problems are binary integer programs. Both [19] and [20] limit OD paths to a single hub stop, so there is no discounted inter-hub travel (i.e., no hub arcs). There has also been some research involving competitive models for locating a single intercontinental gateway (hub) airport, but these works focus on issues such as airline alliances and mergers [1] and setting intercontinental service frequencies[13], more than hub network design.

In this paper, we present a more general discrete Stackelberg hub location problem using the hub arc location model [5, 6] that locates hub arcs whose endpoints are hub nodes, rather than the hub median model that locates fully connected hub nodes as in [14, 23]. The hub arc model allows OD paths with one or two stops at hubs and helps concentrate flows on the discounted hub arcs, by relaxing the restriction in hub median models that every flow between two hubs is discounted, whether or not it is warranted by the level of flow. Furthermore, in our model each firm seeks to maximize the revenue (not traffic) captured and we employ a flexible customer allocation mechanism to model the different customer behaviors that might arise in applications ranging from passenger airlines, to express parcel delivery, to ground (truck) freight transportation. Unlike other models that implicitly treat all customers (e.g., passengers) equivalently by maximizing the traffic captured, in our model some customers are more valuable since they generate higher revenues (e.g., from higher fares for longer trips). We assume the same revenue applies to each competitor for each OD pair, which may be realistic in the long run due to competitive pressures. Because firms in our model do not compete based on revenues (e.g., by offering different fares), we focus on the competition from the different levels of service offered by the OD paths through the hub networks of each firm. We adopt a flexible passenger allocation function that allocates a fraction of demand to each competitor based on the relative utility of the OD paths for each competitor. This allows an "all-or-nothing" allocation and the fractional allocations as in [14, 23].

The remainder of this paper is organized as follows. In Section 2, we provide some background and a formulation of the model. Section 3 describes the solution algorithm and Section 4 includes computational results using real airlines' data. In Section 5, we give concluding remarks and mention some future work.

2 Model description

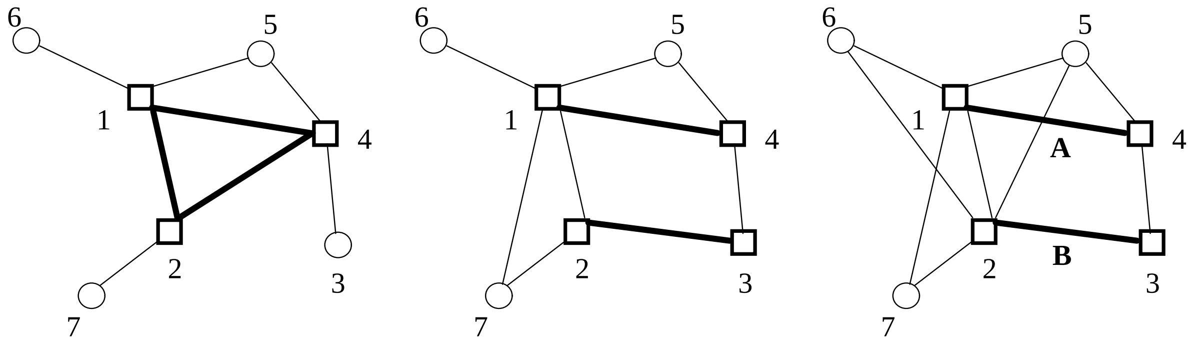
In the hub arc location model, each firm locates a given number of hub arcs. We assume that the firms' hub sets are disjoint, so they do not share any hubs or hub arcs. Without this assumption, the follower can capture at least 50% of the market by using the same hub arcs as the leader. Given the large resources required for a passenger airline hub, the assumption is reasonable; for other applications, (e.g., trucking) this assumption could be relaxed. OD paths for our hub arc location model are limited to three arcs, where the first arc is for collection from the origin to a hub and the last arc is for distribution from a hub to the destination. Each of these may be a degenerate arc (from a node to itself) if the node is also a hub. OD paths may also include a central hub arc for transfer between two hubs.

Figure 1 shows three possible multiple allocation hub networks that provide service for flows among seven origin/destination nodes. Figure 1(a) shows a 3-hub median solution with hubs at nodes 1, 2 and 4 (shown as squares) and three hub arcs (in bold) connecting the hubs. This is the type of model analyzed in previous competitive hub location research [10, 14, 23] where each firm uses a fully connected network of hub arcs. Figure 1(b) shows a 2-hub arc solution (for one firm) with hubs at nodes 1, 2, 3 and 4 connected by two hub arcs between nodes 1 and 4 and nodes 2 and 3. Because each OD pair is joined by a path with at most three arcs where the central arc, if it exists, is a hub arc, the path from node 6 to node 3 is 6-1-4-3, not 6-1-2-3. Figure 1(c) shows a competitive hub arc solution where Firms A and B each locate one hub arc. Each firm provides service for all OD pairs via one-stop or two-stop routes through its own hubs. Thus, the path from node 6 to node 3 for Firm A is 6-1-4-3 and for Firm B is 6-2-3. Similarly, the path from node 4 to node 5 for Firm A is 4-5 and for Firm B is 4-3-2-5.

2.1 Notation

Let Firm A be the leader firm and Firm B be the follower firm. We employ the following notation:

- V: the set of demand nodes, $|V| = n$.
- W_{ij}: the trip demand between node $i \in V$ and node $j \in V (j > i)$.
- F_{ij}: the revenue (e.g. airfare) per unit demand between node $i \in V$ and node $j \in V (j > i)$.
- d_{ij}: the distance between node $i \in V$ and node $j \in V (j \geq i)$. $d_{ii} = 0 (i \in V)$.
- α: the reduced unit cost for transfer on hub arcs to reflect economies of scale.
- χ: the unit transportation costs for collection (from origin to hub).
- δ: the unit transportation costs for distribution (from hub to destination).
- C_{ijkl}: the unit cost for a path from an origin $i \in V$ to a destination $j \in V (j > i)$ through hubs $k \in V$ and $l \in V$ in this order.



(a) Three hub median model

(b) Non-competitive model with two hub arcs

(c) Competitive model with one hub arc each for Firm A and Firm B

Figure 1: Hub arc networks

q^A: the number of Firm A's hub arcs.
q^B: the number of Firm B's hub arcs.
$\mathcal{A}$: a set of q^A hub arcs used by Firm A.
$\mathcal{B}$: a set of q^B hub arcs used by Firm B, where Firms A and B share no hubs.
$\mathcal{H}^A$: the set of combinations of q^A hub arcs that can be used by Firm A.
$\mathcal{H}^B$: the set of combinations of q^B hub arcs that can be used by Firm B.

We assume that the trip demand is symmetric. Therefore, we define W_{ij} and F_{ij} for all $j > i$ for each i. Note that $C_{ijkl} = \chi d_{ik} + \alpha d_{kl} + \delta d_{lj}$ and $C_{ijkk} = \chi d_{ik} + \delta d_{kj}$ for one-stop paths as $d_{kk} = 0$ for all $k \in V$. By introducing C^A_{ijkl} and C^B_{ijkl} for each firm with specific α, χ, δ, we can easily employ a different cost function for each firm. However, in this paper we assume that both firms employ the same cost function.

The decision variables for Firms A and B are as follows:

$x^A_{ijkl} =$ 1 if the lowest cost path for OD pair i, j with Firm A is through hubs $k \in V$ and $l \in V (l \geq k)$ in this order, and 0 otherwise.
$y^A_{kl} =$ 1 if Firm A locates a hub arc between nodes $k \in V$ and $l \in V (l > k)$, and 0 otherwise.
$z^A_k =$ 1 if node $k \in V$ is a hub for Firm A, and 0 otherwise.
$x^B_{ijkl} =$ 1 if the lowest cost path for OD pair i, j with Firm B is through hubs $k \in V$ and $l \in V (l \geq k)$ in this order, and 0 otherwise.
$y^B_{kl} =$ 1 if Firm B locates a hub arc between nodes $k \in V$ and $l \in V (l > k)$, and 0 otherwise.
$z^B_k =$ 1 if node $k \in V$ is a hub for Firm B, and 0 otherwise.

Note that hub arcs are assumed to be undirected and no degenerate hub arc, i.e. (k, k), is allowed to be established. Therefore, it is sufficient to define variables y^A_{kl} and y^B_{kl} for all $l > k$ for each k. Because each firm provides service for each OD pair via a single (lowest cost) path, the competition is between two paths.

2.2 Customer allocation function

Because of the competitive environment, the given demand for each OD pair is divided between Firms A and B using a customer allocation function. This allocation function should reflect the customers' preferences and will likely differ among the many applications for hub models (e.g., between passenger and freight transportation). Previous competitive hub location models have addressed air passenger transportation where passengers are allocated between competitors based on the relative path *costs* [14, 23] or path costs and travel times [10], with an objective of maximizing the number of passengers captured. Having passengers choose among carriers based on the travel cost via the hub network provides an incentive for passengers to travel circuitous paths via the lower cost hub arcs, especially for small values of α. While these circuitous paths are desirable from the carrier's perspective to reduce their own costs, they are undesirable from the passenger's perspective as they lead to longer OD travel times.

In general, air passengers will select among competing carriers that offer different services (e.g., flight frequencies, flight paths, departure and arrival times) and fares. Due to the strategic nature of hub location models, we do not specifically address detailed scheduling issues, such as departure and arrival times, and instead focus on service in terms of the OD path length via the hub network. The OD path length can be viewed as a proxy for the OD travel time, which is the key concern to passengers, given that in our model both firms have the same revenues (i.e., fares) for each OD pair. Thus, our first customer allocation function is based on the relative differences in OD travel distances, which may be most useful for passenger transportation applications. Further refinements to include the number of hub stops or total travel time including airport time, as in [10], are an area of future research.

For freight transportation hub networks, the situation is different because the customer is the freight shipper, who is not the traveler. Therefore, the customer is not particularly concerned with the length of the OD path the freight follows, as long as the freight is delivered when scheduled (e.g., next-day by 10:30 am). Because on-time delivery could occur via a variety of OD paths, a freight carrier has more flexibility than a passenger carrier to route freight via a lower cost, but more circuitous, path that exploits the reduced travel costs on hub arcs. (In contrast, the passengers would prefer more direct OD paths that avoid hub arcs and additional stops at hubs.) Because in our model the customer (e.g., freight *shipper*) pays the same cost (i.e., carrier revenue) with both competing carriers for a given shipment (i.e., OD pair), the freight *carrier* can choose to route freight along the minimum cost path via the hub network, assuming this provides adequate service. (For non-competitive hub location models with service level constraints, see [3, 7, 16].) So, our second customer allocation function is based on the relative differences in OD travel costs, which may be most useful for freight transportation applications.

Because customers for a given OD pair do not in general all select the same service, we model customer allocation as a step function with five levels of capture (as in [14, 23]) depending on the relative utility of the paths via Firm A's and Firm B's hub network. To develop our customer allocation functions, we first define $C_{ij}^{\mathcal{A}}$ as the minimum cost for a trip from origin i to destination j using hub arc set $\mathcal{A}$ for Firm A:

$$C_{ij}^{\mathcal{A}} = \min_{(k,l)\in\mathcal{A}} \min\{C_{ijkl}, C_{ijlk}, C_{ijkk}, C_{ijll}\}.$$

We define $C_{ij}^{\mathcal{B}}$ analogously for Firm B:

$$C_{ij}^{\mathcal{B}} = \min_{(k,l)\in\mathcal{B}} \min\{C_{ijkl}, C_{ijlk}, C_{ijkk}, C_{ijll}\}.$$

Similarly, let $D_{ij}^{\mathcal{A}}$ and $D_{ij}^{\mathcal{B}}$ be the *distance* of the path that corresponds to the minimum *cost* path from i to j for for Firms A and B, respectively. Thus, if the minimum cost path from i to j for Firm A with hub arc set $\mathcal{A}$ is $i-k-l-j$, then $C_{ij}^{\mathcal{A}} = \chi d_{ik} + \alpha d_{kl} + \delta d_{lj}$ and $D_{ij}^{\mathcal{A}} = d_{ik} + d_{kl} + d_{lj}$. We now define the customer allocation distance ratio $DR_{ij}^{\mathcal{A},\mathcal{B}}$ as

$$DR_{ij}^{\mathcal{A},\mathcal{B}} = (D_{ij}^{\mathcal{A}} - D_{ij}^{\mathcal{B}})/(D_{ij}^{\mathcal{A}} + D_{ij}^{\mathcal{B}}).$$

We define an analogous customer allocation cost ratio $CR_{ij}^{\mathcal{A},\mathcal{B}}$ as

$$CR_{ij}^{\mathcal{A},\mathcal{B}} = (C_{ij}^{\mathcal{A}} - C_{ij}^{\mathcal{B}})/(C_{ij}^{\mathcal{A}} + C_{ij}^{\mathcal{B}}).$$

Each of these is the ratio of the difference in path distances (or costs) to the sum of the path distances (or costs). Both $CR_{ij}^{\mathcal{A},\mathcal{B}}$ and $DR_{ij}^{\mathcal{A},\mathcal{B}}$ range from -1 to $+1$ and represent the relative advantage of Firm B over Firm A. A ratio of zero indicates equivalent distances or costs for the paths with Firm A and Firm B. A large ratio (close to $+1$) indicates that the best path via Firm B's hub network is much preferred (shorter or less expensive) over the best path via Firm A's hub network.

To allocate customers among the competitors, we define $\phi_{ij}^{A}(x^A, x^B)$ and $\phi_{ij}^{B}(x^A, x^B)$ as the fraction of demand between origin i and destination j captured by Firm A and Firm B, respectively, where $x^A = [x_{ijkl}^A]$ and $x^B = [x_{ijkl}^B]$. We set $\phi_{ij}^{A}(x^A, x^B) + \phi_{ij}^{B}(x^A, x^B) = 1$ for all OD pairs, so all demand is transported. Note that the allocation functions ϕ_{ij}^{A} and ϕ_{ij}^{B} depend on the distance or cost ratio ($DR_{ij}^{\mathcal{A},\mathcal{B}}$ or $CR_{ij}^{\mathcal{A},\mathcal{B}}$), which depends on the path distances or costs ($D_{ij}^{\mathcal{A}}$ and $D_{ij}^{\mathcal{B}}$ or $C_{ij}^{\mathcal{A}}$ and $C_{ij}^{\mathcal{B}}$), which in turn depends on the decision variables x_{ijkl}^A and x_{ijkl}^B. To link the cost or distance ratios $CR_{ij}^{\mathcal{A},\mathcal{B}}$ and $DR_{ij}^{\mathcal{A},\mathcal{B}}$ to the fraction of flow captured we adopt a five-level step function as shown in Table 1. This function assigns a fraction of the demand for each OD pair of 100%, 75%, 50%, 25% or 0% to each competitor based on the values r_1 and r_2, where $r_1 \geq r_2 \geq 0$. This step function essentially allocates the customers equally (50% to each firm) when the path costs or distances are similar (between $-r_2$ and r_2), allocates

Table 1: The fraction of flow captured

$CR_{ij}^{\mathcal{A},\mathcal{B}}$ or $DR_{ij}^{\mathcal{A},\mathcal{B}}$	$\phi_{ij}^{A}(x^A, x^B)$
$\leq -r_1$	100%
$-r_1$ to $-r_2$	75%
$-r_2$ to r_2	50%
r_2 to r_1	25%
$> r_1$	0%

three times as many customers to the firm with better performance when the absolute value of the cost or distance ratio is between r_2 and r_1, and allocates all customers to the firm with better performance when the absolute value of the cost or distance ratio exceeds r_1. By adjusting the values of r_1 and r_2, a range of different customer allocation schemes can be evaluated. In the extreme case of $r_1 = r_2 = 0$, the firm offering the better (lower distance or cost) OD path will capture all the demand. If the paths for Firm A and Firm B have equal lengths (or costs), then $DR_{ij}^{\mathcal{A},\mathcal{B}} = 0$ (or $CR_{ij}^{\mathcal{A},\mathcal{B}} = 0$) and we assume the demand is split equally between the two firms. (This is slightly different than in [14, 23] where in the event of equal paths, all demand is allocated to a single competitor.)

2.3 Formulation

We can formulate our Stackelberg hub arc location problem as a bilevel programming problem in which each firm locates a given number of its own hub arcs to maximize the revenue it captures. First we consider Firm B's problem in which Firm A's decision variables x_{ijkl}^A, y_{kl}^A and z_k^A are all given. Given Firm A's hub arcs, Firm B will establish its hub network (i.e., hub arcs) so as to maximize its total revenue captured. This is formulated as for the hub arc model in [3] based on [12]. Thus, Firm B's hub arc location problem in a competitive environment, denoted HALCE-B, is as follows:

[HALCE-B]

$$\text{maximize}_{x^B, y^B, z^B} \quad \sum_{i \in V} \sum_{j \geq i} F_{ij} W_{ij} (1 - \phi_{ij}^{A}(x^A, x^B)) \tag{1}$$

$$\text{subject to} \quad \sum_{k \in V} \sum_{l > k} y_{kl}^B = q^B, \tag{2}$$

$$z_k^B \leq 1 - z_k^A \qquad k \in V, \tag{3}$$

$$z_k^B \leq \sum_{l > k} y_{kl}^B + \sum_{l < k} y_{lk}^B \qquad k \in V, \tag{4}$$

$$x_{ijkl}^B \leq y_{kl}^B \qquad i, j, k, l \in V,\ j > i,\ l > k, \tag{5}$$

$$x_{ijlk}^B \leq y_{kl}^B \qquad i, j, k, l \in V,\ j > i,\ l > k, \tag{6}$$

$$x^B_{ijkk} + \sum_{m \in V \setminus \{k\}} (x^B_{ijkm} + x^B_{ijmk}) \le z^B_k \qquad i,j,k,l \in V,\ j > i, \tag{7}$$

$$\sum_{k \in V} \sum_{l \in V} x^B_{ijkl} = 1 \qquad i,j \in V, j > i \tag{8}$$

$$x^B_{ijkl} \in \{0,1\} \qquad i,j,k,l \in V,\ j > i,\ l \ge k,$$

$$y^B_{kl} \in \{0,1\} \qquad k,l \in V,\ l > k,$$

$$z^B_k \in \{0,1\} \qquad k \in V.$$

The objective function (1) is the total revenue captured by Firm B. Constraint (2) ensures that exactly q^B hub arcs are selected. Constraints (3) ensure that Firm B cannot select a node as a hub if it is a hub for Firm A. Constraints (4) ensure that hub nodes are established only at the endpoints of hub arcs. Constraints (5), (6) and (7) ensure that hub arcs and hub nodes are established for every path. Constraints (8) ensure there is a path with Firm B for all OD pairs. Note that constraint (3) and the objective function link HALCE-B to the hub arc location problem for Firm A.

Firm A's problem is stated as the following bilevel programming problem:

[HALCE-A]

$$\text{maximize}_{x^A, y^A, z^A} \quad \sum_{i \in V} \sum_{j \ge i} F_{ij} W_{ij} \phi^A_{ij}(x^A, x^B) \tag{9}$$

$$\text{subject to} \quad \sum_{k \in V} \sum_{l > k} y^A_{kl} = q^A, \tag{10}$$

$$z^A_k \le \sum_{l > k} y^A_{kl} + \sum_{l < k} y^A_{lk} \qquad k \in V, \tag{11}$$

$$x^A_{ijkl} \le y^A_{kl} \qquad i,j,k,l \in V,\ j > i,\ l > k, \tag{12}$$

$$x^A_{ijlk} \le y^A_{kl} \qquad i,j,k,l \in V,\ j > i,\ l > k, \tag{13}$$

$$x^A_{ijkk} + \sum_{m \in V \setminus \{k\}} (x^A_{ijkm} + x^A_{ijmk}) \le z^A_k \qquad i,j,k,l \in V,\ j > i, \tag{14}$$

$$\sum_{k \in V} \sum_{l \in V} x^A_{ijkl} = 1 \qquad i,j \in V,\ j > i, \tag{15}$$

$$[x^B, y^B, z^B] = \xi(x^A, y^A, z^A), \tag{16}$$

$$x^A_{ijkl} \in \{0,1\} \qquad i,j,k,l \in V,\ j > i,\ l \ge k,$$

$$y^A_{kl} \in \{0,1\} \qquad k,l \in V,\ j > i,\ l > k,$$

$$z^A_k \in \{0,1\} \qquad k \in V.$$

The objective function (9) is the total revenue for Firm A. Constraint (10) ensures that exactly q^A hub arcs are selected. Constraints (11) ensure that hub nodes are established only at the endpoints of hub arcs. Constraints (12), (13) and (14) ensure that hub arcs and hub nodes are established for every path. Constraints (15) ensure there is a possible path with Firm A for all OD pairs. Constraint (16) indicates that HALCE-B is solved optimally for any value of (x^A, y^A, z^A), where $\xi(x^A, y^A, z^A)$ denotes an optimal solution for HALCE-B. Thus, Firm A solves its own problem subject to the condition that (x^B, y^B, z^B) is an optimal solution of HALCE-B.

Given the interdependent nature of the two problems HALCE-A and HALCE-B, we do not solve the formulations directly. In the next section we present an optimal algorithm that uses two bounding procedures within an enumeration scheme.

3 Algorithm

We could obtain an optimal solution by enumerating every possible combination of hub arcs for Firms A and B. However, since HALCE-A is a bilevel optimization problem that includes constraints where Firm B also finds optimal hub arcs, the complete enumeration approach could solve problems of small size only. Therefore, we develop a much more efficient "smart" enumeration scheme by using bounding techniques to reduce the number of Firm B's hub arc combinations that need to be enumerated for a given set of Firm A's hub arcs, and to reduce the number of OD pairs that need to be evaluated for a given set of hub arcs for Firms A and B.

Recall that $\mathcal{H}^A$ and $\mathcal{H}^B$ are the sets of all possible combinations of q^A and q^B hub arcs for Firms A and B, respectively. Let $f^A(\mathcal{A}, \mathcal{B})$ denote the revenue of Firm A, where $\mathcal{A} \in \mathcal{H}^A$ and $\mathcal{B} \in \mathcal{H}^B$ are the hub arcs employed by Firm A and Firm B, respectively. In the same manner, let $f^B(\mathcal{A}, \mathcal{B})$ denote Firm B's revenue with given $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{B}^*_{\mathcal{A}}$ be the optimal set of hub arcs for Firm B for a given set of Firm A's hub arcs: $\mathcal{B}^*_{\mathcal{A}} = \text{argmax}_{\mathcal{B}} f^B(\mathcal{A}, \mathcal{B})$. Note that

$$f^A(\mathcal{A}, \mathcal{B}^*_{\mathcal{A}}) = W - \max_{\mathcal{B} \in \mathcal{H}^B} f^B(\mathcal{A}, \mathcal{B}),$$

where W is the total revenue for both firms: $W = \sum_{i \in V} \sum_{j>i} W_{ij} F_{ij}$. The idea behind the upper bounding procedure is as follows: For a given set of Firm A's hub arcs, denoted $\bar{\mathcal{A}}$, if we find some set of Firm B's hub arcs, say $\bar{\mathcal{B}}$, such that $f^A(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ is worse (i.e., smaller) than a known solution for Firm A, then we need not consider any other sets of Firm B's hub arcs with $\bar{\mathcal{A}}$. More formally,

Proposition Let f^* be the best known value of HALCE-A. Consider a given set of Firm A's hub arcs, denoted $\bar{\mathcal{A}} \in \mathcal{H}^A$. If $f^A(\bar{\mathcal{A}}, \mathcal{B}) < f^*$, for some $\mathcal{B} \in \mathcal{H}^B$, then $f^A(\bar{\mathcal{A}}, \mathcal{B}^*_{\bar{\mathcal{A}}}) < f^*$.

Proof Assume that $\bar{\mathcal{B}} \in \mathcal{H}^B$ satisfies the above inequality, i.e., $f^A(\bar{\mathcal{A}}, \bar{\mathcal{B}}) < f^*$ holds. Since $f^A(\bar{\mathcal{A}}, \bar{\mathcal{B}}) + f^B(\bar{\mathcal{A}}, \bar{\mathcal{B}}) = W$ then $W - f^B(\bar{\mathcal{A}}, \bar{\mathcal{B}}) < f^*$. Hence, $f^A(\bar{\mathcal{A}}, \mathcal{B}^*_{\bar{\mathcal{A}}}) = W - f^B(\bar{\mathcal{A}}, \mathcal{B}^*_{\bar{\mathcal{A}}}) < W - f^B(\bar{\mathcal{A}}, \bar{\mathcal{B}}) < f^*$. □

Thus, to encourage early cutting in the algorithm we would like to enumerate the hub arc combinations for Firm B in a sequence such that the algorithm quickly finds very good (high revenue) solutions for

Firm B. Similarly, we seek to enumerate the hub arc combinations for Firm A in sequence such that the algorithm quickly finds very good (high revenue) solutions for Firm A. Our algorithm also uses upper bounds to avoid enumerating all OD pairs when evaluating a candidate set of hub arcs for Firms A and B, as shown below in Step 2 below. In the algorithm, $\bar{g}^A$ is an upper bound for Firm A's revenue and $\bar{g}^B$ is used as the best known solution for Firm B with a given $\mathcal{A}$. The smart enumeration (SE) solution algorithm can be described as follows.

[SE Algorithm]

Step 0: Set $f^* := -\infty, \mathcal{A}^* = \emptyset$ and $\mathcal{B}^* = \emptyset$.

Step 1: If $\mathcal{H}^A$ is empty, go to Step 4. Select $\mathcal{A} \in \mathcal{H}^A$. Set $\mathcal{H}^A := \mathcal{H}^A \backslash \{\mathcal{A}\}$, and $\bar{g}^A := \infty$. Create $\mathcal{H}^B_{\mathcal{A}}$ as the set of all hub arc combinations from $\mathcal{H}^B$ that do not include a hub node used in $\mathcal{A}$.

Step 2:

- **(a)** If $\mathcal{H}^B_{\mathcal{A}}$ is empty, then go to Step 3. Select $\mathcal{B} \in \mathcal{H}^B_{\mathcal{A}}$. Set $\mathcal{H}^B_{\mathcal{A}} := \mathcal{H}^B_{\mathcal{A}} \backslash \{\mathcal{B}\}$.
- **(b)** $g^A := 0$, $g^B := 0$, $\bar{g}^B := 0$, and Π is the set of all OD pairs $\{i, j\}$, where $j > i$.
- **(c)** If $\Pi = \emptyset$, then go to Step 2(e). Select OD pair $\{i, j\} \in \Pi$. $\Pi := \Pi \backslash \{i, j\}$. $g^A := g^A + F_{ij} W_{ij} \phi^A_{ij}(x^A, x^B)$. $g^B := g^B + F_{ij} W_{ij} \phi^B_{ij}(x^A, x^B)$.
- **(d)** If $W - g^A < \bar{g}^B$ then go to Step 2(f). Otherwise, go to Step 2(c).
- **(e)** If $g^B > \bar{g}^B$ then $\bar{g}^B := g^B$.
- **(f)** If $g^A < \bar{g}^A$, set $\bar{g}^A := g^A$. If $\bar{g}^A < f^*$, go to Step 1. Otherwise go to Step 2(a).

Step 3: If $f^* < \bar{g}^A$, then set $f^* := \bar{g}^A$, $\mathcal{A}^* := \mathcal{A}$, $\mathcal{B}^* := \mathcal{B}$. Go to Step 1.

Step 4: The optimal hub arcs for Firms A and B are $\mathcal{A}^*$ and $\mathcal{B}^*$, respectively. The optimal objective value of Firms A and B are f^* and $W - f^*$, respectively.

Note that $W - g^A$ in Step 2(d) is an upper bound on Firm B's revenue. If this is less than Firm B's best known solution, then we need not consider any more OD pairs for the current set of Firm B's hub arcs. Otherwise, the algorithm returns to Step 2(c) to consider another OD pair. To make this efficient, we select the OD pairs in decreasing order of OD demand×distance. Based on experimentation over a wide range of problems and prior results for a similar algorithm in [6], we select the hub arc combinations $\mathcal{A}$ and $\mathcal{B}$ in Steps 1 and 2(a) by first sorting the potential hub arcs in decreasing order of OD demand×revenue, and then selecting $\mathcal{A}$ and $\mathcal{B}$ from the sorted lists as described in [6].

4 Results

The section reports computational results from the optimal solutions for 540 problem instances using the standard CAB hub location data set derived from air passenger traffic between 25 major cities in the US. As is common with the CAB data, we set $\chi = \delta = 1$ and vary α from 0.2 to 1.0. These problems were solved using the SE algorithm coded in C++ on a DELL OPTIPLEX GX620 computer with a 3.4 GHz Intel Pentium 4 processor operated under Windows XP Professional with 2.0 GB DDR2-SDRAM memory. We set the candidate sets of hub arcs for Firms A and B to be the entire set of arcs connecting all pairs of the 25 cities, so there were 300 candiate hub arcs for each firm.

4.1 Problem scenarios

We utilize a variety of scenarios to consider a range of transportation systems with different OD revenues and different customer behaviors. For the OD revenues, we prepared two data sets, denoted *Airfare* and *Distance*. The revenue set *Airfare* is based on IATA Y class standard airfare between the city pairs collected by the lead author in 1998 from http://www.airfare.com/. For each OD pair, the lowest Y class airfare was selected from the many fares provided. Thus, the *Airfare* data set represents realistic revenues at a particular point in time. (Of course, the long run patterns of airfares may differ due to the dynamic nature of the industry and its pricing structures.) A plot of the airfares versus the direct OD distance showed a strong correlation for some origins (cities) individually ($R^2 > 0.7$), but a weaker correlation in aggregate ($R^2 = 0.189$). More details on the charateristics of the *Airfare* revenue set are in Appendix A.

For comparison purposes we created a second revenue set, denoted *Distance*, that is the direct OD distance for each city pair. This could represent a situation where revenues are strongly correlated with OD distance, as is the case for aircraft operating costs [1]. Also, if the carrier profit per unit of demand (i.e., per passenger or per shipment) is strongly correlated with the OD distance, then this revenue data set could also be used to explore profit maximizing solutions. Using the *Airfare* and *Distance* revenue sets, our objective is to maximize the total revenue captured. (If the demand is in units of passengers, then we could easily maximize the number of passengers captured by using a revenue of 1.0 for all OD pairs.)

Note that since the demand and the revenue appear only in the problem objective, the product $F_{ij} \times W_{ij}$ can be viewed as a new demand set. Thus, the problem of maximizing the revenue captured is equivalent to maximizing the demand captured when the demand W_{ij} is replaced by $F_{ij} \times W_{ij}$. Because the *Airfare* and *Distance* revenue sets are not strongly correlated with the demand in the CAB data set, they do not create a very large change in the relative sizes of the demand for each city. Table 2 shows a list of the 25 CAB cities with the percentages of the total demand, demand$\times$*Airfare*, and demand$\times$*Distance*, respectively, originating and terminating at each city. For example, the "largest" city is 17 which has 17% of the total originating and terminating demand, 16.9% of the the total originating and terminating demand weighted by the *Airfare* revenues, and 16.3% of the total originating and terminating demand weighted by the *Distance* revenues. One effect of the *Distance* revenue set is to effectively increase the demand of the west coast cities; for example, 12 (Los Angeles) and 22 (San Francisco) have 7.3% and 5.1%, respectively, of the total demand, but are 12.5% and 8.7% of the total when demand is weighted by the *Distance* revenue set. Correspondingly, central city 4 (Chicago), which has 10.0% of the total originating and terminating demand, has only 8.5% of the total when demand is weighted by the *Distance* revenue set. Note that the *Airfare* revenue set does not generally provide such large changes (except for city 4).

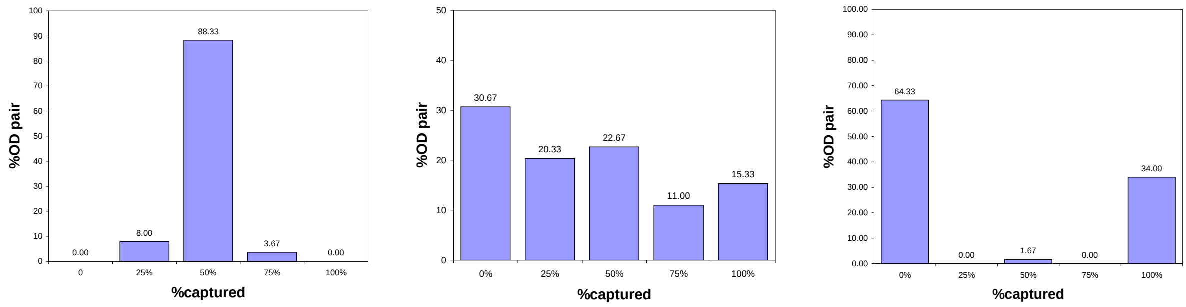
Table 2: Cities in the CAB data set

#	City	% Demand	% Demand $\times$*Airfare*	% Demand $\times$*Distance*
1	Atlanta	2.8	2.9	2.3
2	Baltimore	1.7	1.5	1.4
3	Boston	6.1	5.3	4.5
4	Chicago	10.0	8.0	8.5
5	Cincinnati	1.6	2.1	1.0
6	Cleveland	3.0	3.0	2.1
7	Dallas	3.1	3.8	3.0
8	Denver	2.4	3.2	2.7
9	Detroit	4.3	5.0	3.3
10	Houston	2.4	2.9	2.4
11	Kansas City	2.0	1.8	1.7
12	Los Angeles	7.3	7.2	12.5
13	Memphis	1.2	1.4	0.9
14	Miami	5.5	5.2	6.6
15	Minneapolis	2.5	3.5	2.3
16	New Orleans	1.8	2.0	1.7
17	New York	17.0	16.9	16.3
18	Philadelphia	3.6	3.6	3.3
19	Phoenix	1.5	1.3	1.8
20	Pittsburgh	2.8	3.0	1.8
21	St. Louis	2.9	2.6	2.2
22	San Francisco	5.1	4.9	8.7
23	Seattle	1.9	2.1	3.1
24	Tampa	1.9	1.8	1.8
25	Washington DC	5.7	4.7	4.1

Our computational results include three different settings for r_1 and r_2 in the customer allocation function introduced in Section 2.2 to explore a range of customer behaviors in selecting between competing firms. Our base case, denoted *low* customer selectivity, corresponds to $r_1 = 0.75$ and $r_2 = 0.25$, and reflects customers being rather insensitive to the relative service of the competitors. With these large values of r_1 and r_2, a very large relative difference in distance (or cost) is needed to strongly prefer one firm over the other. Specifically, the five level customer allocation step function with low customer selectivity implies the following relative performance characteristics for each firm's hub network: to capture all demand for an OD pair it must provide a travel distance (or cost) of less than one-seventh (14%) of the competitor's value; to capture 75% of the demand it must provide a distance (or cost) of 14% to 60% of the competitor's value; to share the demand equally (capture 50%), it must provide a distance (or cost) of 60% to 167% of the competitor's value; to capture 25% of the demand, it must provide a distance (or cost) of 167% to 700% of the competitor's value; and to capture none of the demand, it must provide a distance (or cost) of over 700% of the competitor's value. In our preliminary tests using low customer selectivity with the CAB data, both firms generally share the demand equally for about 80% of all OD pairs, with the remaining OD pairs shared 25% and 75%. As an example with low customer selectivity, Figure 2(a) shows the distribution of the percentage of OD pairs captured by Firm A in each level of the five-level step function for an instance with $q^A = q^B = 2$ and $\alpha = 0.2$. Note that this is close to the extreme case of very large values for r_1 and r_2, when each firm would capture 50% of the demand for every OD pair (and therefore in aggregate).

For our second customer allocation pattern, we selected values for r_1 and r_2 to create a more even distribution of OD pairs among the five different levels of allocation (100%, 75%, 50%, etc.). This is not straightforward as different problem parameters (e.g., number of hub arcs, revenue sets, α values, etc.) produce different allocation patterns for the same settings of r_1 and r_2. Based on our experimention we denote $r_1 = 0.083$ and $r_2 = 0.015$ as *medium* customer selectivity. Figure 2(b) shows the more even distribution of the percentage of OD pairs captured by Firm A with medium customer selectivity for the same instance shown in Figure 2(a). Here, each of the five levels of capture includes between 11% and 31% of the OD pairs.

For our third customer allocation pattern, denoted *high* customer selectivity, we set $r_1 = r_2 = 0$. In this case, the firm providing better service captures all passengers and the other firm captures none, even when the travel difference through the two hub networks is extremely small. When the two hub networks provide identical service (i.e., identical path distances when using $DR_{ij}^{\mathcal{A},\mathcal{B}}$ or identical path costs when using $CR_{ij}^{\mathcal{A},\mathcal{B}}$), we split the demand equally between the competitors. Figure 2(c) shows the distribution of the percentage of OD pairs captured by Firm A with high customer selectivity for the same instance shown in Figure 2(a). Although Firm A captures 100% of the revenues for only about one-third of the OD pairs in this instance, this corresponds to 52.6% of the total revenue, since Firm A captures more higher revenue customers. Note that Figure 2(c) shows 1.67% of OD pairs being split equally between the two firms for five OD pairs with OD paths of equal distance. Figure 2 clearly shows the flexibility of our customer allocation function with its ability to model different levels of customer selectivity. This is just one example, but similar patterns were observed in other problem instances.



(a) low selectivity (b) medium selectivity (c) high selectivity

Figure 2: The distribution of the percentage of OD pairs captured by Firm A with low, medium and high customer selectivity (using customer allocation based on the distance ratio, the *Airfare* revenue set, $q^A = q^B = 2$ and $\alpha = 0.2$)

Our results use the low, medium and high levels of customer selectivity to explore how the optimal solutions depend on the customer allocation. If a carrier knows the customer response pattern for a particular market, then appropriate values of r_1 and r_2 should be selected to reflect the customer behavior. To simplify the notation in presenting and discussing results, we will adopt the notation HALCE(RS,AL,SL,q^A,q^B,α) where RS indicates the revenue set (RA=*Airfare*, RD=*Distance*), AL indicates the allocation type (PD=path distance, PC=path cost), and SL indicates the level of customer selectivity (low, medium or high).

The remainder of this section includes results from 540 instances using the two revenue sets (*Airfare* and *Distance*), three levels of customer selectivity (low, medium and high), five values of α (0.2, 0.4, 0.6, 0.8, 1.0) and nine $\langle q^A, q^B \rangle$ combinations with up to three hub arcs for each firm: $\langle q^A, q^B \rangle$ = $\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 2,3 \rangle, \langle 3,1 \rangle, \langle 3,2 \rangle, \langle 3,3 \rangle$. Each problem is solved to optimality using the SE algorithm from Section 3. We first consider customer allocation based on the distance ratio $DR_{ij}^{\mathcal{A},\mathcal{B}}$. As noted earlier, these might best represent passenger transportation systems since the customers prefer shorter distance paths.

4.2 Base Case

As a base case we present detailed results using low customer selectivity in Table 3 for the *Airfare* revenue set and in Table 4 for the *Distance* revenue set. The results show that Firms A and B capture similar revenue shares in all problems, even when one firm has three hub arcs and the other has only one. Firm A's share ranges from 46.12% to 53.75% in Table 3, with a similar, but slightly smaller range in Table 4. The results also show the share captured is rather insensitive to α for each particular number of hub arcs - and the same solutions for a fixed $\langle q^A, q^B \rangle$ with large α values in many cases. Note that with $\alpha = 1$, the hub arcs provide no advantage so any combination of the hub nodes shown is optimal. Another interesting finding is that the "first entry paradox" [9] occurs in which Firm A does not take advantage of being the leader (it has less than 50% of the revenue) with $q^A = q^B = 1$. When $q^A < q^B$, it is not surprising that Firm A captures less than 50% of the revenue, though it is interesting that it is always near 50%. Comparing the solutions in Tables 3 and 4 for the same problem shows the influence of using different revenue sets. In many cases, Firm A uses the same hubs arcs with both revenue sets, but Firm B does not (e.g., see $q^A = q^B = 2$ with α=0.6, 0.8 and 1.0). In other cases, such as $q^A = 2, q^B = 3$ with α=0.4, 0.6, 0.8 and 1.0, both firms use the same optimal hub arcs with both revenue sets, even though the revenue captured differs due to the different revenue data.

As expected, the computational effort shown in Tables 3 and 4 increases with increasing numbers of hub arcs. The maximum cpu time (for $q^A = q^B = 3$ and $\alpha = 0.2$) is nearly 100 minutes. In many cases, but not all, the cpu times decreased somewhat (often around 25%) as α increased from 0.2 to 1.0.

Figure 3 shows how Firm A's share increases as it adds hub arcs with q^B=1 for various values of α. As expected, there are marginally decreasing returns with added hub arcs. The figure also shows that Firm A's share is smaller with the *Distance* revenue set, but the sensitivity to α varies between the revenue sets. When $q^A = 1$ Firm A's share seems more sensitive to α using the *Airfare* revenue set, but the opposite is true for $q^A = 3$. Another interesting observation is that with both revenue sets, when $q^A = 1$, Firm A's share is the worst when $\alpha = 0.2$ and the best when $\alpha = 1.0$, while with $q^A = 3$, the worst case occurs when $\alpha = 1.0$ and the best case occurs when $\alpha = 0.2$.

4.3 Customer selectivity

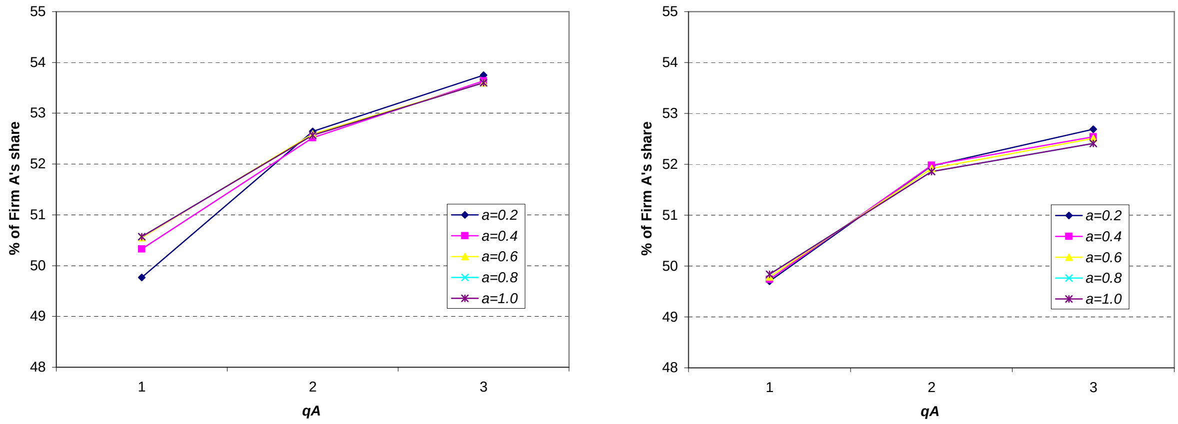
To explore the role of customer preferences we compare results with the low, medium and high levels of customer selectivity. Complete results for these problems are available in Appendix B. Figure 4 shows the percentage of Firm A's share for the problems of $\langle q^A, q^B \rangle = \langle 3, 1 \rangle, \langle 2, 2 \rangle$, and $\langle 1, 3 \rangle$ with low, medium and high selectivity. The solid lines and the dashed lines show the results with $\alpha = 0.8$ and $\alpha = 0.2$, respectively. (Other values of α produce similar results.) A comparison of the two charts shows that the share captured is rather insensitive to the revenue set for all levels of selectivity. Combined with Figure 2, some interesting relations between captured share and the distribution of demand for different levels of selectivity can be observed. When the selectivity level is low, both firms share the customers almost

Table 3: Optimal results for HALCE(RA,PD,low,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.11	2-21	4-17	49.77	50.23
		0.4	0.08	4-17	8-20	50.33	49.67
		0.6	0.06	4-17	8-25	50.55	49.45
		0.8	0.08	4-17	8-25	50.57	49.43
		1.0	0.06	4-17	8-25	50.57	49.43
1	2	0.2	2.70	4-20	1-18 5-8	47.12	52.88
		0.4	1.12	4-17	10-25 5-8	48.14	51.86
		0.6	1.14	4-17	10-25 8-9	48.40	51.60
		0.8	1.12	4-17	9-25 8-10	48.41	51.59
		1.0	1.14	4-17	9-25 8-10	48.41	51.59
1	3	0.2	295.72	4-20	7-15 1-18 5-8	46.12	53.88
		0.4	109.06	4-17	8-25 5-7 1-3	46.91	53.09
		0.6	107.31	4-17	3-9 7-25 1-12	47.14	52.86
		0.8	107.48	4-17	3-9 7-25 1-12	47.15	52.85
		1.0	108.28	4-17	3-9 7-25 1-12	47.15	52.85
2	1	0.2	4.36	17-20 4-8	11-25	52.64	47.36
		0.4	4.42	12-17 4-7	11-25	52.52	47.48
		0.6	4.64	12-17 1-4	11-25	52.59	47.41
		0.8	4.30	1-17 4-12	7-25	52.57	47.43
		1.0	4.28	1-17 4-12	7-25	52.57	47.43
2	2	0.2	23.91	12-17 4-8	9-18 7-22	50.64	49.36
		0.4	16.50	17-20 4-8	10-12 5-25	50.67	49.33
		0.6	15.56	17-22 4-7	9-25 11-12	50.69	49.31
		0.8	17.84	7-17 4-22	9-25 11-12	50.69	49.31
		1.0	17.69	7-17 4-22	9-25 11-12	50.69	49.31
2	3	0.2	1189.12	17-20 4-8	7-22 5-25 1-3	49.32	50.68
		0.4	636.25	12-17 4-7	3-22 9-11 16-25	49.46	50.54
		0.6	767.64	17-22 4-7	3-9 11-25 12-16	49.60	50.40
		0.8	907.18	7-17 4-22	3-9 11-25 12-16	49.60	50.40
		1.0	907.35	7-17 4-22	3-9 11-25 12-16	49.60	50.40
3	1	0.2	482.28	17-20 7-12 1-4	2-21	53.75	46.25
		0.4	498.92	17-20 4-7 1-12	2-13	53.64	46.36
		0.6	492.65	1-17 4-7 12-20	2-13	53.60	46.40
		0.8	504.31	1-17 4-7 12-20	13-25	53.60	46.40
		1.0	489.90	1-17 4-7 12-20	13-25	53.60	46.40
3	2	0.2	847.58	17-20 4-8 7-22	9-25 11-19	51.74	48.26
		0.4	755.32	17-20 4-8 16-22	7-12 5-25	51.73	48.27
		0.6	657.08	1-17 4-7 20-22	3-8 5-25	51.73	48.27
		0.8	640.69	1-17 4-7 20-22	3-8 5-25	51.73	48.27
		1.0	640.55	1-17 4-7 20-22	3-8 5-25	51.73	48.27
3	3	0.2	4809.90	3-17 4-20 7-22	12-18 5-8 13-15	50.72	49.28
		0.4	4516.08	3-17 4-7 20-22	9-12 11-15 1-18	50.64	49.36
		0.6	3813.45	1-17 4-7 20-22	3-9 15-25 12-13	50.67	49.33
		0.8	3901.92	1-17 4-7 20-22	3-9 15-25 12-13	50.67	49.33
		1.0	3877.50	1-17 4-7 20-22	3-9 15-25 12-13	50.67	49.33

Table 4: Optimal results for HALCE(RD,PD,low,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.16	7-25	5-19	49.70	50.30
		0.4	0.12	8-20	7-18	49.74	50.26
		0.6	0.14	6-22	4-18	49.79	50.21
		0.8	0.12	6-22	2-12	49.84	50.16
		1.0	0.11	6-22	2-12	49.84	50.16
1	2	0.2	9.42	5-8	17-22 4-11	47.92	52.08
		0.4	2.06	4-17	8-25 5-7	47.99	52.01
		0.6	4.08	13-17	12-25 4-7	48.05	51.95
		0.8	4.09	13-17	7-12 4-25	48.07	51.93
		1.0	4.09	13-17	7-12 4-25	48.07	51.93
1	3	0.2	580.65	5-8	17-23 4-11 13-19	47.28	52.72
		0.4	466.26	2-11	12-25 1-17 4-7	47.36	52.64
		0.6	524.34	13-17	7-12 3-25 4-24	47.49	52.51
		0.8	524.26	13-17	7-12 3-25 4-24	47.51	52.49
		1.0	537.18	13-17	7-12 3-25 4-24	47.51	52.49
2	1	0.2	5.95	12-17 4-8	13-25	51.97	48.03
		0.4	5.78	4-18 1-12	2-11	51.98	48.02
		0.6	4.67	12-17 4-7	9-22	51.92	48.08
		0.8	4.72	4-22 1-17	9-12	51.86	48.14
		1.0	4.77	4-22 1-17	9-12	51.86	48.14
2	2	0.2	26.47	12-17 4-8	7-22 21-25	50.19	49.81
		0.4	19.25	12-17 4-7	22-25 9-11	50.25	49.75
		0.6	16.22	17-22 4-7	12-21 3-25	50.43	49.57
		0.8	18.23	4-22 7-17	12-21 3-25	50.43	49.57
		1.0	18.12	4-22 7-17	12-21 3-25	50.43	49.57
2	3	0.2	804.91	12-17 4-8	7-22 1-25 19-21	49.49	50.51
		0.4	618.11	12-17 4-7	3-22 16-25 9-11	49.64	50.36
		0.6	550.78	17-22 4-7	12-16 3-9 11-25	49.87	50.13
		0.8	763.33	4-22 7-17	12-16 3-9 11-25	49.87	50.13
		1.0	767.14	4-22 7-17	12-16 3-9 11-25	49.87	50.13
3	1	0.2	644.18	17-25 7-12 4-8	2-21	52.69	47.31
		0.4	666.97	4-17 1-12 7-20	2-13	52.54	47.46
		0.6	544.43	7-17 12-20 1-4	9-22	52.51	47.49
		0.8	562.97	17-24 4-8 7-22	11-25	52.41	47.59
		1.0	570.32	17-24 4-8 7-22	11-25	52.41	47.59
3	2	0.2	1355.77	12-17 4-8 1-7	21-25 19-22	50.96	49.04
		0.4	1249.52	17-22 1-12 4-7	3-25 19-21	50.97	49.03
		0.6	983.57	4-22 1-17 7-12	3-21 19-25	50.97	49.03
		0.8	938.79	4-22 1-17 7-12	3-21 19-25	50.97	49.03
		1.0	951.49	4-22 1-17 7-12	3-21 19-25	50.97	49.03
3	3	0.2	5887.58	4-8 7-22 17-20	10-12 9-25 3-21	50.37	49.63
		0.4	4077.64	17-22 1-12 4-7	3-19 2-16 9-11	50.42	49.58
		0.6	3212.09	4-22 1-17 7-12	9-19 2-16 3-11	50.43	49.57
		0.8	3568.35	4-22 1-17 7-12	9-19 2-16 3-11	50.43	49.57
		1.0	3588.21	4-22 1-17 7-12	9-19 2-16 3-11	50.43	49.57



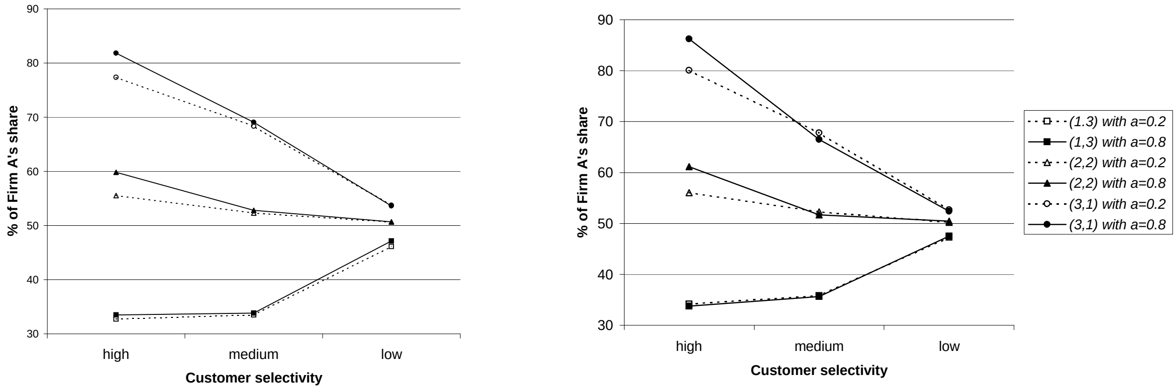
(a) HALCE(RA,PD,low,q^A,1,α) (b) HALCE(RD,PD,low,q^A,1,α)

Figure 3: Share increase with use of additional hub arcs for Firm A

equally regardless of the number of hub arcs selected, since the demand for vast majority of OD pairs is split equally (see Figure 2). The other extreme is high selectivity, where a firm may gain a considerable advantage when it has more hub arcs than its competitor - and where results are more sensitive to α. In the case of $\langle q^A, q^B\rangle = \langle 3, 1\rangle$, Firm A captures over 77% of the revenue, and for $\langle q^A, q^B\rangle = \langle 1, 3\rangle$ Firm B similarly captures over 65% of the revenue. However, the figure shows a small advantage for the leader, as Firm A captures more revenue with $\langle q^A, q^B\rangle = \langle 3, 1\rangle$, than does Firm B with $\langle q^A, q^B\rangle = \langle 1, 3\rangle$. As the level of selectivity changes from medium to high, there does seem to be a benefit for the leader (Firm A) from having more hub arcs: with $\langle q^A, q^B\rangle = \langle 3, 1\rangle$ Firm A increases its share of the revenue about 12% as the level of selectivity changes from medium to high, while with $\langle q^A, q^B\rangle = \langle 1, 3\rangle$ Firm B increases its share of the revenue only about 2%. Also note that when the number of hub arcs is the same for each competitor, the leader has only a rather small advantage regardless of how selective are the customers, except with large α and high selectivity.

4.4 Hub and hub arc use

Tables 5 and 6 summarize the results of 270 instances using customer allocation based on the distance ratio $DR_{ij}^{\mathcal{A},\mathcal{B}}$ for the six problem scenarios (using either the *Airfare* or *Distance* revenue set and either the low, medium or high level of customer selectivity). Complete results for all these problems are in Tables 3 and 4 and Appendix B. For each of the six problem scenario we solved 45 instances with different combinations of hub arcs and α values, analogous to the two problem scenarios shown in Tables 3 and 4. Table 5 shows summary data for optimal hubs and Table 6 shows summary data for the optimal hub



(a) HALCE(RA,PD,SL,q^A,q^A,α)

(b) HALCE(RD,PD,SL,q^A,q^A,α)

Figure 4: The percentage of captured share by Firm A with different levels of customer selectivity

arcs. Both tables have the same labels as follows. The first column identifies the hub (city) or the hub arc. The second column is the percentage of times that a city or hub arc appears in the 270 problems. The remaining 18 columns appear in groups of three for each of the six problem scenarios. Each group of three columns is headed by a three-part description of the problem scenario with the revenue set first (RA or RD), the customer allocation type second (PD), and the level of selectivity third. The columns labeled “A” and “B” are the percentage of times a city (or hub arc) appears in the optimal solutions for Firm A and Firm B, respectively. The column labeled “Total” is the sum of the percentages in columns “A” and “B”, which is the percentage of times a city (or hub arc) appears for either Firm A or B. The tables are sorted in descending order of the “%” column.

Table 5 shows that the top two cities (4=Chicago, 17=New York) are used as hubs in nearly every problem. Then there is a significant break before the next most frequently used city (12=Los Angeles), which is used in 57.0% of instances overall. The next most frequently used cities (1 and 8) show similar aggregate usage, though it derives from different patterns with city 1 being used more evenly across the six problem scenarios and city 8 varying in usage from 15.6% to 84.4%. Collectively, the results in Table 5 show that Firm A tends to use the same cities as hubs more often than Firm B, and uses fewer cities as hubs overall. Across the six problem scenarios, Firm A uses a hub at the 40% or greater level 24 times and uses a hub at the 5% or greater level 66 times. In contrast, Firm B uses a hub at the 40% or greater level 13 times and uses a hub at the 5% or greater level 112 times. The heavily used hubs (over 40% for at least one problem scenario) for Firm A are cities 1, 4, 7, 8, 12, 17, 20 and 22, and for Firm B are cities 3, 6, 9, 11, 12, 18, 21 and 25. Only city 12 is heavily used as a hub by both firms; the remaining cities tend to be used predominantly by a single firm. It is also interesting that a city that is used infrequently overall, can be heavily used in one problem scenario, such as city 2. The variation in hub usage for different problem scenarios highlights the importance of the customer selectivity in modeling different transport systems. For example, city 14 is heavily used with high selectivity, but not at all with low selectivity. Conversely, city 7 is predominantly used with low selectivity, and city 11 is predominantly used with medium selectivity.

Naturally, there are some geographic patterns in the hub usage, with west coast cities 12 and 22 used frequently overall (with slightly more usage by Firm A). Also, some centrally located cites seem to enjoy a geographical advantage over more peripheral cities with the CAB data set. For example, cities 1, 8, 10 and 15 have very similar demands and revenue patterns (see Table 2), but cities 1 and 8 are among the cities most often used as hubs, while cities 10 and 15 are rarely used as hubs. Cities 1 (Atlanta) and 8 (Denver) seem to have a geographical advantage from being more centrally located, while cities 10 (Houston) and 15 (Minneapolis) are disadvantaged by being located on the periphery of the CAB data set. Similarly, city 3 (Boston) seems disadvantaged, as it is ranked fourth in terms of demand (see Table 2), but eighth in usage as a hub (Table 5). Thus, while there is a strong link in many cases between the level of demand at a city (i.e. originating and terminating traffic) and its usage as a hub (see the similarities in the top ranked cities in Tables 2 and 5), geography also plays a strong role in selecting hub locations.

Table 6 provides a summary of hub arc usage for the 270 instances. The 21 hub arcs included in this table are the five most frequently used hub arcs in the optimal solutions for each of the six problem scenarios. Not surprisingly, the top eight hub arcs in Table 6 use either city 4 or 17, reflecting their importance as hubs (as shown in Table 5. The most frequently used hub arc is (1,4), which is used in about one-quarter of the problems overall. It connects the first and fifth most used hubs and is used more by Firm A than Firm B. The second and third most frequently used hub arcs in aggregate are (4,8) and (1,17), which are used almost exclusively by Firm A and almost never with high customer selectivity. In contrast, hub arcs (12,22) and (14,17) are used mainly with high customer selectivity. Although the use of individual hub arcs seems low overall (maximum of 26.7% vs. 91.5% for hub use), given the much greater number of hub arcs compared to hubs (300 vs. 25), and that each solution has half as many hub arcs as hubs, such concentrated use of hub arcs is noteworthy.

Table 5: Ranking of cities used as hubs in optimal solutions

#		RA,PD,low			RA,PD,medium			RA,PD,high			RD,PD,low			RD,PD,medium			RD,PD,high		
Hub	%	Total	A	B	Total	A	B	Total	A	B	Total	A	B	Total	A	B	Total	A	B
4	91.5	100.0	97.8	2.2	80.0	73.3	6.7	91.1	71.1	20.0	91.1	68.9	22.2	88.9	82.2	6.7	97.8	82.2	15.6
17	85.2	95.6	93.3	2.2	84.4	68.9	15.6	100.0	95.6	4.4	86.7	80.0	6.7	57.8	46.7	11.1	86.7	75.6	11.1
12	57.0	60.0	24.4	35.6	13.3	0	13.3	68.9	53.3	15.6	82.2	42.2	40.0	35.6	17.8	17.8	82.2	75.6	6.7
8	50.0	46.7	13.3	33.3	84.4	77.8	6.7	42.2	26.7	15.6	26.7	24.4	2.2	84.4	68.9	15.6	15.6	0	15.6
1	49.3	48.9	31.1	17.8	42.2	35.6	6.7	53.3	37.8	15.6	35.6	31.1	4.4	51.1	44.4	6.7	64.4	48.9	15.6
25	43.3	77.8	0	77.8	8.9	0	8.9	20.0	6.7	13.3	64.4	4.4	60.0	33.3	8.9	24.4	55.6	6.7	48.9
7	41.5	71.1	48.9	22.2	35.6	28.9	6.7	20.0	8.9	11.1	80.0	55.6	24.4	28.9	24.4	4.4	13.3	0	13.3
3	40.7	37.8	4.4	33.3	24.4	6.7	17.8	68.9	26.7	42.2	42.2	0	42.2	22.2	15.6	6.7	48.9	11.1	37.8
22	37.0	42.2	35.6	6.7	2.2	0	2.2	31.1	6.7	24.4	66.7	48.9	17.8	13.3	0	13.3	66.7	48.9	17.8
20	35.2	46.7	44.4	2.2	55.6	37.8	17.8	37.8	4.4	33.3	8.9	8.9	0	31.1	17.8	13.3	31.1	0	31.1
11	33.7	26.7	0	26.7	42.2	6.7	35.6	15.6	8.9	6.7	33.3	2.2	31.1	62.2	8.9	53.3	22.2	0	22.2
21	30.4	4.4	2.2	2.2	53.3	8.9	44.4	35.6	2.2	33.3	26.7	0	26.7	20.0	4.4	15.6	42.2	2.2	40.0
9	28.1	42.2	0	42.2	20.0	2.2	17.8	40.0	6.7	33.3	31.1	0	31.1	17.8	0	17.8	17.8	0	17.8
14	27.4	0	0	0	6.7	0	6.7	55.6	20.0	35.6	0	0	0	31.1	13.3	17.8	71.1	37.8	33.3
6	25.6	0	0	0	75.6	24.4	51.1	24.4	0	24.4	6.7	6.7	0	44.4	4.4	40.0	2.2	0	2.2
2	21.5	8.9	2.2	6.7	22.2	6.7	15.6	6.7	0	6.7	22.2	2.2	20.0	64.4	31.1	33.3	4.4	0	4.4
18	15.9	11.1	0	11.1	53.3	6.7	46.7	0	0	0	6.7	2.2	4.4	22.2	2.2	20.0	2.2	0	2.2
5	14.8	24.4	0	24.4	13.3	0	13.3	20.0	4.4	15.6	8.9	4.4	4.4	17.8	0	17.8	4.4	0	4.4
13	13.7	17.8	0	17.8	13.3	8.9	4.4	6.7	6.7	0	20.0	13.3	6.7	24.4	4.4	20.0	0	0	0
15	13.0	13.3	0	13.3	20.0	0	20.0	20.0	2.2	17.8	0	0	0	8.9	0	8.9	15.6	0	15.6
19	12.2	2.2	0	2.2	22.2	0	22.2	0	0	0	26.7	0	26.7	17.8	0	17.8	4.4	0	4.4
16	9.6	11.1	2.2	8.9	17.8	4.4	13.3	4.4	0	4.4	17.8	0	17.8	0	0	0	6.7	0	6.7
23	9.3	0	0	0	6.7	0	6.7	11.1	6.7	4.4	2.2	0	2.2	13.3	4.4	8.9	22.2	11.1	11.1
24	7.4	0	0	0	0	0	0	8.9	0	8.9	11.1	4.4	6.7	8.9	0	8.9	15.6	0	15.6
10	6.7	11.1	0	11.1	2.2	2.2	0	17.8	4.4	13.3	2.2	0	2.2	0	0	0	6.7	0	6.7

Table 6: Ranking of hub arcs used in optimal solutions

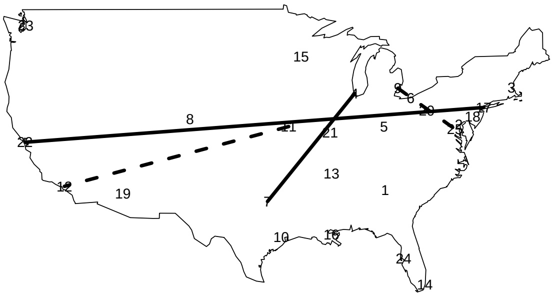
Hub		RA,PD,low			RA,PD,medium			RA,PD,high			RD,PD,low			RD,PD,medium			RD,PD,high		
Arc	%	Total	A	B	Total	A	B	Total	A	B	Total	A	B	Total	A	B	Total	A	B
1-4	26.7	4.4	4.4	0	26.7	22.2	4.4	44.4	33.3	11.1	2.2	2.2	0	26.7	26.7	0	55.6	48.9	6.7
4-8	16.3	13.3	13.3	0	26.7	26.7	0	4.4	4.4	0	17.8	17.8	0	35.6	35.6	0	0	0	0
1-17	13.3	24.4	24.4	0	13.3	13.3	0	4.4	4.4	0	20.0	17.8	2.2	17.8	17.8	0	0	0	0
3-17	11.5	4.4	4.4	0	11.1	6.7	4.4	26.7	26.7	0	0	0	0	8.9	8.9	0	17.8	11.1	6.7
4-17	11.5	28.9	26.7	2.2	8.9	8.9	0	8.9	6.7	2.2	4.4	4.4	0	8.9	6.7	2.2	8.9	8.9	0
4-7	10.4	33.3	33.3	0	2.2	2.2	0	2.2	0	2.2	20.0	15.6	4.4	4.4	4.4	0	0	0	0
12-17	10.4	8.9	8.9	0	0	0	0	20.0	20.0	0	15.6	15.6	0	0	0	0	17.8	17.8	0
4-22	8.9	8.9	8.9	0	0	0	0	2.2	0	2.2	26.7	26.7	0	0	0	0	15.6	13.3	2.2
12-22	7.8	0	0	0	0	0	0	13.3	6.7	6.7	0	0	0	2.2	0	2.2	31.1	28.9	2.2
8-17	6.7	0	0	0	20.0	20.0	0	17.8	15.6	2.2	0	0	0	2.2	2.2	0	0	0	0
12-23	6.7	0	0	0	4.4	0	4.4	8.9	6.7	2.2	0	0	0	13.3	4.4	8.9	13.3	11.1	2.2
3-9	6.3	20.0	0	20.0	0	0	0	8.9	0	8.9	6.7	0	6.7	0	0	0	2.2	0	2.2
14-17	6.3	0	0	0	0	0	0	6.7	6.7	0	0	0	0	0	0	0	31.1	31.1	0
6-11	5.6	0	0	0	15.6	0	15.6	0	0	0	0	0	0	17.8	0	17.8	0	0	0
7-12	5.6	4.4	2.2	2.2	0	0	0	0	0	0	26.7	15.6	11.1	2.2	2.2	0	0	0	0
8-14	4.8	0	0	0	0	0	0	15.6	4.4	11.1	0	0	0	13.3	11.1	2.2	0	0	0
17-20	4.8	15.6	15.6	0	4.4	2.2	2.2	0	0	0	2.2	2.2	0	6.7	2.2	4.4	0	0	0
2-13	4.1	4.4	0	4.4	4.4	2.2	2.2	0	0	0	2.2	0	2.2	13.3	4.4	8.9	0	0	0
4-6	3.0	0	0	0	13.3	11.1	2.2	2.2	0	2.2	0	0	0	2.2	2.2	0	0	0	0
7-8	3.0	0	0	0	13.3	13.3	0	0	0	0	0	0	0	4.4	4.4	0	0	0	0
20-22	3.0	15.6	15.6	0	0	0	0	2.2	0	2.2	0	0	0	0	0	0	0	0	0

The use of hub arcs is much less consistent than the use of hubs across the problem scenarios. Only the top two hub arcs are used at a level of over 20% for two or more problem scenarios. All other frequently used hub arc (used in over 20% of the instances for a problem scenario) are used in a single problem scenario. The results also show a differential intensity of hub arc use by the leader and follower. Firm A uses a small fraction of the hub arcs intensively, with 13 hub arcs used at a level over 20% and 25 hub arcs used at a level over 15%. Hub arcs used most intensively by Firm A include the top ten in Table 6. In contrast, Firm B uses no hub arcs at a level over 20% and only 3 hub arcs at a level over 15%. Hub arcs used most intensively by Firm B include (3,9) and (7,15). Note also the 48 "0" entries in the "Total" columns of Table 6 (38% of the entries), which indicate a hub arc that is not used for any instances in that scenario. In summary, Firm A uses a small fraction of the hub arcs intensively, while Firm B uses a wider range of hub arcs at less intensisity; however, Firm B often manages to capture nearly the same revenue that Firm A does when using the same number of hub arcs.

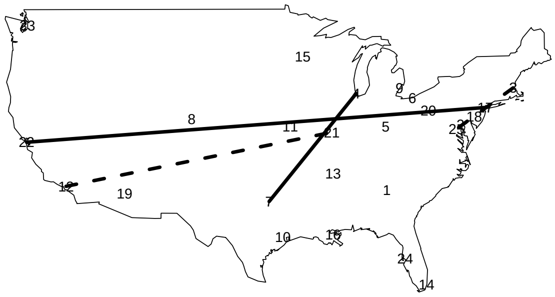
The complex patterns of hub arc use indicate that the optimal hub networks for Firms A and B vary considerably over the different problem scenarios. As an example, Figure 5 shows the optimal hub arcs with the *Airfare* and *Distance* revenue sets with low, medium and high selectivity for $q^A = q^B = 2$ and $\alpha = 0.6$. The solid lines show Firm A's hub arcs and the dashed lines show Firm B's hub arcs. With low selectivity, the pattern of hub arcs is quite similar with both revenue sets, but with medium and high selectivity the patterns are quite different. Of the 24 hub arcs in these six optimal solutions, there are 20 different hub arcs using 20 different hub cities, with some concentrated usage of cities 4 and 17, which are hubs in all six solutions, and cities 1, 7, 8, 12 and 14, which appear in three or four of the six solutions. This shows just one example of how the level of customer selectivity can produce dramatically different optimal hub networks, even though the revenue captured and some of the hub nodes may be very similar.

4.5 Customer allocation using the cost ratio

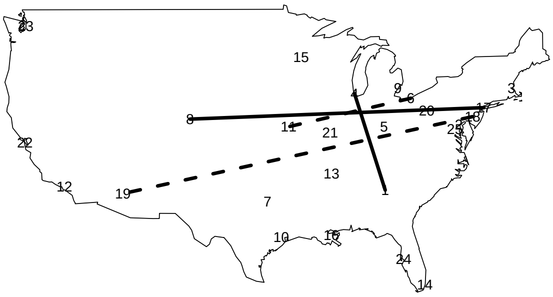
The results above use the customer allocation function based on the ratio of travel distances ($DR_{ij}^{\mathcal{A},\mathcal{B}}$). This seems most appropriate for passenger transportation, since passengers prefer shorter travel times. In this section, we provide results for 270 instances using the customer allocation function based on the ratio of travel costs ($CR_{ij}^{\mathcal{A},\mathcal{B}}$). These problems may be most appropriate for freight transportation where the freight carriers have flexibility to route the freight efficiently to minimize *cost*, and the travel *distance* may be less important. In these results we use the same two revenue sets, five α values, and nine $\langle q^A, q^B \rangle$ combinations as earlier. We present an illustration of the results here using the high level of customer selectivity, which corresponds to the situation in [14]. For complete results with the customer allocation



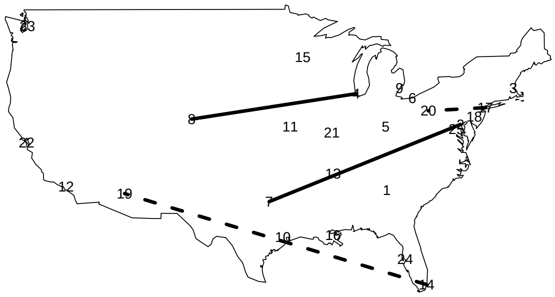
(a) HALCE(RA,PD,low,2,2,0.6)



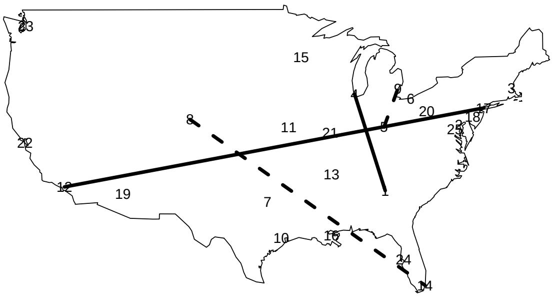
(b) HALCE(RD,PD,low,2,2,0.6)



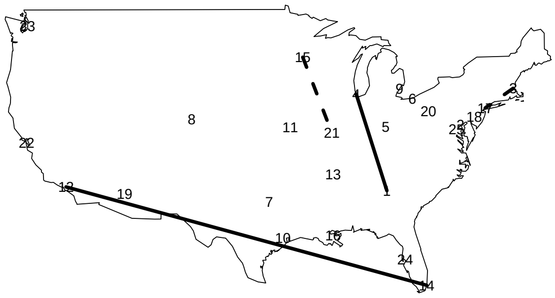
(c) HALCE(RA,PD,medium,2,2,0.6)



(d) HALCE(RD,PD,medium,2,2,0.6)



(e) HALCE(RA,PD,high,2,2,0.6)



(f) HALCE(RD,PD,high,2,2,0.6)

Figure 5: Optimal solutions for $q_A = q_B = 2$ and $\alpha = 0.6$

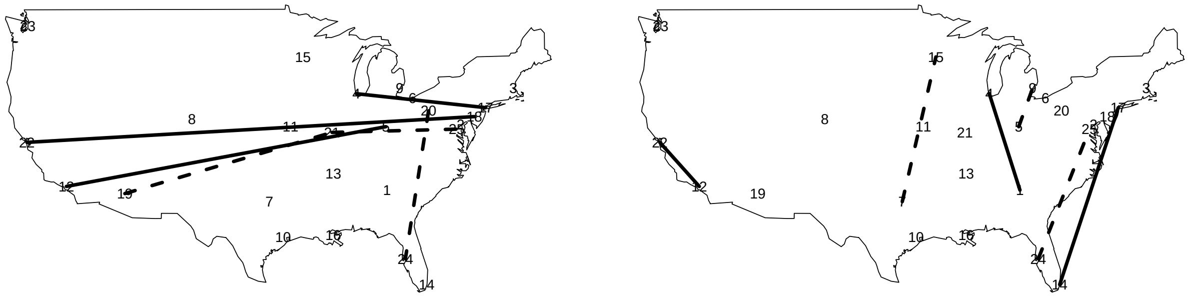
function based on the cost ratio, see Appendix C.

As an example of the results with the cost and distance ratios, in Table 7 we provide the optimal solutions with $\langle q^A, q^B \rangle = \langle 3, 3 \rangle$. The top half of Table 7 is results with customer allocation based on the cost ratio, and the lower half of Table 7 is the corresponding results for customer allocation based on the distance ratio. The first column in this table indicates the revenue set and the customer allocation type. A comparison of the first ten rows and the last ten rows shows that Firm A generally captures a smaller share of revenue with the cost ratio, especially for small values of α. This implies that use of the cost ratio CR_{ij} gives Firm B some advantages to capture more revenue. (Note that for $\alpha = 1.0$ the cost ratio and distance ratio become equivalent.) However, this behavior is not true in generall. The results also show that although many hubs are common between the solutions using the cost and distance ratios (cities 4, 12 and 17 are in all solutions in Table 7), different hub arcs are being used.

Table 7: Optimal results for HALCE(RS,AL,high,3,3,α)

Problem	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
RA, PC	0.2	967.92	17-21 4-12 18-24	22-25 2-14 6-7	54.96	45.04
	0.4	820.38	6-17 4-12 7-25	9-24 19-20 2-21	56.73	43.27
	0.6	861.01	4-12 17-20 13-18	15-22 6-21 24-25	58.12	41.88
	0.8	1043.79	4-17 9-12 7-25	14-21 6-8 2-18	58.81	41.19
	1.0	709.49	7-17 12-25 4-9	3-21 6-14 2-22	63.09	36.91
RD, PC	0.2	467.36	4-22 12-18 17-21	8-25 1-20 13-19	62.37	37.63
	0.4	654.88	4-17 9-12 19-22	1-18 8-21 6-25	59.49	40.51
	0.6	929.15	4-17 18-22 5-12	21-25 20-24 19-21	58.95	41.05
	0.8	502.90	4-12 7-17 18-22	8-11 24-25 9-20	63.42	36.58
	1.0	341.00	14-17 4-22 12-25	3-18 9-21 1-8	71.47	28.53
RA, PD	0.2	1111.08	3-17 12-23 1-4	14-25 20-24 16-21	62.82	37.18
	0.4	1051.97	3-17 12-22 1-4	5-9 8-14 10-15	60.52	39.48
	0.6	1011.01	3-17 12-14 1-4	22-23 5-9 10-15	61.56	38.44
	0.8	779.09	14-17 12-22 1-4	9-25 3-10 15-21	62.93	37.07
	1.0	1087.78	7-17 12-25 4-9	3-21 6-14 2-22	63.09	36.91
RD, PD	0.2	648.97	3-17 12-23 1-4	14-25 20-24 16-21	67.80	32.20
	0.4	634.64	14-17 12-22 1-4	3-25 19-23 15-21	69.22	30.78
	0.6	629.72	14-17 12-22 1-4	24-25 7-15 5-9	69.34	30.66
	0.8	700.72	4-14 3-17 12-22	23-25 1-9 7-11	72.18	27.82
	1.0	827.12	14-17 4-22 12-25	1-18 3-21 8-9	71.47	28.53

Figure 6 provides an example of the optimal hub networks using the customer allocation function based on the cost ratio and the distance ratio for the *Distance* revenue set with $q^A = q^B = 3$ and $\alpha = 0.6$. These two maps show strikingly different patterns that result from the quite different underlying motivations reflected in the cost and distance ratios. With the distance ratio (Figure 6(b)) the customers (e.g., passengers) prefer shorter travel distances, so one-stop paths (origin-hub-destination) have high utility and use of the hub arcs is discouraged. Thus, the hub arc networks in Figure 6(b)) are designed to create more one-stop trips, so the locations of the hub nodes are more important than the locations of the hub



(a) HALCE(RD,PC,high,3,3,0.6) (b) HALCE(RD,PD,high,3,3,0.6)

Figure 6: Optimal solutions for $q^A = q^B = 3$ and $\alpha = 0.6$

arcs. This is reflected in the hub arcs use, where in Figure 6(b) less than 15% of the total revenues are derived from trips that utilize a hub arc (Firms A and B derive only 9.1% and 5.2%, respectively, of their total revenues from trips that utilize a hub arc). In contrast, with the cost ratio (Figure 6(a)), the demand (e.g., freight) is routed via the hub arcs to take advantage of the cost savings. In this solution, two-thirds of the total revenues result from trips that utilize a hub arc, with Firms A and B deriving 40.9% and 24.7% of the total revenues, respectively, from trips that utilize a hub arc. Thus, the hub arcs in Figure 6(a) are designed to attract customers to the hub arcs, so they tend to reflect the underlying demand patterns (east-west and northeast-Florida).

These results in Table 7 and Figure 6 reflect some interesting behaviors that suggest strong differences in freight and passenger hub networks. These results highlight how optimal network design can be very sensitive to the details of the customer allocation mechanism in competitive problems.

4.6 Computation times

All the instances in this paper were solved with the SE algorithm described in Section 3. Details on the cpu times for all instances are available in Appendix B and Appendix C. In general, the cpu times with medium selectivity were greatest with a maximum of 9110 seconds for $q^A = q^B = 3$ and $\alpha = 0.6$. The cpu times with high selectivity were smallest and the maximum was only 1111 seconds for $q^A = q^B = 3$ and $\alpha = 0.2$. The cpu times for problems with customer allocation based on the cost ratio with high selectivity were similar to those based on the distance ratio with a maximum of 1044 seconds for $q^A = q^B = 3$ and $\alpha = 0.8$. The efficiency of the SE algorithm can be gauged from the cpu times as well as the reduced set of enumerations. For example, with the HALCE(RD,PD,high,3,3,0.6) problem shown in Figure 6(b) complete enumeration would require evaluating 4.28×10^{12} hub arc combinations for Firms A and B; the SE algorithm evaluated only 1.78×10^{7} of these combinations. Furthermore, for each hub arc combination the algorithm evaluated on average only 132 of the 300 OD pairs due to the upper bounding procedure for Firm B's revenue. Thus, the net effect is that the evaluation effort of the SE algorithm was about 1.83×10^{-4}% of that required for complete enumeration.

5 Conclusions

This paper provides a model and results for competitive hub arc location as a step towards the design of competitive large-scale transportation systems. We examined how the optimal solutions are affected by different customer allocation functions, different revenue sets, the number of hub arcs, and the degree of discount for hub arc travel, and we highlighted some interesting differences between the leader's and follower's optimal hub arcs and hubs. The results show a strong link in many cases between the level of demand at a city (i.e. originating and terminating traffic) and its usage as a hub. This is due in large part to the wide disparity in city sizes in the CAB data. However, the results also show how geography plays an important role in designing hub networks, with centrally located cities having a locational advantage over more peripheral cities. Our results highlight how optimal network design can be very sensitive to the details of the customer allocation mechanism in competitive problems - even though the amount of business captured in aggregate may be relatively insensitive to the allocation mechanism.

The optimal competitive hub networks with customer allocation based on the ratio of path distances reflect the passenger's desire for shorter routes with fewer stops at hubs. These networks may be very different than the optimal hub networks designed to minimize transportation cost, both in competitive revenue maximizing environments (e.g., with customer allocation based on the ratio of path distances) and in non-competitive cost minimizing environments (e.g., hub median and hub arc location models).

Research on competitive hub location problems is a new area, so there are considerable opportunities for future research. A few important ones are to develop better solution algorithms; to explore different customer allocation functions (especially based on real world data); to compare solutions with different revenue functions and different levels of demand; and to use other data sets. Extensions of our models include developing better bounding strategies in the SE algorithm and exploring variations of the customer allocation function. This could include different percentages as levels of the step function, different levels of customer selectivity (i.e., values of r_1 and r_2) and different measures of service to include the impact of stopovers at hubs. Research could also be directed at extending models to include revenues and costs.

References

[1] N. Adler and K. Smilowitz. Hub-and-spoke network alliances and mergers: Price-location competition in the airline industry. *Transportation Research Part B*, 41:394–409, 2007.

[2] S. Alumur and B. Kara. Network hub location problems: The state of the art. *European Journal of Operational Research*, 190:1–21, 2008.

[3] J.F. Campbell. Hub location for time definite transportation. *Computers and Operations Research*, 2009. doi:10.1016/j.cor.2008.01.009.

[4] J.F. Campbell, A.T. Ernst, and M. Krishnamoorthy. *Facility Location: Applications and Theory*, chapter Hub location problems, pages 373–407. Springer, Berlin, 2002.

[5] J.F. Campbell, A.T. Ernst, and M. Krishnamoorthy. Hub arc location problems: Part i–introduction and results. *Management Science*, 51:1540–1555, 2005.

[6] J.F. Campbell, A.T. Ernst, and M. Krishnamoorthy. Hub arc location problems: Part ii–formulations and optimal algorithms. *Management Science*, 51:1556–1571, 2005.

[7] H. Chen and A.M. Campbell. Network design for time-constrained delivery. *Naval Research Logistics*, 55:493–515, 2008.

[8] H.A. Eiselt and G. Laporte. Sequential location problems. *European Journal of Operational Research*, 96:217–231, 1996.

[9] H.A. Eiselt, G. Laporte, and J.-F. Thisse. Competitive location models: A framework and bibliography. *Transportation Science*, 27:44–54, 1993.

[10] H.A. Eiselt and V. Marianov. A conditional p-hub location problem with attraction functions. *Computers and Operations Research*, 2008. doi:10.1016/j.cor.2008.11.014.

[11] S.L. Hakimi. On locating new facilities in a competitive environment. *European Journal of Operational Research*, 12:29–35, 1983.

[12] A. Marín, L. Cánovas, and M. Landete. New formulations for the uncapacitated multiple allocation hub location problem. *European Journal of Operational Research*, 172:274–292, 2006.

[13] J.C. Marín. Analyzing competition for hub location in intercontinental aviation markets. *Transportation Research Part E*, 40:135–150, 2004.

[14] V. Marianov, D. Serra, and C. ReVelle. Location of hubs in a competitive environment. *European Journal of Operational Research*, 114:363–371, 1999.

[15] M.E. O'Kelly. A quadratic integer program for the location of interacting hub facilities. *European Journal of Operational Research*, 32:393–404, 1987.

[16] J. Pirkul and S. Soni. New formulations and solution procedures for the hop constrained network design problem. *European Journal of Operational Research*, 148:126–140, 2003.

[17] F. Plastria. Static competitive facility location: An overview of optimization approaches. *European Journal of Operational Research*, 129:461–470, 2001.

[18] F. Plastria and L. Vanhaverbeke. Discrete models for competitive location with foresight. *Computers and Operations Research*, 35:683–700, 2008.

[19] M. Sasaki. Hub network design model in a competitive environment with flow threshold. *Journal of the Operations Research Society of Japan*, 48:158–171, 2005.

[20] M. Sasaki and M. Fukushima. Stackelberg hub location problem. *Journal of the Operations Research Society of Japan*, 44:390–402, 2001.

[21] D. Serra and C. ReVelle. Market capture by two competitors: The pre-emptive location problem. *Journal of Regional Science 34*, 34:549–561, 1994.

[22] D. Serra and C. ReVelle. *Facility Location: A Survey of Applications and Methods*, chapter Competitive location in discrete space, pages 367–386. Springer-Verlag, New York, Berlin, 1995.

[23] B. Wagner. A note on "location of hubs in a competitive environment". *European Journal of Operational Research*, 184:57–62, 2008.

Appendix A Airfare vs. distance

Figure 7 shows a scatter plot of airfare vs. distance for all 300 OD pairs. The associated linear regression coefficient of determination is $R^2 = 0.189$. The outlier appearing near the Y axis corresponds to OD pair {Cleveland,Pittsburgh}. Five of 300 OD pairs did not appear in the original data, so their airfares were set to zero. The OD pairs with zero airfare are: (2,18), (2,25), (6,9), (13,21) and (14,24).

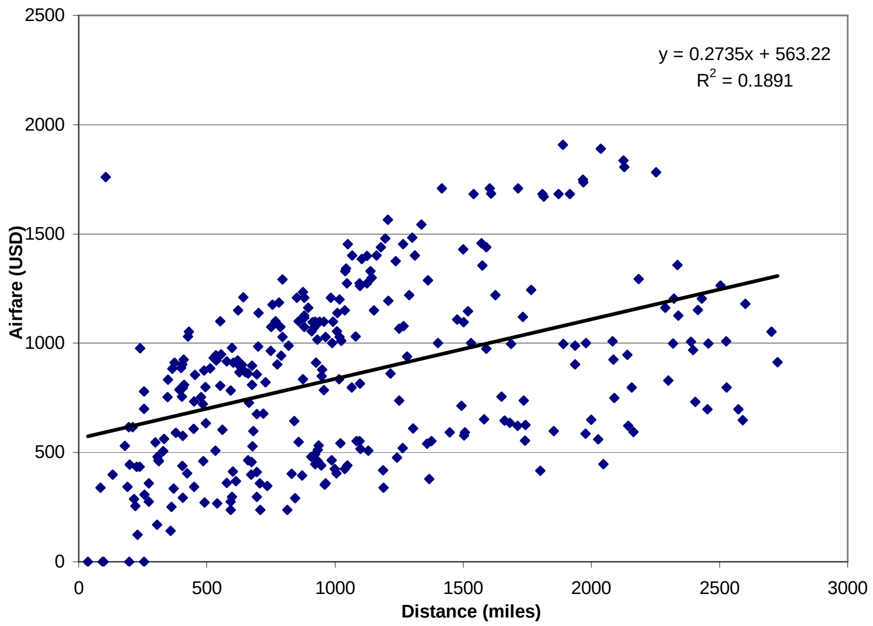


Figure 7: Airfare vs. distance

Appendix B Complete results with customer allocation based on the distance ratio

Tables 8 and 9 show detailed results using medium customer selectivity for the *Airfare* revenue set and for the *Distance* revenue set, respectively. Similarly, Tables **??** and **??** show detailed results using high customer selectivity for the *Airfare* revenue set and for the *Distance* revenue set, respectively.

Appendix C Complete results with customer allocation based on the cost ratio

Tables 12 - 17 show detailed results with customer allocation based on the cost ratio using either the *Airfare* or *Distance* revenue set and either the low, medium or high level of customer selectivity. Tables 12 and 13 show the results using low customer selectivity for the *Airfare* revenue set and for the *Distance* revenue set, respectively. Tables 14 and 15 show the results using medium customer selectivity for the *Airfare* revenue set and for the *Distance* revenue set, respectively. Tables 16 and 17 show the results using high customer selectivity for the *Airfare* revenue set and for the *Distance* revenue set, respectively.

Table 8: Optimal results for HALCE(RA,PD,medium,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.11	20-21	2-8	51.83	48.17
		0.4	0.11	20-21	6-11	50.87	49.13
		0.6	0.10	11-20	17-21	51.44	48.56
		0.8	0.10	11-20	17-21	51.11	48.89
		1.0	0.08	11-20	17-21	50.35	49.65
1	2	0.2	3.05	8-20	3-17 1-4	39.30	60.70
		0.4	3.15	8-20	16-18 6-11	38.87	61.13
		0.6	2.14	8-20	18-19 6-11	37.29	62.71
		0.8	1.07	4-17	15-21 2-6	37.29	62.71
		1.0	0.56	4-17	6-18 8-21	38.32	61.68
1	3	0.2	132.57	8-20	3-17 12-23 1-4	33.51	66.49
		0.4	120.56	8-20	16-18 5-7 6-11	34.21	65.79
		0.6	78.79	8-17	3-6 11-15 18-19	33.88	66.12
		0.8	58.04	4-17	3-18 11-15 5-6	33.82	66.18
		1.0	43.73	4-17	6-18 11-15 3-5	35.53	64.47
2	1	0.2	5.39	18-20 8-21	4-6	62.40	37.60
		0.4	5.40	4-8 13-18	20-21	62.85	37.15
		0.6	6.19	8-17 1-4	20-21	64.23	35.77
		0.8	4.85	8-17 1-4	11-20	63.82	36.18
		1.0	5.47	17-20 4-7	6-8	64.32	35.68
2	2	0.2	21.44	3-17 4-8	6-21 2-18	52.29	47.71
		0.4	30.27	8-17 1-4	2-16 6-11	52.27	47.73
		0.6	28.53	8-17 1-4	18-19 6-11	52.91	47.09
		0.8	22.27	8-17 1-4	7-11 2-6	52.81	47.19
		1.0	16.12	1-17 4-8	18-19 6-21	52.06	47.94
2	3	0.2	641.64	3-17 4-8	11-21 2-18 6-9	46.43	53.57
		0.4	744.22	8-17 1-4	3-18 5-9 7-11	46.84	53.16
		0.6	697.11	8-17 1-4	20-25 15-21 14-19	47.44	52.56
		0.8	541.17	8-17 1-4	6-25 15-21 14-19	47.40	52.60
		1.0	528.66	1-17 4-8	16-18 15-19 6-21	47.60	52.40
3	1	0.2	396.37	4-6 7-8 2-18	17-21	68.33	31.67
		0.4	413.81	4-8 6-21 2-13	17-20	68.20	31.80
		0.6	415.15	16-17 4-8 7-20	6-19	69.40	30.60
		0.8	424.14	4-8 13-17 6-20	11-18	69.05	30.95
		1.0	440.57	4-8 13-17 6-20	9-18	68.80	31.20
3	2	0.2	701.20	3-17 4-6 7-8	18-20 15-21	58.58	41.42
		0.4	1097.39	16-17 4-8 7-20	3-6 1-21	57.72	42.28
		0.6	929.31	7-17 1-4 6-8	5-9 18-19	58.42	41.58
		0.8	708.59	1-17 4-8 6-7	12-18 9-13	58.98	41.02
		1.0	787.38	1-17 4-6 7-8	9-18 12-21	58.07	41.93
3	3	0.2	5837.17	4-9 2-17 7-8	3-25 12-23 6-21	52.90	47.10
		0.4	8952.53	10-17 4-8 7-20	19-23 2-13 6-11	52.66	47.34
		0.6	6684.64	7-17 1-4 6-8	12-14 20-25 15-21	52.99	47.01
		0.8	3551.94	1-17 4-6 7-8	9-22 16-18 5-11	53.22	46.78
		1.0	2771.33	1-17 4-6 7-8	18-21 12-16 9-20	53.52	46.48

Table 9: Optimal results for HALCE(RD,PD,medium,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.14	2-21	11-25	50.30	49.70
		0.4	0.13	11-20	2-8	50.62	49.38
		0.6	0.13	8-20	2-4	51.00	49.00
		0.8	0.13	11-20	2-8	50.31	49.69
		1.0	0.11	8-20	2-4	49.27	50.73
1	2	0.2	2.47	1-4	11-21 18-25	38.52	61.48
		0.4	3.38	8-20	12-23 2-5	41.25	58.75
		0.6	2.16	8-20	4-17 14-19	40.19	59.81
		0.8	2.05	2-8	19-25 11-20	38.66	61.34
		1.0	1.61	2-4	1-9 11-18	39.00	61.00
1	3	0.2	92.60	1-4	6-17 5-25 11-13	35.81	64.19
		0.4	168.00	1-4	8-14 5-9 13-15	34.78	65.22
		0.6	87.38	4-17	3-18 11-15 5-9	35.24	64.76
		0.8	60.41	4-17	3-18 8-9 1-21	35.66	64.34
		1.0	67.08	4-17	8-18 3-21 6-24	36.95	63.05
2	1	0.2	5.28	4-8 2-3	6-21	61.19	38.81
		0.4	5.58	4-8 2-13	6-11	62.46	37.54
		0.6	4.47	4-8 2-7	11-25	62.15	37.85
		0.8	4.31	1-17 4-8	2-11	62.82	37.18
		1.0	3.92	1-17 4-8	2-11	62.58	37.42
2	2	0.2	22.84	4-8 2-3	13-25 6-11	52.27	47.73
		0.4	32.95	1-4 8-14	12-23 5-9	52.07	47.93
		0.6	22.30	4-8 2-7	17-20 14-19	52.04	47.96
		0.8	17.02	8-17 1-4	2-22 6-11	51.69	48.31
		1.0	13.91	4-12 1-17	6-7 2-8	51.66	48.34
2	3	0.2	759.48	4-8 2-3	12-23 5-21 17-18	46.37	53.63
		0.4	1021.93	1-4 8-14	12-23 7-24 5-9	46.33	53.67
		0.6	921.63	4-8 2-7	17-20 11-15 14-19	46.29	53.71
		0.8	461.71	1-17 4-8	6-25 11-15 14-19	46.66	53.34
		1.0	313.37	4-12 1-17	21-22 6-14 8-25	47.85	52.15
3	1	0.2	421.65	4-8 11-21 2-25	5-20	67.80	32.20
		0.4	457.62	4-6 7-8 2-18	9-11	67.27	32.73
		0.6	436.96	4-8 11-20 2-13	12-18	66.24	33.76
		0.8	428.71	1-4 7-8 2-17	12-25	66.50	33.50
		1.0	446.10	6-17 4-8 7-25	12-20	65.80	34.20
3	2	0.2	934.25	3-17 12-23 1-4	6-11 2-13	57.50	42.50
		0.4	1059.43	3-17 1-4 8-14	6-11 2-13	57.65	42.35
		0.6	1043.84	17-25 4-8 7-14	11-20 19-24	56.77	43.23
		0.8	935.49	1-17 8-12 4-7	2-22 6-11	56.26	43.74
		1.0	973.44	12-25 7-17 4-8	11-18 1-6	56.25	43.75
3	3	0.2	3380.91	3-17 12-23 1-4	13-21 6-9 2-18	53.75	46.25
		0.4	9014.08	3-17 1-4 8-14	12-22 6-11 2-13	51.70	48.30
		0.6	9109.82	17-20 1-4 8-14	6-11 19-24 2-13	51.48	48.52
		0.8	3579.95	1-17 8-12 4-7	14-22 6-25 11-13	51.86	48.14
		1.0	3719.48	1-17 7-12 4-8	6-14 11-22 19-25	51.90	48.10

Table 10: Optimal results for HALCE(RA,PD,high,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.09	20-21	8-17	51.61	48.39
		0.4	0.11	11-20	4-17	48.67	51.33
		0.6	0.08	8-17	1-4	49.64	50.36
		0.8	0.08	11-17	4-7	52.30	47.70
		1.0	0.06	11-17	5-8	54.86	45.14
1	2	0.2	0.91	8-17	4-6 12-23	36.37	63.63
		0.4	1.05	8-17	3-6 1-4	37.83	62.17
		0.6	0.83	8-17	12-22 3-20	39.03	60.97
		0.8	0.69	8-17	12-22 3-20	38.65	61.35
		1.0	0.36	4-17	3-21 9-20	43.03	56.97
1	3	0.2	26.84	4-17	14-20 6-14 21-24	32.71	67.29
		0.4	35.39	12-17	3-6 1-4 8-14	31.97	68.03
		0.6	50.56	12-17	3-6 1-4 14-22	34.42	65.58
		0.8	38.25	12-17	3-14 4-22 2-9	33.47	66.53
		1.0	27.09	4-17	3-9 14-20 12-21	38.71	61.29
2	1	0.2	4.09	3-17 11-15	1-4	65.65	34.35
		0.4	4.14	8-17 1-4	3-6	67.74	32.26
		0.6	4.23	12-17 4-13	1-6	70.05	29.95
		0.8	3.78	10-17 4-5	7-20	74.39	25.61
		1.0	3.78	1-17 4-12	20-21	75.82	24.18
2	2	0.2	10.75	12-17 4-8	3-9 2-21	55.55	44.45
		0.4	13.31	8-17 1-4	3-6 7-14	54.98	45.02
		0.6	10.44	12-17 1-4	5-9 8-14	58.61	41.39
		0.8	8.28	14-17 4-13	3-9 10-21	59.85	40.15
		1.0	7.08	1-17 4-12	11-14 9-20	62.06	37.94
2	3	0.2	283.68	12-17 4-8	3-9 11-15 5-25	51.43	48.57
		0.4	277.86	3-17 1-4	14-25 15-21 20-24	50.06	49.94
		0.6	214.33	12-17 1-4	3-6 8-14 7-15	51.10	48.90
		0.8	139.69	12-17 4-14	22-25 1-9 3-11	52.46	47.54
		1.0	201.68	7-17 4-12	9-22 3-21 10-25	53.26	46.74
3	1	0.2	400.92	3-17 12-23 1-4	6-21	77.38	22.62
		0.4	400.79	3-17 1-4 8-14	12-20	78.20	21.80
		0.6	400.14	3-17 1-4 8-14	12-20	78.12	21.88
		0.8	395.37	14-17 4-5 8-10	12-22	81.82	18.18
		1.0	396.01	7-17 12-25 4-9	20-21	83.44	16.56
3	2	0.2	460.69	3-17 12-23 1-4	20-24 16-21	67.14	32.86
		0.4	458.08	3-17 12-22 1-4	5-9 8-14	66.66	33.34
		0.6	477.95	3-17 12-14 1-4	5-9 10-15	67.13	32.87
		0.8	445.13	3-17 12-14 4-13	20-22 7-15	69.97	30.03
		1.0	440.38	7-17 12-25 4-9	3-14 21-22	71.16	28.84
3	3	0.2	1111.08	3-17 12-23 1-4	14-25 20-24 16-21	62.82	37.18
		0.4	1051.97	3-17 12-22 1-4	5-9 8-14 10-15	60.52	39.48
		0.6	1011.01	3-17 12-14 1-4	22-23 5-9 10-15	61.56	38.44
		0.8	779.09	14-17 12-22 1-4	9-25 3-10 15-21	62.93	37.07
		1.0	1087.78	7-17 12-25 4-9	3-21 6-14 2-22	63.09	36.91

Table 11: Optimal results for HALCE(RD,PD,high,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.08	1-4	20-21	54.40	45.60
		0.4	0.08	1-4	11-20	53.64	46.36
		0.6	0.08	1-4	11-20	52.14	47.86
		0.8	0.08	1-4	11-17	49.58	50.42
		1.0	0.06	4-17	20-21	56.73	43.27
1	2	0.2	0.59	4-17	1-9 11-14	36.51	63.49
		0.4	0.75	1-4	12-22 14-25	36.31	63.69
		0.6	0.31	12-17	14-22 1-4	37.08	62.92
		0.8	0.31	12-17	4-22 14-20	37.11	62.89
		1.0	0.44	4-17	12-20 3-21	43.62	56.38
1	3	0.2	23.25	4-17	12-23 6-14 21-24	34.19	65.81
		0.4	17.03	12-17	14-22 8-25 3-8	29.83	70.17
		0.6	14.52	12-17	4-14 8-22 20-25	33.21	66.79
		0.8	15.49	12-17	3-14 2-22 4-10	33.77	66.23
		1.0	15.19	12-17	3-14 4-25 7-22	38.75	61.25
2	1	0.2	3.89	12-23 1-4	5-25	68.22	31.78
		0.4	3.86	12-22 1-4	8-25	69.03	30.97
		0.6	3.81	12-14 1-4	17-22	70.60	29.40
		0.8	3.78	14-17 4-22	11-25	74.56	25.44
		1.0	3.69	17-22 4-12	21-25	76.75	23.25
2	2	0.2	8.30	3-17 1-4	20-24 16-21	56.06	43.94
		0.4	7.27	12-22 1-4	3-17 21-24	56.58	43.42
		0.6	6.36	12-14 1-4	3-17 15-21	58.57	41.43
		0.8	5.70	12-17 4-22	14-20 3-11	61.12	38.88
		1.0	5.84	17-22 4-12	14-20 3-21	66.77	33.23
2	3	0.2	124.49	12-23 1-4	3-17 3-25 11-15	51.53	48.47
		0.4	107.90	14-17 12-22	1-4 24-25 19-23	51.17	48.83
		0.6	94.94	14-17 12-22	1-4 10-23 20-25	54.32	45.68
		0.8	107.83	12-17 4-22	14-20 8-10 1-9	55.21	44.79
		1.0	138.21	17-22 4-12	8-25 3-9 14-21	60.61	39.39
3	1	0.2	409.86	3-17 12-23 1-4	11-14	80.07	19.93
		0.4	402.71	14-17 12-22 1-4	21-25	80.74	19.26
		0.6	410.71	14-17 12-22 1-4	21-25	82.99	17.01
		0.8	402.21	14-17 12-22 1-4	21-25	86.25	13.75
		1.0	400.78	14-17 4-22 12-25	9-21	84.75	15.25
3	2	0.2	435.55	3-17 12-23 1-4	20-24 16-21	72.15	27.85
		0.4	433.08	14-17 12-22 1-4	3-25 7-15	73.69	26.31
		0.6	433.07	14-17 12-22 1-4	3-25 7-15	74.58	25.42
		0.8	433.41	14-17 12-22 4-21	3-25 7-15	77.17	22.83
		1.0	432.75	14-17 4-22 12-25	2-3 9-11	76.56	23.44
3	3	0.2	648.97	3-17 12-23 1-4	14-25 20-24 16-21	67.80	32.20
		0.4	634.64	14-17 12-22 1-4	3-25 19-23 15-21	69.22	30.78
		0.6	629.72	14-17 12-22 1-4	24-25 7-15 5-9	69.34	30.66
		0.8	700.72	4-14 3-17 12-22	23-25 1-9 7-11	72.18	27.82
		1.0	827.12	14-17 4-22 12-25	1-18 3-21 8-9	71.47	28.53

Table 12: Optimal results for HALCE(RA,PC,low,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.05	12-20	18-22	49.83	50.17
		0.4	0.05	4-17	12-20	50.62	49.38
		0.6	0.03	4-17	2-9	50.55	49.45
		0.8	0.05	4-17	8-25	50.57	49.43
		1.0	0.05	4-17	8-25	50.57	49.43
1	2	0.2	0.63	4-17	12-20 9-18	45.23	54.77
		0.4	0.69	4-17	9-18 12-21	47.13	52.87
		0.6	0.74	4-17	7-12 2-9	47.93	52.07
		0.8	0.77	4-17	7-12 9-25	48.28	51.72
		1.0	0.8	4-17	8-25 9-10	48.41	51.59
1	3	0.2	52.19	4-17	9-18 6-12 24-25	42.42	57.58
		0.4	58.55	4-17	9-18 12-21 2-14	44.75	55.25
		0.6	65.37	4-17	7-12 2-9 1-3	46.58	53.42
		0.8	70.48	4-17	3-25 7-12 1-9	46.97	53.03
		1.0	71.53	4-17	3-12 7-9 1-2	47.15	52.85
2	1	0.2	2.16	6-17 4-12	20-22	55.90	44.10
		0.4	2.24	4-17 12-13	21-25	54.71	45.29
		0.6	2.34	4-17 11-12	5-25	53.57	46.43
		0.8	2.53	4-17 7-12	11-25	52.63	47.37
		1.0	2.55	1-17 4-12	7-25	52.57	47.43
2	2	0.2	4.69	6-17 4-12	9-18 20-22	51.80	48.20
		0.4	7.33	4-17 7-12	9-18 11-22	50.99	49.01
		0.6	9.08	4-12 17-20	9-18 7-19	50.79	49.21
		0.8	10.97	7-17 4-22	9-25 11-12	50.64	49.36
		1.0	11.05	7-17 4-22	12-25 9-11	50.69	49.31
2	3	0.2	156.97	6-17 4-12	9-18 14-20 13-22	49.16	50.84
		0.4	309.29	6-17 4-12	14-18 7-20 19-21	49.25	50.75
		0.6	438.2	4-17 7-12	2-9 11-22 1-3	49.51	50.49
		0.8	450.9	7-17 4-22	3-12 1-9 2-11	49.55	50.45
		1.0	550.34	7-17 4-22	3-12 9-25 11-16	49.60	50.40
3	1	0.2	250.76	6-17 4-12 14-20	7-25	59.15	40.85
		0.4	267.8	4-17 14-25 11-12	8-20	57.12	42.88
		0.6	286.11	4-17 7-12 5-24	21-25	54.87	45.13
		0.8	301.81	4-17 7-12 1-20	11-25	53.86	46.14
		1.0	314.29	1-17 4-7 12-20	13-25	53.60	46.40
3	2	0.2	295.91	4-17 12-20 14-18	2-7 6-19	54.43	45.57
		0.4	315.63	4-17 7-12 14-18	1-25 6-22	53.23	46.77
		0.6	378.23	1-17 7-12 4-20	3-25 8-21	52.15	47.85
		0.8	454.15	17-20 4-22 7-24	12-18 5-16	51.76	48.24
		1.0	446.61	1-17 4-7 20-22	3-8 5-25	51.73	48.27
3	3	0.2	673.73	4-17 14-18 2-12	3-21 19-25 1-20	52.11	47.89
		0.4	1145	1-17 4-18 9-12	7-25 3-6 19-21	51.24	48.76
		0.6	1279.06	1-17 7-12 4-20	3-25 15-22 5-10	51.00	49.00
		0.8	2452.72	7-17 20-22 1-4	8-12 3-5 13-25	50.65	49.35
		1.0	2151.81	1-17 4-7 20-22	9-12 3-15 2-13	50.67	49.33

Table 13: Optimal results for HALCE(RD,PC,low,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.05	12-20	6-25	52.29	47.71
		0.4	0.03	12-20	6-25	52.79	47.21
		0.6	0.05	12-20	4-25	50.28	49.72
		0.8	0.03	8-20	12-17	49.21	50.79
		1.0	0.03	4-17	12-20	53.31	46.69
1	2	0.2	1.14	12-20	18-22 6-24	45.99	54.01
		0.4	1.58	12-25	4-22 2-14	45.73	54.27
		0.6	1.27	4-17	7-12 5-25	47.54	52.46
		0.8	2.17	4-17	7-12 21-25	47.81	52.19
		1.0	2.92	13-17	7-12 4-25	48.07	51.93
1	3	0.2	32.66	12-17	4-18 21-22 24-25	42.91	57.09
		0.4	181.81	12-20	4-22 7-18 2-24	44.11	55.89
		0.6	118.44	4-17	11-12 1-7 2-9	46.60	53.40
		0.8	406.06	2-11	7-12 6-17 4-24	47.11	52.89
		1.0	347.78	13-17	4-12 3-25 7-24	47.51	52.49
2	1	0.2	2.19	12-17 6-14	20-22	55.42	44.58
		0.4	2.11	4-12 14-18	8-9	54.37	45.63
		0.6	2.12	14-17 4-12	6-19	53.43	46.57
		0.8	2.81	4-17 7-12	8-20	51.96	48.04
		1.0	2.87	4-22 1-17	9-12	51.86	48.14
2	2	0.2	2.97	12-17 4-22	8-14 5-18	54.80	45.20
		0.4	2.91	12-17 4-22	8-14 5-25	55.31	44.69
		0.6	3.17	4-17 5-12	9-22 1-18	54.46	45.54
		0.8	2.64	17-22 4-12	14-21 9-18	57.93	42.07
		1.0	2.97	17-22 4-12	3-25 14-21	57.94	42.06
2	3	0.2	172.33	14-17 4-12	6-22 1-25 11-18	48.90	51.10
		0.4	109.36	14-17 4-12	18-24 19-21 7-20	49.01	50.99
		0.6	258.87	4-17 7-12	14-25 11-22 9-18	49.66	50.34
		0.8	355.76	4-22 7-17	8-12 14-25 3-21	49.73	50.27
		1.0	445.36	4-22 7-17	12-16 3-9 11-25	49.87	50.13
3	1	0.2	249.86	17-22 5-12 8-19	4-18	59.12	40.88
		0.4	248.76	4-17 11-12 14-25	6-19	56.09	43.91
		0.6	250.39	14-17 4-17 7-12	8-25	54.66	45.34
		0.8	338.65	1-17 7-12 4-8	19-20	52.67	47.33
		1.0	367.01	17-24 4-8 7-22	11-25	52.41	47.59
3	2	0.2	283.07	17-22 5-12 8-19	21-23 2-24	55.56	44.44
		0.4	313.76	12-18 4-19 8-22	2-14 11-20	52.79	47.21
		0.6	407.21	14-17 4-17 7-12	11-22 1-18	51.61	48.39
		0.8	685.99	17-22 4-12 1-7	18-20 13-19	50.97	49.03
		1.0	681.83	4-22 1-17 7-12	3-21 19-25	50.97	49.03
3	3	0.2	479.61	4-17 12-20 21-22	1-18 3-19 6-8	60.07	39.93
		0.4	536.59	4-22 12-25 9-17	7-19 20-24 2-21	57.47	42.53
		0.6	685.98	4-17 6-12 8-22	9-23 19-21 18-25	56.44	43.56
		0.8	385.52	4-12 21-22 17-18	9-15 24-25 7-19	59.82	40.18
		1.0	345.47	14-17 4-22 12-25	3-18 8-9 7-21	60.55	39.45

Table 14: Optimal results for HALCE(RA,PC,medium,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.03	17-21	8-20	52.30	47.70
		0.4	0.05	17-21	11-18	52.11	47.89
		0.6	0.03	17-21	8-20	49.74	50.26
		0.8	0.05	11-20	17-21	50.29	49.71
		1.0	0.06	11-20	17-21	50.35	49.65
1	2	0.2	0.45	17-21	13-18 6-8	31.95	68.05
		0.4	0.47	17-21	8-9 13-18	33.65	66.35
		0.6	0.41	4-17	18-21 6-8	33.96	66.04
		0.8	0.44	4-17	18-21 6-8	36.29	63.71
		1.0	0.44	4-17	8-18 6-21	38.32	61.68
1	3	0.2	13.92	4-17	6-18 18-21 8-9	23.33	76.67
		0.4	14.88	4-17	18-20 18-21 8-9	26.25	73.75
		0.6	17.08	4-17	5-12 18-21 6-15	29.09	70.91
		0.8	22.33	4-17	6-8 15-20 2-21	32.92	67.08
		1.0	31.44	4-17	6-18 11-15 3-5	35.53	64.47
2	1	0.2	2.08	17-21 6-22	4-18	70.99	29.01
		0.4	2.05	17-21 6-12	9-18	70.69	29.31
		0.6	2.13	5-17 4-8	18-21	68.55	31.45
		0.8	2.22	4-17 8-20	18-21	66.79	33.21
		1.0	2.34	17-20 4-7	6-8	64.32	35.68
2	2	0.2	4.52	4-12 5-17	18-21 6-22	55.35	44.65
		0.4	5.69	17-21 5-12	2-4 7-20	54.57	45.43
		0.6	5.30	4-12 17-20	5-18 6-8	54.29	45.71
		0.8	8.55	1-17 4-12	13-18 6-8	52.10	47.90
		1.0	11.64	1-17 4-8	18-19 6-21	52.06	47.94
2	3	0.2	93.27	4-17 5-12	9-22 18-21 1-20	45.58	54.42
		0.4	116.30	4-17 12-20	18-21 5-22 2-6	45.65	54.35
		0.6	121.78	4-12 17-20	21-22 5-18 6-8	47.30	52.70
		0.8	211.68	4-12 17-20	9-18 5-7 6-8	46.96	53.04
		1.0	377.91	1-17 4-8	16-18 15-19 6-21	47.60	52.40
3	1	0.2	222.35	4-12 6-18 13-17	2-22	81.01	18.99
		0.4	235.13	6-17 4-12 2-13	5-18	79.28	20.72
		0.6	229.00	4-12 17-20 2-13	6-18	77.16	22.84
		0.8	232.83	4-17 8-20 13-18	6-12	73.73	26.27
		1.0	255.46	4-8 13-17 6-20	9-18	68.80	31.20
3	2	0.2	275.08	16-17 4-18 6-12	20-22 2-21	65.97	34.03
		0.4	274.62	4-12 6-18 13-17	11-22 2-5	65.56	34.44
		0.6	300.07	6-17 4-12 13-18	8-9 1-2	63.94	36.06
		0.8	357.26	6-17 4-8 12-13	18-21 19-20	61.10	38.90
		1.0	516.29	1-17 4-6 7-8	9-18 12-21	58.07	41.93
3	3	0.2	606.22	7-17 4-12 6-18	9-24 21-22 2-5	57.07	42.93
		0.4	635.00	6-17 4-12 7-18	9-22 20-24 2-21	58.04	41.96
		0.6	784.30	4-12 17-20 7-18	21-22 6-8 1-2	56.91	43.09
		0.8	1759.29	6-17 4-12 1-8	15-22 5-18 7-20	54.57	45.43
		1.0	1611.48	1-17 4-6 7-8	12-18 9-21 16-20	53.52	46.48

Table 15: Optimal results for HALCE(RD,PC,medium,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.05	12-20	5-18	54.33	45.67
		0.4	0.03	12-20	4-18	54.82	45.18
		0.6	0.05	12-20	2-4	52.59	47.41
		0.8	0.05	8-20	2-4	53.84	46.16
		1.0	0.06	8-20	2-4	49.27	50.73
1	2	0.2	0.27	12-20	6-22 5-18	32.60	67.40
		0.4	0.55	5-12	21-22 19-25	34.52	65.48
		0.6	0.52	12-20	17-22 5-19	35.06	64.94
		0.8	0.98	12-20	11-22 5-18	36.71	63.29
		1.0	1.23	2-4	1-9 11-18	39.00	61.00
1	3	0.2	10.38	12-20	4-19 6-22 5-18	23.90	76.10
		0.4	21.03	5-12	9-22 18-19 2-21	25.98	74.02
		0.6	14.39	4-12	9-22 18-19 8-21	27.99	72.01
		0.8	35.38	4-12	21-22 8-9 19-20	30.66	69.34
		1.0	49.02	4-17	8-18 3-21 6-24	36.95	63.05
2	1	0.2	1.86	17-21 12-20	4-22	72.67	27.33
		0.4	1.86	17-21 6-12	4-22	72.47	27.53
		0.6	1.92	4-8 2-12	5-17	69.65	30.35
		0.8	2.00	4-12 2-8	18-21	66.07	33.93
		1.0	2.11	1-17 4-8	2-11	62.58	37.42
2	2	0.2	3.58	4-22 2-12	5-17 19-21	55.38	44.62
		0.4	3.83	4-12 1-17	6-22 24-25	55.14	44.86
		0.6	3.67	4-12 1-17	6-22 18-21	55.66	44.34
		0.8	4.41	4-12 1-17	14-18 6-8	53.87	46.13
		1.0	9.03	4-12 1-17	6-7 2-8	51.66	48.34
2	3	0.2	61.22	4-22 2-12	9-19 18-21 20-24	48.61	51.39
		0.4	80.52	12-17 4-22	9-18 20-24 19-21	46.43	53.57
		0.6	84.05	4-12 5-17	14-25 21-22 6-19	46.65	53.35
		0.8	90.14	17-22 4-12	14-18 8-21 6-19	46.67	53.33
		1.0	157.06	4-12 1-17	22-25 8-14 6-21	47.85	52.15
3	1	0.2	217.06	4-22 17-21 12-20	1-2	79.56	20.44
		0.4	217.17	4-22 17-21 2-12	5-18	79.59	20.41
		0.6	218.44	4-8 2-12 5-17	18-22	78.11	21.89
		0.8	226.80	4-12 1-17 8-20	2-21	73.32	26.68
		1.0	252.08	6-17 4-8 7-25	12-20	65.80	34.20
3	2	0.2	265.83	4-22 17-21 2-12	1-18 6-8	69.95	30.05
		0.4	260.58	4-17 12-20 21-22	1-25 18-19	67.14	32.86
		0.6	274.03	4-17 22-25 12-21	2-19 13-20	63.91	36.09
		0.8	316.88	4-12 6-22 13-17	19-20 2-21	60.46	39.54
		1.0	601.97	12-25 7-17 4-8	1-18 6-11	56.25	43.75
3	3	0.2	471.49	12-17 4-22 18-21	1-2 2-9 13-19	62.52	37.48
		0.4	500.71	4-17 12-20 21-22	1-25 18-19 11-19	59.60	40.40
		0.6	608.80	4-17 12-21 20-22	19-25 6-8 13-18	57.34	42.66
		0.8	695.01	14-17 4-22 12-21	18-24 6-19 8-13	55.62	44.38
		1.0	2356.00	1-17 7-12 4-8	6-14 11-22 19-25	51.90	48.10

Table 16: Optimal results for HALCE(RA,PC,high,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.05	17-21	2-11	50.14	49.86
		0.4	0.03	17-21	2-11	51.32	48.68
		0.6	0.05	17-21	8-20	49.79	50.21
		0.8	0.03	4-17	11-20	51.72	48.28
		1.0	0.05	11-17	5-8	54.86	45.14
1	2	0.2	0.39	17-21	4-18 3-7	31.36	68.64
		0.4	0.39	17-21	4-8 1-25	31.65	68.35
		0.6	0.22	4-17	1-18 6-8	34.07	65.93
		0.8	0.22	4-17	8-9 1-25	37.32	62.68
		1.0	0.25	4-17	3-9 20-21	43.03	56.97
1	3	0.2	9.97	4-17	3-9 18-21 2-18	21.07	78.93
		0.4	9.06	4-17	18-21 8-9 18-25	21.87	78.13
		0.6	9.67	4-17	8-9 20-21 2-18	23.97	76.03
		0.8	11.05	4-17	8-9 20-21 2-18	28.41	71.59
		1.0	19.42	4-17	3-9 14-20 12-21	38.71	61.29
2	1	0.2	1.92	17-21 12-20	1-25	75.11	24.89
		0.4	1.91	17-21 12-20	4-25	74.19	25.81
		0.6	1.91	4-12 5-17	13-25	73.49	26.51
		0.8	1.89	6-17 12-21	5-8	74.00	26.00
		1.0	1.89	1-17 4-12	20-21	75.82	24.18
2	2	0.2	4.34	17-21 4-12	22-25 13-18	56.38	43.62
		0.4	4.78	17-21 9-12	4-22 18-20	55.89	44.11
		0.6	5.28	4-17 5-12	8-25 1-18	55.66	44.34
		0.8	4.67	6-17 4-12	18-25 5-8	57.07	42.93
		1.0	4.28	1-17 4-12	11-14 9-20	62.06	37.94
2	3	0.2	86.54	4-17 5-12	18-21 6-22 1-3	42.87	57.13
		0.4	90.64	4-12 5-17	18-21 1-25 19-20	43.76	56.24
		0.6	87.63	9-17 4-12	15-22 20-21 2-18	45.07	54.93
		0.8	112.41	4-12 17-20	8-9 5-18 24-25	47.59	52.41
		1.0	104.33	7-17 4-12	3-22 21-25 9-10	53.26	46.74
3	1	0.2	218.85	6-17 4-12 13-25	1-20	82.20	17.80
		0.4	216.55	6-17 4-12 13-25	2-7	82.68	17.32
		0.6	215.27	4-17 6-12 13-25	8-9	82.92	17.08
		0.8	218.19	17-20 4-8 1-12	13-25	81.23	18.77
		1.0	211.07	7-17 12-25 4-9	20-21	83.44	16.56
3	2	0.2	278.92	4-17 6-12 1-25	18-21 20-24	67.04	32.96
		0.4	284.57	6-17 4-12 13-25	21-22 1-2	67.52	32.48
		0.6	289.43	9-12 13-17 4-20	8-25 1-18	68.02	31.98
		0.8	300.57	17-20 4-8 12-13	1-25 9-19	67.58	32.42
		1.0	261.20	7-17 12-25 4-9	3-14 21-22	71.16	28.84
3	3	0.2	967.92	17-21 4-12 18-24	22-25 2-14 6-7	54.96	45.04
		0.4	820.38	6-17 4-12 7-25	9-24 19-20 2-21	56.73	43.27
		0.6	861.01	4-12 17-20 13-18	15-22 6-21 24-25	58.12	41.88
		0.8	1043.79	4-17 9-12 7-25	14-21 6-8 2-18	58.81	41.19
		1.0	709.49	7-17 12-25 4-9	3-21 6-14 2-22	63.09	36.91

Table 17: Optimal results for HALCE(RD,PC,high,q^A,q^B,α)

qA	qB	α	CPU sec.	A's hub arcs	B's hub arcs	A's share(%)	B's share(%)
1	1	0.2	0.05	12-20	6-25	52.27	47.73
		0.4	0.03	12-20	6-25	52.81	47.19
		0.6	0.05	12-20	4-25	50.44	49.56
		0.8	0.03	8-20	12-17	49.80	50.20
		1.0	0.03	4-17	20-21	56.73	43.27
1	2	0.2	0.34	5-12	21-22 2-19	27.25	72.75
		0.4	0.17	12-17	20-22 18-25	28.48	71.52
		0.6	0.23	4-12	19-25 8-21	31.40	68.60
		0.8	0.13	12-17	4-22 6-18	37.57	62.43
		1.0	0.30	4-17	3-12 20-21	43.62	56.38
1	3	0.2	12.42	12-20	4-22 6-18 1-19	20.96	79.04
		0.4	5.02	12-17	4-19 20-22 18-25	21.40	78.60
		0.6	9.02	12-17	4-22 18-20 2-18	21.91	78.09
		0.8	4.78	12-17	22-25 6-18 11-19	29.27	70.73
		1.0	10.19	12-17	4-22 3-25 7-14	38.75	61.25
2	1	0.2	1.89	4-17 5-12	2-19	72.54	27.46
		0.4	1.89	4-17 12-20	5-22	72.48	27.52
		0.6	1.88	4-17 5-12	8-25	72.97	27.03
		0.8	1.84	17-22 4-12	21-25	73.43	26.57
		1.0	1.83	17-22 4-12	21-25	76.75	23.25
2	2	0.2	3.11	12-17 4-22	19-21 18-25	55.56	44.44
		0.4	3.13	12-17 4-22	18-20 19-21	55.89	44.11
		0.6	3.05	17-22 4-12	8-21 18-25	56.61	43.39
		0.8	2.66	17-22 4-12	14-21 6-18	61.62	38.38
		1.0	2.91	17-22 4-12	3-14 20-21	66.77	33.23
2	3	0.2	40.89	12-17 4-22	9-23 19-21 18-25	48.39	51.61
		0.4	48.55	12-17 4-22	18-20 24-25 19-21	46.36	53.64
		0.6	48.38	17-22 4-12	14-21 18-20 19-25	44.91	55.09
		0.8	35.31	17-22 4-12	14-21 18-19 6-18	52.37	47.63
		1.0	69.77	17-22 4-12	3-25 8-14 9-21	60.61	39.39
3	1	0.2	217.42	17-21 12-20 1-22	4-14	82.07	17.93
		0.4	216.37	4-17 12-25 1-22	2-24	82.20	17.80
		0.6	225.85	4-17 12-20 1-22	14-25	82.00	18.00
		0.8	214.92	4-17 12-25 21-22	2-24	83.12	16.88
		1.0	215.62	14-17 4-22 12-25	9-21	84.75	15.25
3	2	0.2	264.51	12-17 4-22 18-21	16-25 2-9	69.50	30.50
		0.4	265.73	12-17 4-22 13-25	18-21 6-24	69.25	30.75
		0.6	274.35	4-22 12-20 13-17	11-19 2-6	67.89	32.11
		0.8	260.00	4-12 21-22 17-18	24-25 9-11	71.20	28.80
		1.0	249.61	14-17 4-22 12-25	3-9 2-11	76.56	23.44
3	3	0.2	467.36	4-22 12-18 17-21	8-25 1-20 13-19	62.37	37.63
		0.4	654.88	4-17 9-12 19-22	1-18 8-21 6-25	59.49	40.51
		0.6	929.15	4-17 18-22 5-12	21-25 20-24 19-21	58.95	41.05
		0.8	502.90	4-12 7-17 18-22	8-11 24-25 9-20	63.42	36.58
		1.0	341.00	14-17 4-22 12-25	3-18 9-21 1-8	71.47	28.53